Quantum Condensed Matter Physics - Lecture Notes

Chetan Nayak

November 5, 2004

Contents

Part I

Preliminaries

CHAPTER 1

Conventions, Notation, Reminders

1.1 Mathematical Conventions

Vectors will be denoted in boldface, $\mathbf{x}, \mathbf{E}$, or with a Latin subscript x_i, E_i, $i = 1, 2, \ldots, d$. Unless otherwise specified, we will work in $d = 3$ dimensions. Occasionally, we will use Greek subscripts, e.g. j_μ, $\mu = 0, 1, \ldots, d$ where the 0-component is the time-component as in $x_\mu = (t, x, y, z)$. Unless otherwise noted, repeated indices are summed over, e.g. $a_i b_i = a_1 b_1 + a_2 b_2 + a_3 b_3 = \mathbf{a} \cdot \mathbf{b}$

We will use the following Fourier transform convention:

$$
\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \frac{d\omega}{(2\pi)^{1/2}} \tilde{f}(\omega)\, e^{-i\omega t} \\
\tilde{f}(\omega) &= \int_{-\infty}^{\infty} \frac{dt}{(2\pi)^{1/2}} f(t)\, e^{i\omega t}
\end{aligned}
\tag{1.1}
$$

1.2 Plane Wave Expansion

A standard set of notations for Fourier transforms does not seem to exist. The diversity of notations appear confusing. The problem is that the normalizations are often chosen differently for transforms defined on the real space continuum and transforms defined on a real space lattice. We shall do the same, so that the reader is not confused when confronted with varied choices of normalizations.

1.2.1 Transforms defined on the continuum in the interval $[-L/2, L/2]$

Consider a function $f(x)$ defined in the interval $[-L/2, L/2]$ which we wish to expand in a Fourier series. We shall restrict ourselves to the commonly used periodic boundary condition, i. e., $f(x) = f(x + L)$. We can write,

$$f(x) = \frac{1}{\sqrt{L}} \sum_q f_q e^{iqx}, \tag{1.2}$$

Because the function has the period L, q must be given by $2\pi n/L$, where the integer $n = 0, \pm 1, \pm 2, \ldots$. Note that n takes all integer values between $-\infty$ and $+\infty$. The plane waves form a complete orthogonal set. So the inverse is

$$f_q = \frac{1}{\sqrt{L}} \int_{-L/2}^{L/2} dx e^{iqx} f(x). \tag{1.3}$$

Let us now take the limit $L \to \infty$, so that the interval between the successive values of q, $\Delta q = 2\pi/L$ then tend to zero, and we can convert the q-sum to an integral. For the first choice of the normalization we get

$$f(x) = \lim_{L\to\infty} \sqrt{L} \int_{-\infty}^{\infty} \frac{dq}{2\pi} f_q e^{iqx}, \tag{1.4}$$

and

$$\lim_{L\to\infty} \sqrt{L} f_q = \int_{-\infty}^{\infty} dx e^{iqx} f(x). \tag{1.5}$$

If we define $\tilde{f}(q) = \lim_{L\to\infty} \sqrt{L} f_q$, everything is fine, but note the asymmetry: the factor $(1/2\pi)$ appears in one of the integrals but not in the other, although we could have arranged, with a suitable choice of the normalization at the very beginning, so that both integrals would symmetrically involve a factor of $(1/\sqrt{2\pi})$. Note also that

$$\lim_{L\to\infty} L\delta_{q,q'} \to 2\pi\delta(q - q'). \tag{1.6}$$

These results are simple to generalize to the multivariable case.

1.2.2 Transforms defined on a real-space lattice

Consider now the case in which the function f is specified on a periodic lattice in the real space, of spacing a, i. e., $x_n = na$; $x_{N/2} = L/2$, $x_{-N/2} =$

$-L/2$, and $Na = L$. The periodic boundary condition implies that $f(x_n) = f(x_n + L)$. Thus, the Fourier series now reads

$$f(x_n) = \frac{1}{L}\sum_q f_q e^{iqx_n}. \tag{1.7}$$

Note that the choice of the normalization in Eq. (1.7) and Eq. (1.2) are different. Because of the periodic boundary condition, q is restricted to

$$q = \frac{2\pi m}{Na}, \tag{1.8}$$

but the integers m constitute a finite set. To see this note that our complete set of functions are invariant with respect to the shift $q \to q + G$, where the smallest such reciprocal vectors, G, are $\pm(2\pi/a)$. Thus the distinct set of q's can be chosen to be within the 1st Brillouin zone $-(\pi/a) < q \leq (\pi/a)$; accordingly, the distinct set of integers m can be restricted in the interval $-N/2 < m \leq N/2$. Therefore the number of distinct q's is equal to N, exactly the same as the number of the lattice sites in the real space. What about the orthogonality and the completeness of these set of plane waves? It is easy to see that

$$\sum_{n=0}^{N} e^{i(q-q')x_n} = N\delta_{q,q'} \overset{N\to\infty}{\longrightarrow} \frac{2\pi}{a}\delta(q-q'). \tag{1.9}$$

Note the consistency of Eq. (1.6) and Eq. (1.9). The completeness can be written as

$$\sum_{q\in 1stBZ} e^{iqx_n} = N\delta_{n,0}. \tag{1.10}$$

In the limit that $N \to \infty$, this equation becomes

$$\int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dq}{2\pi} e^{iqx_n} = \frac{1}{a}\delta_{n,0}. \tag{1.11}$$

The integration runs over a finite range of q, despite the fact that the lattice is infinitely large. Why shouldn't it? No matter how large the lattice is, the lattice periodicity has not disappeared. It is only in the limit $a \to 0$ that we recover the results of the continuum given above. To summarize, we started with a function which was *only* defined on a discrete set of lattice points; in the limit $N \to \infty$, this discreteness does not go away but the set $[q]$ approaches a bounded continuum. The function f_q is periodic with respect

to the reciprocal lattice vectors, i.e., the entire q space can be divided up into periodic unit cells, but clearly not in an unique manner.

Finally, the inverse Fourier series is given by

$$f_q = a \sum_n e^{-iqx_n} f(x_n). \tag{1.12}$$

In the limit $N \to \infty$,

$$f(x_n) = \int_{-\frac{\pi}{a}}^{\frac{\pi}{a}} \frac{dq}{2\pi} e^{iqx_n} f_q, \tag{1.13}$$

$$f_q = a \sum_n e^{-iqx_n} f(x_n). \tag{1.14}$$

The prefactor a in front of this sum is actually the volume of the unit cell in real space. You can now generalize all this to three dimensions and work out the consequences of various normalizations.

1.3 Quantum Mechanics

A quantum mechanical system is defined by a Hilbert space, $\mathcal{H}$, whose vectors are states, $|\psi\rangle$. There are linear operators, $\mathcal{O}_i$ which act on this Hilbert space. These operators correspond to physical observables. Finally, there is an inner product, which assigns a complex number, $\langle\chi|\psi\rangle$, to any pair of states, $|\psi\rangle$, $|\chi\rangle$. A state vector, $|\psi\rangle$ gives a complete description of a system through the expectation values, $\langle\psi|\mathcal{O}_i|\psi\rangle$ (assuming that $|\psi\rangle$ is normalized so that $\langle\psi|\psi\rangle = 1$), which would be the average values of the corresponding physical observables if we could measure them on an infinite collection of identical systems each in the state $|\psi\rangle$.

If for all vectors $|\psi\rangle$ and $|\chi\rangle$,

$$\langle\chi|\mathcal{L}|\psi\rangle = \langle\psi|\mathcal{O}|\chi\rangle^* . \tag{1.15}$$

then the operator $\mathcal{L}$ is the Hermitian adjoint of $\mathcal{O}$ and will be denoted by $\mathcal{O}^\dagger$. Here c^* is the complex conjugate of the complex number c. The notation follows Dirac and tacitly uses the dual vector space of bras $\{\, \langle\psi \mid \,\}$ corresponding to vector space of kets $\{\, \mid \psi\rangle \,\}$. Although the introduction of the dual vector space could be avoided, it is a very elegant and useful concept. Just see how ugly it would be if we were to define the scalar product of two vectors as $(|\chi\rangle, |\psi\rangle) = (|\psi\rangle, |\chi\rangle)^*$.

An Hermitian operator satisfies

$$\mathcal{O} = \mathcal{O}^\dagger \tag{1.16}$$

while a unitary operator satisfies

$$\mathcal{O}\mathcal{O}^\dagger = \mathcal{O}^\dagger\mathcal{O} = 1 \tag{1.17}$$

If $\mathcal{O}$ is Hermitian, then

$$e^{i\mathcal{O}} \tag{1.18}$$

is unitary. Given an Hermitian operator, $\mathcal{O}$, its eigenstates are orthogonal,

$$\langle\lambda'|\mathcal{O}|\lambda\rangle = \lambda\,\langle\lambda'|\lambda\rangle = \lambda'\,\langle\lambda'|\lambda\rangle \tag{1.19}$$

For $\lambda \neq \lambda'$,

$$\langle\lambda'|\lambda\rangle = 0 \tag{1.20}$$

If there are n states with the same eigenvalue, then, within the subspace spanned by these states, we can pick a set of n mutually orthogonal states. Hence, we can use the eigenstates $|\lambda\rangle$ as a basis for Hilbert space. Any state $|\psi\rangle$ can be expanded in the basis given by the eigenstates of $\mathcal{O}$:

$$|\psi\rangle = \sum_\lambda c_\lambda|\lambda\rangle \tag{1.21}$$

with

$$c_\lambda = \langle\lambda|\psi\rangle. \tag{1.22}$$

The Hamiltonian, or total energy, which we will denote by H, is a particularly important operator. Schrödinger's equation tells us that H determines how a state of the system will evolve in time.

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \tag{1.23}$$

If the Hamiltonian is independent of time, then we can define energy eigenstates,

$$H|E\rangle = E|E\rangle \tag{1.24}$$

which evolve in time according to:

$$|E(t)\rangle = e^{-i\frac{Et}{\hbar}}|E(0)\rangle \tag{1.25}$$

An arbitrary state can be expanded in the basis of energy eigenstates:

$$|\psi\rangle = \sum_i c_i|E_i\rangle. \tag{1.26}$$

It will evolve according to:

$$|\psi(t)\rangle = \sum_j c_j e^{-i\frac{E_j t}{\hbar}} |E_j\rangle. \tag{1.27}$$

The usual route for constructing the quantum mechanical description of a physical system (Hilbert space, inner product, operators corresponding to physical observables) leans heavily on the classical description. The classical variables p, q are promoted to quantum operators and the Poisson bracket relation $[p, q]_{\text{P.B.}} = 1$ becomes the commutator of the corresponding operators p, q: $[p, q] = -i$. Hilbert space is then constructed as the representation space for the algebra of the operators p, q. The theory is then "solved" by finding the eigenstates and eigenvalues of the Hamiltonian. With these in hand, we can determine the state of the system at an arbitrary time t, given its state at some initial time t_0, according to (1.26) and (1.27). This procedure is known as canonical quantization.

Let us carry this out explicitly in the case of a simple harmonic oscillator. The solution of the harmonic oscillator will be useful preparation for the Fock space construction of quantum field theory.

The harmonic oscillator is defined by the Hamiltonian,

$$H = \frac{1}{2}\,\omega\,\left(p^2 + q^2\right) \tag{1.28}$$

and the commutation relations,

$$[p, q] = -i \tag{1.29}$$

We define raising and lowering operators:

$$\begin{aligned} a &= (q + ip)/\sqrt{2} \\ a^\dagger &= (q - ip)/\sqrt{2} \end{aligned} \tag{1.30}$$

The Hamiltonian and commutation relations can now be written:

$$\begin{aligned} H &= \omega\left(a^\dagger a + \frac{1}{2}\right) \\ [a, a^\dagger] &= 1 \end{aligned} \tag{1.31}$$

We construct the Hilbert space of the theory by noting that (1.31) implies the commutation relations,

$$[H, a^\dagger] = \omega a^\dagger$$

$$[H, a] = -\omega a \tag{1.32}$$

These, in turn, imply that there is a ladder of states,

$$\begin{aligned} Ha^\dagger|E\rangle &= (E+\omega)\, a^\dagger|E\rangle \\ Ha|E\rangle &= (E-\omega)\, a|E\rangle \end{aligned} \tag{1.33}$$

This ladder will continue down to negative energies (which it can't since the Hamiltonian is manifestly positive definite) unless there is an $E_0 \geq 0$ such that

$$a|E_0\rangle = 0 \tag{1.34}$$

To find E_0, we need to find the precise action of a, $a^\dagger$ on energy eigenstates $|E\rangle$. From the commutation relations, we know that $a^\dagger|E\rangle \propto |E+\omega\rangle$. To get the normalization, we write $a^\dagger|E\rangle = c_E|E+\omega\rangle$. Then,

$$\begin{aligned} |c_E|^2 &= \langle E|aa^\dagger|E\rangle \\ &= E + \frac{\omega}{2} \end{aligned} \tag{1.35}$$

Hence,

$$\begin{aligned} a^\dagger|E\rangle &= \sqrt{E+\frac{\omega}{2}}\,|E+\omega\rangle \\ a|E\rangle &= \sqrt{E-\frac{\omega}{2}}\,|E-\omega\rangle \end{aligned} \tag{1.36}$$

From the second of these equations, we see that $a|E_0\rangle = 0$ if $E_0 = \omega/2$.

Thus, we can label the states of a harmonic oscillator by their integral $a^\dagger a$ eigenvalues, $|n\rangle$, with $n \geq 0$ such that

$$H|n\rangle = \omega\left(n+\frac{1}{2}\right)|n\rangle \tag{1.37}$$

and

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned} \tag{1.38}$$

These relations are sufficient to determine the probability of any physical observation at time t given the state of the system at time t_0.

In this book, we will be concerned with systems composed of many particles. At the most general and abstract level, they are formulated in precisely the same way as any other system, i.e. in terms of a Hilbert space with an

inner product acted on by operators corresponding to observables. However, there is one feature of this description which is peculiar to many-particle systems composed of *identical particles* and has no real classical analog: Hilbert space must furnish an irreducible representation of the permutation group acting on identical particles. We will briefly review this aspect of quantum many-particle systems.

When we have a system with many particles, we must now specify the states of all of the particles. If we have two distinguishable particles whose Hilbert spaces are spanned by the bases

$$|i, 1\rangle \tag{1.39}$$

where $i = 0, 1, \ldots$ are the states of particle 1 and

$$|\alpha, 2\rangle \tag{1.40}$$

where $\alpha = 0, 1, 2, \ldots$ are the states of particle 2. Then the two-particle Hilbert space is spanned by the set:

$$|i, 1; \alpha, 2\rangle \equiv |i, 1\rangle \otimes |\alpha, 2\rangle \tag{1.41}$$

Suppose that the two single-particle Hilbert spaces are identical, e.g. the two particles are in the same box. Then the two-particle Hilbert space is:

$$|i, j\rangle \equiv |i, 1\rangle \otimes |j, 2\rangle \tag{1.42}$$

If the particles are identical, however, we must be more careful because $|i, j\rangle$ and $|j, i\rangle$ must be physically the same state, i.e.

$$|i, j\rangle = e^{i\alpha}|j, i\rangle. \tag{1.43}$$

Applying this relation twice implies that

$$|i, j\rangle = e^{2i\alpha}|i, j\rangle \tag{1.44}$$

so $e^{i\alpha} = \pm 1$. The former corresponds to bosons, while the latter corresponds to fermions. The two-particle Hilbert spaces of bosons and fermions are respectively spanned by:

$$|i, j\rangle + |j, i\rangle \tag{1.45}$$

and

$$|i, j\rangle - |j, i\rangle \tag{1.46}$$

The n-particle Hilbert spaces of bosons and fermions are respectively spanned by:

$$\sum_{\pi} |i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle \tag{1.47}$$

and

$$\sum_{\pi} (-1)^{\pi} |i_{\pi(1)}, \ldots, i_{\pi(n)}\rangle \tag{1.48}$$

Here π denotes a permutation of the particles. In position space, this means that a bosonic wavefunction must be completely symmetric:

$$\psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = \psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n) \tag{1.49}$$

while a fermionic wavefunction must be completely antisymmetric:

$$\psi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n) = -\psi(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n) \tag{1.50}$$

1.4 Statistical Mechanics

The concept of partition function is central to equilibrium statistical mechanics. For a canonical ensemble that we shall frequently use, it is given by Z,

$$Z = \sum_{n} e^{-\beta E_n}. \tag{1.51}$$

where the temperature of the ensemble, T, is $1/k_B\beta$, and k_B is the Botzmann constant. Here E_n are the energy eigenvalues of the Hamiltonian. Given the partition function, the macroscopic properties can be calculated from the free energy, F,

$$F = -\frac{1}{\beta} \ln Z. \tag{1.52}$$

To make sure that a system is in equilibrium, we must make the scale of observation considerably greater than all the relevant time scales of the problem; however, in some cases it is not clear if we can reasonably achieve this condition.

Alternatively, we may, following Boltzmann, define entropy, S, in terms of the available phase space volume, $\Gamma(E)$, which is

$$S = k_B \ln \Gamma(E). \tag{1.53}$$

But how do we find $\Gamma(E)$? We must solve the equations of motion, that is, we must know the dynamics of the system, and the issue of equilibration

must be addressed. In contrast, in the canonical ensemble, the calculation of the partition function is a counting problem.

The Boltzmann formula can be reconciled with the ensemble approach of Gibbs. We must determine $\Gamma(E)$. In general, this is impossible without computing the trajectory of the system in the phase space. The recourse is to assume that $\Gamma(E)$ is the entire volume of the phase space allowed by the conservation laws. No matter how complicated the motion may be, *if* the system, in the course of time, visits every point in the phase space, all we need to do is to calculate the measure in the phase space corresponding to the conserved quantities. It is convenient to introduce quantum mechanics at this step to simplify the argument. According to quantum mechanics each point in the phase space corresponds to a quantum state. So, we simply have to count the number of states, and we write

$$\Gamma(E) = \sum_n \delta(E - E_n). \tag{1.54}$$

Equation (1.53), combined with Eq. (1.54), defines the microcannonical ensemble of Gibbs. But in "deriving" it, we did not have to invoke the notion of an ensemble.

We can go further and ask what would happen if we replaced the above formula by the following:

$$\Gamma'(E) = \sum_n e^{-\beta(E_n - E)}, \tag{1.55}$$

where β, for the moment, is an unknown positive number. You can show that the entropy defined by $\Gamma'(E)$ leads to the same thermodynamics as the one defined by $\Gamma(E)$, provided $\beta = 1/k_B T$. We have now arrived at the cannonical ensemble. This is curious; in Eq. (1.54) we only sum over states of energy E, but in (1.55) we seem to sum over all states. The reason for this miracle is the extensive nature of E and S. They are of order N ($\sim 10^{23}$). Consequently, the sum is so sharply peaked that practically all the weight is concentrated at E. Now, Eq. (1.55), combined with Eq. 1.53, leads to the same thermodynamics as you would obtain from a canonical ensemble.

Although the ensemble approach is quite elegant and convenient, uncritical use of it can be misleading. Suppose that you are given a Hamiltonian which has two widely separated scales, a very fast one and a very slow one. If the observation scale is longer than the shorter time scale, but smaller than the longer time scale, the slow degrees of freedom can be assumed to be constant. They cannot wander very much in the phase space. Thus, in

calculating the relevant volume of the phase space we must ignore the slow degrees of freedom, otherwise we would get an answer that will not agree with observations.

A simple well known example of two distinct time scales is the problem of ortho- and para-hydrogen. The spins of the nuclei in a hydrogen molecule can be either in a triplet state, or in a singlet state. The interaction between the nuclei is negligible and so is the interaction between the nuclei and the electronic spins that are in a singlet state. Thus, the ortho-para conversion takes time, on the order of days, while the momenta of the molecules equilibrate in a microscopic time scale. Therefore, the number of nuclei in the singlet state and the number of nuclei in the triplet state are separately constants of motion on the time scale of a typical experiment, and the free energies of these two subsytems must be added rather than the partition functions. Experimental observations strikingly confirm this fact.

When there are a few discrete set of widely separated scales, it is easy to apply our formulae, because it is clear what the relevant region of the phase space is. There are instances, however, where this is not the case, and there is a continuum of of time scales, extending from very short microscopic scales to very long macroscopic scales. The common amorphous material, window glass, falls into this category. If glass is to be described by a Hamiltonian, it is not sufficient to know all the states and sum over all of them; we must examine the actual dynamics of the system. Glass is known to exhibit many anomalous thermal properties, including a time dependent specific heat. In this respect, the Boltzmann formula, Eq. (1.53), can still be used. In principle, we could calculate the actual trajectories to determine the volume of the phase space sampled during the observation time. There is no need to use the hypothesis that $\Gamma(E)$ is the *total* volume allowed by the conservation laws. Of course, as far as we know, this formula is a postulate as well and is not derived from any other known laws of physics.

We still have to understand what we mean by an ensemble average when experiments are done on a single system. The ensemble average of an observable $\mathcal{O}$ is defined to be

$$\langle \mathcal{O} \rangle = \mathrm{Tr}\, \hat{\rho} \mathcal{O}, \tag{1.56}$$

where the density matrix ρ is given by

$$\begin{aligned} \hat{\rho} &= \sum_n w_n \left| n \right\rangle \left\langle n \right|, \\ &= \frac{1}{Z} \sum_n e^{-\beta E_n} \left| n \right\rangle \left\langle n \right|. \end{aligned} \tag{1.57}$$

It is more likely, however, that an experiment yields the most probable value of $\mathcal{O}$,that is, the value shared by most members of the ensemble. However, the distribution of the members in the ensemble is so strongly peaked for a macroscopic system that roughly only one member matters; fluctuations are insignificant in the thermodynamic limit defined by $N \to \infty, V \to \infty$ such that $\rho = \frac{N}{V}$ is a given number. The relative fluctuations in $\mathcal{O}$ is given by

$$\sqrt{\frac{< \mathcal{O}^2 > - < \mathcal{O} >^2}{< \mathcal{O} >^2}} \sim O\left(\frac{1}{\sqrt{N}}\right) \tag{1.58}$$

which is insignificant when $N \sim 10^{23}$. Thus, the most probable value is the only value, hence the mean value.

Another useful partition function is the grand canonical partition function Z_G defined by,

$$Z_G = \mathrm{Tr}\ e^{-\beta(H-\mu N)}. \tag{1.59}$$

In this ensemble, the number of particles is not fixed, and the system is assumed to be in contact with a particle bath as well as a heat bath. In the definition of the trace one must also include a sum over a number of particles. The average number of particles is determined by the chemical potential μ. It is convenient to think of chemical potential as a "force" and the number of particles as a "coordinate", similar to a mechanical system in which a force fixes the conjugate coordinate. As in mechanical equilibrium, in which all the forces must balance, in a statistical equilibrium the chemical potentials for all the components must balance, that is, must be equal. It is also possible to give a similar interpretation to our formula for the canonical ensemble where we can take the temperature as the "force" and the entropy as the corresponding "coordinate". For the grand canonical ensemble, we define the grand potential, Ω:

$$\begin{aligned} \Omega &= -\frac{1}{\beta} \ln Z_G \\ &= F - \mu N. \end{aligned} \tag{1.60}$$

All thermodynamic quantities can be calculated from these definitions.

Actually, we could go on, and define more and more ensembles. For example, we may assume that, in addition, pressure P is not constant and define a pressure ensemble, in which we add a term $-PV$ in the exponent. For every such extension, we would add a "force" multiplied by the corresponding conjugate "coordinate". We could also consider an ensemble in which the linear momentum is not fixed etc.

The definition of the free energies allow us to calculate various thermodynamic quantities. Since

$$F = E - TS \tag{1.61}$$

and

$$dE = TdS - PdV + \mu dN, \tag{1.62}$$

we get

$$dF = -SdT - PdV + \mu dN. \tag{1.63}$$

Then,

$$S = -\left(\frac{\partial F}{\partial T}\right)_{V,N}, \tag{1.64}$$

$$P = -\left(\frac{\partial F}{\partial V}\right)_{T,N}, \tag{1.65}$$

$$\mu = -\left(\frac{\partial F}{\partial N}\right)_{T,V}. \tag{1.66}$$

Similarly, from the definition of the definition of the thermodynamic potential Ω, we can derive the same relations as

$$S = -\left(\frac{\partial \Omega}{\partial T}\right)_{V,\mu}, \tag{1.67}$$

$$P = -\left(\frac{\partial \Omega}{\partial V}\right)_{T,\mu}, \tag{1.68}$$

$$\mu = -\left(\frac{\partial \Omega}{\partial N}\right)_{T,V}. \tag{1.69}$$

Part II

Basic Formalism

CHAPTER 2

Phonons and Second Quantization

2.1 Classical Lattice Dynamics

Consider the lattice of ions in a solid. Suppose the equilibrium positions of the ions are the sites $\vec{R}_i$. Let us describe small displacements from these sites by a displacement field $\vec{u}(\vec{R}_i)$. We will imagine that the crystal is just a system of masses connected by springs of equilibrium length a.

At length scales much longer than its lattice spacing, a crystalline solid can be modelled as an elastic medium. We replace $\vec{u}(\vec{R}_i)$ by $\vec{u}(\vec{r})$ (i.e. we replace the lattice vectors, $\vec{R}_i$, by a continuous variable, $\vec{r}$). Such an approximation is valid at length scales much larger than the lattice spacing, a, or, equivalently, at wavevectors $q \ll 2\pi/a$.

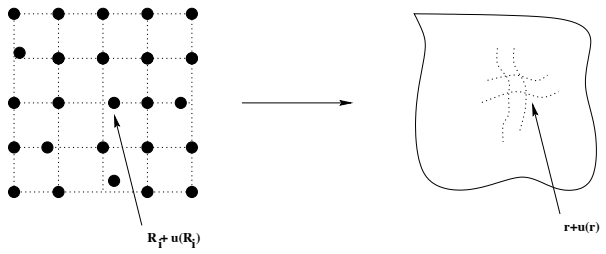


Figure 2.1: A crystalline solid viewed as an elastic medium.

The potential energy of the elastic medium must be translationally and rotationally invariant (at shorter distances, these symmetries are broken to discrete lattice symmetries, but let's focus on the long-wavelength physics for now). Translational invariance implies $V[\vec{u}+\vec{u}_0] = V[\vec{u}]$, so V can only be a function of the derivatives, $\partial_i u_j$. Rotational invariance implies that it can only be a function of the symmetric combination,

$$u_{ij} \equiv \frac{1}{2}\left(\partial_i u_j + \partial_j u_i\right) \tag{2.1}$$

There are only two possible such terms, $u_{ij}u_{ij}$ and u_{kk}^2 (repeated indices are summed). A third term, u_{kk}, is a surface term and can be ignored. Hence, the action of a crystalline solid to quadratic order, viewed as an elastic medium, is:

$$S_0 = \int dt d^3\vec{r}\,\mathcal{L} = \frac{1}{2}\int dt d^3\vec{r}\left[\rho(\partial_t u_i)^2 \,-\, 2\mu u_{ij}u_{ij} \,-\, \lambda u_{kk}^2\right] \tag{2.2}$$

where ρ is the mass density of the solid and μ and λ are the Lamé coefficients. Under a dilatation, $\vec{u}(\vec{r}) = \alpha\vec{r}$, the change in the energy density of the elastic medium is $\alpha^2(\lambda + 2\mu/3)/2$; under a shear stress, $u_x = \alpha y, u_y = u_z = 0$, it is $\alpha^2\mu/2$. In a crystal – which has only a discrete rotational symmetry – there may be more parameters than just μ and λ, depending on the symmetry of the lattice. In a crystal with cubic symmetry, for instance, there are, in general, three independent parameters. We will make life simple, however, and make the approximation of full rotational invariance.

2.2 The Normal Modes of a Lattice

Let us expand the displacement field in terms of its normal-modes. The equations of motion which follow from (2.2) are:

$$\rho\partial_t^2 u_i = (\mu+\lambda)\,\partial_i\partial_j u_j + \mu\partial_j\partial_j u_i \tag{2.3}$$

The solutions,

$$u_i(\vec{r},t) = \epsilon_i\, e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{2.4}$$

where is a unit polarization vector, satisfy

$$-\rho\omega^2\epsilon_i = -\,(\mu+\lambda)\,k_i\,(k_j\epsilon_j) - \mu k^2\epsilon_i \tag{2.5}$$

For longitudinally polarized waves, $k_i = k\epsilon_i$,

$$\omega_k^l = \pm\sqrt{\frac{2\mu+\lambda}{\rho}}\,k \equiv \pm v_l k \tag{2.6}$$

while transverse waves, $k_j \epsilon_j = 0$ have

$$\omega_k^t = \pm \sqrt{\frac{\mu}{\rho}}\, k \equiv \pm v_s k \tag{2.7}$$

Hence, the general solution of (2.3) is of the form:

$$u_i(\vec{r},t) = \sum_{k,s} \frac{1}{\sqrt{2\rho\omega_k^s}}\, \epsilon_i^s \left(a_{\vec{k},s} e^{i(\vec{k}\cdot\vec{r}-\omega_k^s t)} \,+\, a_{\vec{k},s}^\dagger e^{-i(\vec{k}\cdot\vec{r}-\omega_k^s t)} \right) \tag{2.8}$$

$s = 1, 2, 3$ corresponds to the longitudinal and two transverse polarizations. The normalization factor, $1/\sqrt{2\rho\omega_k^s}$, was chosen for later convenience.

The allowed $\vec{k}$ values are determined by the boundary conditions in a finite system. For periodic boundary conditions in a cubic system of size $V = L^3$, the allowed $\vec{k}$'s are $\frac{2\pi}{L}(n_1, n_2, n_3)$. Hence, the $\vec{k}$-space volume per allowed $\vec{k}$ is $(2\pi)^3/V$. Hence, we can take the infinite-volume limit by making the replacement:

$$\begin{aligned} \sum_k f(\vec{k}) &= \frac{1}{(\Delta\vec{k})^3} \sum_k f(\vec{k})\, (\Delta\vec{k})^3 \\ &= \frac{V}{(2\pi)^3} \int d^3\vec{k}\, f(\vec{k}) \end{aligned} \tag{2.9}$$

It would be natural to use this in defining the infinite-volume limit, but we will, instead, use the following, which is consistent with our Fourier transform convention:

$$u_i(\vec{r},t) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sum_s \frac{1}{\sqrt{2\rho\omega_k^s}}\, \epsilon_i^s \left(a_{\vec{k},s} e^{i(\vec{k}\cdot\vec{r}-\omega_k^s t)} \,+\, a_{\vec{k},s}^\dagger e^{-i(\vec{k}\cdot\vec{r}-\omega_k^s t)} \right) \tag{2.10}$$

2.3 Canonical Formalism, Poisson Brackets

The *canonical conjugate* to our classical field, u_i, is

$$\pi_i \equiv \frac{\partial \mathcal{L}}{\partial(\partial_t u_i)} = \rho \partial_t u_i \tag{2.11}$$

The Hamiltonian is given by

$$H = \int d^3\vec{r}\, \pi_i \partial_t u_i - \mathcal{L}$$

$$= \frac{1}{2}\int d^3\vec{r}\left[\rho(\partial_t u_i)^2 \,+\, 2\mu u_{ij}u_{ij} \,+\, \lambda u_{kk}^2\right]$$
$$= \frac{1}{2}\int d^3\vec{r}\left[\frac{1}{\rho}\,\pi_i^2 \,+\, 2\mu u_{ij}u_{ij} \,+\, \lambda u_{kk}^2\right] \tag{2.12}$$

Let us define the functional derivative,

$$\frac{\delta}{\delta\eta}\int d^3\vec{r}\,\mathcal{F}(\vec{r}) = \frac{\partial\mathcal{F}}{\partial\eta} \,-\, \frac{\partial\mathcal{F}}{\partial(\partial_i\eta)} \tag{2.13}$$

Then the equation of motion for π_i can be written

$$\partial_t\pi_i = -\,\frac{\delta H}{\delta u_i} \tag{2.14}$$

while

$$\partial_t u_i = \frac{\delta H}{\delta\pi_i} \tag{2.15}$$

From these equations, we see that it is natural to define the Poisson brackets:

$$[U,V]_{\rm PB} = \int d^3\vec{r}\,\left(\frac{\delta U}{\delta u_i}\frac{\delta V}{\delta\pi_i} \,-\, \frac{\delta U}{\delta\pi_i}\frac{\delta V}{\delta u_i}\right) \tag{2.16}$$

With this definition,

$$[u_j(\vec{r}),\pi_i(\vec{r'})]_{\rm PB} = \delta\left(\vec{r}-\vec{r'}\right) \tag{2.17}$$

and

$$\partial_t\pi_i = [\pi_i,H]_{\rm PB}$$
$$\partial_t u_i = [u_i,H]_{\rm PB} \tag{2.18}$$

As we will see shortly, the normalization chosen above for the normal mode expansion of $u_i(\vec{r})$ is particularly convenient since it leads to:

$$[a_{\vec{k},s},a^\dagger_{\vec{k'},s'}]_{\rm PB} \;=\; -i\,\delta_{ss'}\,\delta\left(\vec{k}-\vec{k'}\right) \tag{2.19}$$

When we quantize a classical field theory, we will promote the Poisson brackets to commutators, $[,]_{\rm PB} \to i\,[,]$.

2.4 Motivation for Second Quantization

The action (2.2) defines a classical field theory. It has 3 degrees of freedom *per spatial point* – i.e. it has infinitely many degrees of freedom. This is a consequence of the continuum limit which we took. A real finite-size sample of a solid has a finite number of degrees of freedom: if there are N ions, there are $3N$ degrees of freedom, $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$. However, it is extremely convenient to take the continuum limit and ignore the difference between $3N$ and ∞. Furthermore, we will also be concerned with the electromagnetic field, $\vec{E} = -\nabla\varphi - \partial_t \vec{A}$, $\vec{B} = \nabla \times \vec{A}$, which *does* have infinitely many degrees of freedom (2 per spatial point when gauge invariance is taken into account). By going to the continuum limit, we can handle the electromagnetic field and an elastic medium in a parallel fashion which greatly facilitates calculations. We thereby make a transition from classical particle mechanics (with a discrete number of degrees of freedom) to classical field theory (with continuously many degrees of freedom):

$$\begin{aligned} \vec{r}_a &\leftrightarrow \vec{u}(\vec{x}, t) \\ t &\leftrightarrow t \\ a &\leftrightarrow \vec{x} \end{aligned} \tag{2.20}$$

At the quantum level, we will be dealing with wavefunctionals of the form $\Psi[\vec{u}(\vec{r})]$ or $\Psi[\vec{A}(\vec{r})]$ rather than $\psi(\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N)$. The coordinates $\vec{r}$ are no more than indices (but continuous ones) on the fields. Hence, the operators of the theory will be $\vec{u}(\vec{r})$, $\partial_t \vec{u}(\vec{r})$ or $\vec{A}(\vec{r})$, $\partial_t \vec{A}(\vec{r})$ rather than $\vec{r}_a, \vec{p}_a$.

In this approach, the basic quantities will be the normal modes of the displacement field, rather than the ionic coordinates. As we will see below, the collective excitations of an elastic medium are particle-like objects – phonons – *whose number is not fixed*. Phonons are an example of the *quasiparticle* concept. In order to deal with particles whose number is not fixed (in contrast with the ions themselves, whose number is fixed), we will have to develop the formalism of second quantization. [1]

2.5 Canonical Quantization of Continuum Elastic Theory: Phonons

2.5.1 Review of the Simple Harmonic Oscillator

No physics course is complete without a discussion of the simple harmonic oscillator. Here, we will recall the operator formalism which will lead naturally to the Fock space construction of quantum field theory.

The harmonic oscillator is defined by the Hamiltonian,

$$H = \frac{1}{2}\,\omega\,\left(p^2 + q^2\right) \tag{2.21}$$

and the commutation relations,

$$[p, q] = -i \tag{2.22}$$

We define raising and lowering operators:

$$\begin{aligned} a &= (q + ip)/\sqrt{2} \\ a^\dagger &= (q - ip)/\sqrt{2} \end{aligned} \tag{2.23}$$

The Hamiltonian and commutation relations can now be written:

$$\begin{aligned} H &= \omega\left(a^\dagger a + \frac{1}{2}\right) \\ [a, a^\dagger] &= 1 \end{aligned} \tag{2.24}$$

The commutation relations,

$$\begin{aligned} [H, a^\dagger] &= \omega a^\dagger \\ [H, a] &= -\omega a \end{aligned} \tag{2.25}$$

imply that there is a ladder of states,

$$\begin{aligned} Ha^\dagger|E\rangle &= (E + \omega)\,a^\dagger|E\rangle \\ Ha|E\rangle &= (E - \omega)\,a|E\rangle \end{aligned} \tag{2.26}$$

This ladder will continue down to negative energies (which it can't since the Hamiltonian is manifestly positive definite) unless there is an $E_0 \geq 0$ such that

$$a|E_0\rangle = 0 \tag{2.27}$$

Such a state has $E_0 = \omega/2$.

We label the states by their $a^\dagger a$ eigenvalues. We have a complete set of H eigenstates, $|n\rangle$, such that

$$H|n\rangle = \omega\left(n + \frac{1}{2}\right)|n\rangle \tag{2.28}$$

and $(a^\dagger)^n|0\rangle \propto |n\rangle$. To get the normalization, we write $a^\dagger|n\rangle = c_n|n+1\rangle$. Then,

$$|c_n|^2 = \langle n|aa^\dagger|n\rangle$$

$$= n + 1 \tag{2.29}$$

Hence,

$$\begin{aligned} a^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \\ a|n\rangle &= \sqrt{n}|n-1\rangle \end{aligned} \tag{2.30}$$

2.5.2 Fock Space for Phonons

A quantum theory is made of the following ingredients:

- A Hilbert Space $\mathcal{H}$ of states $|\psi\rangle \in \mathcal{H}$.
- Operators $\mathcal{O}_i$ on $\mathcal{H}$, corresponding to physical observables.
- An Inner Product $\langle\chi|\psi\rangle$ which must be defined so that $\mathcal{O}_i$ is Hermitian with respect to it if the corresponding physical observable is real.

In order to construct these objects for an elastic medium – thereby *quantizing our classical field theory* – we employ the following procedure. We replace the classical variables, u_i, π_i by quantum operators satisfying the canonical commutation relations:

$$\begin{aligned} \left[u_i\left(\vec{x},t\right), u_j\left(\vec{x}',t\right)\right] = \left[\pi_i\left(\vec{x},t\right), \pi_j\left(\vec{x}',t\right)\right] &= 0 \\ \left[u_i\left(\vec{x},t\right), \pi_j\left(\vec{x}',t\right)\right] &= i\,\delta_{ij}\delta\left(\vec{x}-\vec{x}'\right) \end{aligned} \tag{2.31}$$

We can now define the operators $a_{\vec{k},s}$, $a^\dagger_{\vec{k},s}$ according to:

$$\begin{aligned} a_{\vec{k},s} &= \frac{1}{2}\,\epsilon_i^s\sqrt{2\rho\omega_k^s}\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\left(u_i\left(\vec{x},0\right) + \frac{i}{\omega_k^s}\,\partial_t u_i\left(\vec{x},0\right)\right)e^{i\vec{k}\cdot\vec{x}} \\ a^\dagger_{\vec{k},s} &= \frac{1}{2}\,\epsilon_i^s\sqrt{2\rho\omega_k^s}\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\left(u_i\left(\vec{x},0\right) - \frac{i}{\omega_k^s}\,\partial_t u_i\left(\vec{x},0\right)\right)e^{-i\vec{k}\cdot\vec{x}} \end{aligned} \tag{2.32}$$

These expressions can be inverted to give the normal-mode expansion, (8.72). Using $\pi_i = \rho\partial_t u_i$, and the above commutation relations, we see that $a_{\vec{k},s}$ and $a^\dagger_{\vec{k},s}$ satisfy the commutation relations:

$$\begin{aligned} \left[a_{\vec{k},s}, a^\dagger_{\vec{k}',s'}\right] = \frac{1}{2}\rho\sqrt{\omega_k^s\omega_{k'}^{s'}}\,\epsilon_i^s\,\epsilon_j^{s'}\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\int\frac{d^3\vec{x}'}{(2\pi)^{3/2}} \\ \left[\left(u_i\left(\vec{x},0\right) + \frac{i}{\omega_k^s}\,\partial_t u_i\left(\vec{x},0\right)\right)e^{i\vec{k}\cdot\vec{x}}, \left(u_j\left(\vec{x}',0\right) - \frac{i}{\omega_{k'}^{s'}}\,\partial_t u_j\left(\vec{x}',0\right)\right)e^{-i\vec{k}'\cdot\vec{x}'}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\rho\sqrt{\omega_k^s\omega_{k'}^{s'}}\,\epsilon_i^s\,\epsilon_j^{s'}\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\int\frac{d^3\vec{x}'}{(2\pi)^{3/2}}\,e^{i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{x}'}\,(\left[u_i\left(\vec{x},0\right),-\frac{i}{\omega_{k'}^{s'}}\,\partial_t u_j\left(\vec{x'},0\right)\right]+\\
&\qquad\qquad\left[\frac{i}{\omega_k^s}\,\partial_t u_i\left(\vec{x},0\right),u_j\left(\vec{x'},0\right)\right])\\
&= \frac{1}{2}\sqrt{\omega_k^s\omega_{k'}^{s'}}\,\epsilon_i^s\,\epsilon_j^{s'}\int\frac{d^3\vec{x}}{(2\pi)^{3/2}}\int\frac{d^3\vec{x}'}{(2\pi)^{3/2}}\left(\frac{1}{\omega_k^s}+\frac{1}{\omega_{k'}^{s'}}\right)\delta_{ij}\delta\left(\vec{x}-\vec{x'}\right)\,e^{i\vec{k}\cdot\vec{x}-i\vec{k}'\cdot\vec{x}'}\\
&= \delta_{ss'}\,\delta\left(\vec{k}-\vec{k}'\right)
\end{aligned}
\tag{2.33}
$$

We can similarly show that

$$
\left[a_{\vec{k},s},a_{\vec{k}',s'}\right]=\left[a_{\vec{k},s}^\dagger,a_{\vec{k}',s'}^\dagger\right]=0 \tag{2.34}
$$

We can re-write the Hamiltonian, H, in terms of $a_{\vec{k},s}$ and $a_{\vec{k},s}^\dagger$ by substituting (8.72) into (2.12).

$$
\begin{aligned}
H = \frac{1}{2}\int\frac{d^3\vec{k}}{2\rho\omega_k^s}\,(&a_{\vec{k},s}a_{-\vec{k},s}e^{-2i\omega_k^s t}\left(-\rho(\omega_k^s)^2+\mu k^2+\delta_{s1}\Big(\mu+\lambda\Big)k^2\right)\\
&+a_{\vec{k},s}a_{\vec{k},s}^\dagger\left(\rho(\omega_k^s)^2+\mu k^2+\delta_{s1}\Big(\mu+\lambda\Big)k^2\right)\\
&+a_{\vec{k},s}^\dagger a_{\vec{k},s}\left(\rho(\omega_k^s)^2+\mu k^2+\delta_{s1}\Big(\mu+\lambda\Big)k^2\right)\\
&+a_{\vec{k},s}^\dagger a_{-\vec{k},s}^\dagger e^{2i\omega_k^s t}\left(-\rho(\omega_k^s)^2+\mu k^2+\delta_{s1}\Big(\mu+\lambda\Big)k^2\right))\\
=\frac{1}{2}\int d^3\vec{k}\,\omega_k^s\,&\left(a_{\vec{k},s}a_{\vec{k},s}^\dagger+a_{\vec{k},s}^\dagger a_{\vec{k},s}\right)\\
=\int d^3\vec{k}\,\omega_k^s\,&\left(a_{\vec{k},s}^\dagger a_{\vec{k},s}+\frac{1}{2}\delta(0)\right)
\end{aligned}
\tag{2.35}
$$

Hence, the elastic medium can be treated as a set of harmonic oscillators, one for each $\vec{k}$. There is a ground state, or *vacuum state*, $|0\rangle$ which satisfies:

$$
a_{k,s}\,|0\rangle\;=\;0 \tag{2.36}
$$

for all k, s. The full set of energy eigenstates can built on the vacuum state:

$$
\left(a_{k_1,s_1}^\dagger\right)^{n_1}\left(a_{k_2,s_2}^\dagger\right)^{n_2}\ldots\left(a_{k_j,s_j}^\dagger\right)^{n_j}\;|0\rangle \tag{2.37}
$$

The Hilbert space spanned by these states is called *Fock space*. We demand that the states of the form (2.37) are an orthogonal basis of Fock space, thereby defining the inner product. The state (2.37), which has energy

$$
\sum_i n_i\omega_{k_i,s_i} \tag{2.38}
$$

can be thought of as a state with n_1 *phonons* of momentum k_1 and polarization s_1; n_2 phonons of momentum k_2 and polarization s_2;...; n_j phonons of momentum k_j and polarization s_j. The creation operator $a^\dagger_{k_i,s_i}$ creates a phonon of momentum k_i and polarization s_i while the annihilation operator a_{k_1,s_1} annihilates such a phonon. At the quantum level, the normal-mode sound-wave oscillations have aquired a particle-like character; hence the name phonons.

You may have observed that the above Hamiltonian has an infinite constant. This constant is the zero-point energy of the system; it is infinite because we have taken the continuum limit in an infinite system. If we go back to our underlying ionic lattice, we will find that this energy, which is due to the zero-point motion of the ions, is finite. The sum over k really terminates when ω_k is the Debye energy. For the most part, we will not be interested in this energy (see, however, the problem set), so we will drop it. This can be done by introducing the notion of a normal-ordered product, which will be useful later. The normal-ordered product of a set of $a^\dagger_{k_i,s_i}$'s and a_{k_j,s_j}'s is the product with all of the $a^\dagger_{k_i,s_i}$'s to the left and all of the a_{k_i,s_i}'s to the right. It is denoted by a pair of colons. For example,

$$: a_{k_1,s_1} a^\dagger_{k_1,s_1} a_{k_2,s_2} : = a^\dagger_{k_1,s_1} a_{k_1,s_1} a_{k_2,s_2} \tag{2.39}$$

Since creation operators commute with one another and annihilation operators do as well, we do not need to speficy their orderings. Hence in the above example, the ordering of a_{k_1,s_1} and a_{k_2,s_2} above is unimportant. The normal ordered product can be defined for any *free fields*, i.e. for any fields which can be expanded in creation and annihilation operators with time-dependence of the form (8.72). Suppose A is such an operator. Then we can always write $A = A^{(+)} + A^{(-)}$ where $A^{(+)}$ is the part of the expansion of A which contains positive frequencies, $e^{i\omega t}$ and $A^{(-)}$ is the part which contains the negative frequencies. Normal-ordering puts the $A^{(+)}$'s to the left and the $A^{(-)}$'s to the right. If we define the quantum Hamiltonian to be $: H :$, then we eliminate the zero-point energy.

The divergent zero-point energy is the first of many ultra-violet divergences which we will encounter. They occur when we extend the upper limit of k-integrals to infinity. In fact, these integrals are always cutoff at some short length scale. In most of the problems which we will be discussing in this course, this cutoff is the inverse of the lattice scale. In the above example, this is the wavevector corresponding to the Debye energy. When we turn to electrons, the cutoff will be at a scale of electron volts. When field theory is applied to electrodynamics, it must be cutoff at the scale at which

it becomes unified with the weak interactions, approximately $100GeV$.

2.5.3 Fock space for He^4 atoms

We can use the same formalism to discuss a system of bosons, say He^4 atoms. This is particularly convenient when the number of He^4 atoms is not fixed, as for instance in the grand canonical ensemble, where the chemical potential, μ, is fixed and the number of particles, N, is allowed to vary.

Suppose we have a He^4 atom with Hamiltonian

$$H = \frac{P^2}{2m} \tag{2.40}$$

The energy eigenstates $|\vec{k}\rangle$ have energies and momenta

$$\begin{aligned} H|\vec{k}\rangle &= \frac{k^2}{2m}|\vec{k}\rangle \\ \vec{P}|\vec{k}\rangle &= \vec{k}|\vec{k}\rangle \end{aligned} \tag{2.41}$$

They are orthogonal:

$$\langle\vec{k}'|\vec{k}\rangle = \delta(\vec{k}-\vec{k}') \tag{2.42}$$

If we have two particles, we have eigenstates $|\vec{k}_1,\vec{k}_2\rangle$ with

$$\begin{aligned} H|\vec{k}_1,\vec{k}_2\rangle &= \left(\frac{k_1^2}{2m}+\frac{k_2^2}{2m}\right)|\vec{k}_1,\vec{k}_2\rangle \\ \vec{P}|\vec{k}_1,\vec{k}_2\rangle &= \left(\vec{k}_1+\vec{k}_2\right)|\vec{k}_1,\vec{k}_2\rangle \end{aligned} \tag{2.43}$$

satisfying

$$\langle\vec{k}_1,\vec{k}_2|\vec{k}_3,\vec{k}_3\rangle = \delta(\vec{k}_1-\vec{k}_3)\,\delta(\vec{k}_2-\vec{k}_4) \;+\; \delta(\vec{k}_1-\vec{k}_4)\,\delta(\vec{k}_2-\vec{k}_3) \tag{2.44}$$

We can continue in this way to 3-particle, 4-particle,..., etc. states. There is also a no-particle state: the vacuum, $|0\rangle$.

The Hilbert space spanned by all of these states is Fock space. We can define creation and annihilation operators $a_k^\dagger$, a_k satisfying:

$$\begin{aligned} \left[a_{\vec{k}}, a_{\vec{k}'}^\dagger\right] &= \delta(\vec{k}-\vec{k}') \\ \left[a_{\vec{k}}, a_{\vec{k}'}\right] &= \left[a_{\vec{k}}^\dagger, a_{\vec{k}'}^\dagger\right] = 0 \end{aligned} \tag{2.45}$$

so that

$$|\vec{k}\rangle = a_{\vec{k}}^\dagger|0\rangle$$

$$|\vec{k}_1, \vec{k}_2\rangle = a^\dagger_{\vec{k}_1} a^\dagger_{\vec{k}_1} |0\rangle$$
$$\text{etc.} \tag{2.46}$$

Writing

$$H = \int d^3\vec{k}\, \omega_k\, a^\dagger_{\vec{k}} a_{\vec{k}}$$
$$P = \int d^3\vec{k}\, \vec{k}\, a^\dagger_{\vec{k}} a_{\vec{k}} \tag{2.47}$$

where $\omega_k = k^2/2m$, we see that we recover the correct energies and momenta. From the commutation relations, we see that the correct orthonormality properties (2.42) (2.44) are also recovered.

We can now construct the fields, $\psi(x)$, $\psi^\dagger(x)$:

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, a_{\vec{k}}\, e^{-i(\omega_k t - \vec{k}\cdot\vec{x})}$$
$$\psi^\dagger(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, a^\dagger_{\vec{k}}\, e^{i(\omega_k t - \vec{k}\cdot\vec{x})} \tag{2.48}$$

$\psi(x)$ satisfies the equation:

$$i\frac{\partial}{\partial t}\psi(x) = -\frac{1}{2m}\nabla^2\psi(x) \tag{2.49}$$

In other words, suppose we view the Schödinger equation as a *classical wave equation* – analogous to the wave equation of an elastic medium (2.3) – which can be derived from the action

$$S = \int dt\, d^3\vec{r}\, \psi^\dagger \left(i\frac{\partial}{\partial t} + \frac{1}{2m}\nabla^2 \right) \psi \tag{2.50}$$

Then, we can *second quantize* this wave equation and arrive at the Fock space description.

CHAPTER 3

Perturbation Theory: Interacting Phonons

3.1 Higher-Order Terms in the Phonon Lagrangian

The second quantization procedure described in the previous chapter can be immediately applied to any classical field theory which has a Lagrangian which is quadratic in its basic fields. However, most systems have Lagrangians with higher-order terms. For instance, there are certainly terms in the phonon Lagrangian which we have neglected which are cubic, quartic, and higher-order in the displacement fields, u_i. An example of a phonon Lagrangian with such a term included is

$$S = S_0 - \frac{g}{4!} \int dt\, d^3\vec{x}\, (\partial_k u_k)^4 \tag{3.1}$$

The Hamiltonian corresponding to (3.1) is:

$$\begin{aligned} H &= \frac{1}{2} \int d^3\vec{r} \left[\frac{1}{\rho}\, \pi_i^2 \;+\; 2\mu u_{ij} u_{ij} \;+\; \lambda u_{kk}^2 \right] + \frac{g}{4!} \int dt\, d^3\vec{x}\, (\partial_k u_k)^4 \\ &= H_0 \;+\; H' \end{aligned} \tag{3.2}$$

We use this phonon Lagrangian as an illustrative example; it is not intended to be a realistic phonon Lagrangian.

Classically, the presence of such terms means that different solutions can no longer be superposed. Hence, there is no normal mode expansion, and we cannot follow the steps which we took in chapter 2. When g is small, we

can, however, hope to use perturbation theory to solve this Hamiltonian. In this chapter, we develop a perturbation theory for H' using the solution of H_0 presented in chapter 2. As we will see, higher-order terms in the phonon Lagrangian lead to *interactions* between the phonons which cause them to scatter off each other.

In order to facilitate the construction of the perturbation theory, we will need several technical preliminaries: the interaction picture, the time-ordered product, and Wick's theorem.

3.2 Schrödinger, Heisenberg, and Interaction Pictures

In the Schrödinger picture, states evolve in time according to:

$$i\frac{\partial}{\partial t}\left|\psi(t)\right\rangle_S \;=\; H(t)\left|\psi(t)\right\rangle_S \tag{3.3}$$

while operators are time-independent unless they have explicit time dependence. For example, if we have a particle in $1D$, p and x do not depend on time, but we can switch on a time-dependent driving force in which case the Hamiltonian, $H(t) = p^2/2m + x\cos\omega t$, is time-dependent. The time-evolution operator, $U(t,t')$ acts on states in the following way:

$$\left|\psi(t)\right\rangle_S \;=\; U(t,t')\left|\psi(t')\right\rangle_S \tag{3.4}$$

It satisfies the equation

$$i\frac{\partial}{\partial t}U(t,t') = H(t)U(t,t') \tag{3.5}$$

subject to the initial condition, $U(t,t) = 1$. If H is time-independent, then

$$U(t,t') = e^{-i(t-t')H} \tag{3.6}$$

In the Heisenberg picture, on the other hand, states are time-independent,

$$\left|\psi(t)\right\rangle_H \;=\; \left|\psi(0)\right\rangle_S \;=\; \left|\psi(0)\right\rangle_H \tag{3.7}$$

while operators contain all of the time-dependence. Suppose $O_S(t)$ is an operator in the Schrödinger picture (we have allowed for explicit time dependence as in $H(t)$ above). Then the corresponding Heisenberg picture operator is:

$$O_H(t) = U(0,t)\,O_S(t)\,(U(0,t))^\dagger \tag{3.8}$$

Finally, we turn to the interaction picture, which we will use extensively. This picture can be defined when the Hamiltonian is of the form $H = H_0 + H'$ and H_0 has no explicit time-dependence. The interaction picture interpolates between the Heisenberg and Schrödinger pictures. Operators have time-dependence given by H_0:

$$O_I(t) = e^{itH_0}\, O_S(t)\, e^{-itH_0} \tag{3.9}$$

This includes the interaction Hamiltonian, H' which now has time-dependence due to H_0:

$$H_I(t) = e^{itH_0}\, H'_S(t)\, e^{-itH_0} \tag{3.10}$$

(We will drop the prime and simply call it H_I.) The states lack this part of the time-dependence,

$$|\psi(t)\rangle_I = e^{itH_0}\, |\psi(t)\rangle_S \tag{3.11}$$

Hence, states satisfy the differential equation

$$\begin{aligned} i\frac{\partial}{\partial t}\,|\psi(t)\rangle_I \;&=\; i\frac{\partial}{\partial t}\,\left(e^{itH_0}\,|\psi(t)\rangle_S\right) \\ &=\; e^{itH_0}\,(-H_0 + H_S)\,|\psi(t)\rangle_S \\ &=\; e^{itH_0}\;H'_S(t)\;e^{-itH_0}\,|\psi(t)\rangle_I \\ &=\; H_I(t)\,|\psi(t)\rangle_I \end{aligned} \tag{3.12}$$

We can define an Interaction picture time-evolution operator, $U_I(t,t')$, satisfying

$$i\frac{\partial}{\partial t}\,U_I(t,t') \;=\; H_I(t)\,U_I(t,t') \tag{3.13}$$

which evolves states according to

$$|\psi(t)\rangle_I \;=\; U_I(t,t')|\psi(t')\rangle_I \tag{3.14}$$

3.3 Dyson's Formula and the Time-Ordered Product

If we can find $U_I(t,t')$, then we will have solved to full Hamiltonian $H_0 + H'$, since we will know the time dependence of both operators and states. A formal solution was written down by Dyson:

$$U_I(t,t') \;=\; T\left\{e^{-i\int_{t'}^{t} dt''\, H_I(t'')}\right\} \tag{3.15}$$

where the *time-ordered product*, T, of a string of operators, $O_1(t_1)O_2(t_2)\dots O_n(t_n)$, is their product arranged sequentially in the order of their time arguments, with operators with earlier times to the right of operators with later times:

$$T\left\{O_1(t_1)O_2(t_2)\dots O_n(t_n)\right\} = O_{i_1}(t_{i_1})O_{i_2}(t_{i_2})\dots O_{i_n}(t_{i_n}) \quad \text{if } t_{i_1} > t_{i_2} > \dots > t_{i_n} \tag{3.16}$$

There is some ambiguity if $t_i = t_j$ and $O(t_i)$ and $O(t_j)$ do not commute. In (3.15), however, all of the O_i's are H_I, so we do not need to worry about this.

To see that it satisfies the differential equation (3.13), observe that all operators commute under the time-ordering symbol, so we can take the derivative naively:

$$i\frac{\partial}{\partial t}\,T\left\{e^{-i\int_{t'}^{t}dt''\,H_I(t'')}\right\} \;=\; T\left\{H_I(t)e^{-i\int_{t'}^{t}dt''\,H_I(t'')}\right\} \tag{3.17}$$

Since t is the upper limit of integration, it is greater than or equal to any other t'' which appears under the time-ordering symbol. Hence, we can pull it out to the left:

$$i\frac{\partial}{\partial t}\,T\left\{e^{-i\int_{t'}^{t}dt''\,H_I(t'')}\right\} \;=\; H_I(t)\,T\left\{e^{-i\int_{t'}^{t}dt''\,H_I(t'')}\right\} \tag{3.18}$$

With Dyson's formula in hand, we can – at least in principle - compute transition amplitudes. For example, let us suppose that we have a system which is in its ground state, $|0\rangle$. Suppose we perform a neutron scattering experiment in which a neutron is fired into the system with momentum $\vec{P}$ at time t' and then interacts with our system according to H_I. The probability (which is the square of the amplitude) for the system to undergo a transition to an excited state $\langle 1|$ so that the neutron is detected with momentum $\vec{P}'$ at time t is:

$$\left|\langle 1;\vec{P}'|U_I(t,t')|0;\vec{P}\rangle\right|^2 \tag{3.19}$$

Of course, we can rarely evaluate $U_I(t,t')$ exactly, so we must often expand the exponential. The first-order term in the expansion of the exponential is:

$$-i\int_{t'}^{t}dt_1\,H_I(t_1) \tag{3.20}$$

Hence, if we prepare an initial state $|i\rangle$ at $t' = -\infty$, we measure the system in a final state $\langle f|$ at $t = \infty$ with amplitude:

$$\langle f|U_I(\infty,-\infty)|i\rangle = -i\langle f|\int_{-\infty}^{\infty}dt\,H_I(t)|i\rangle \tag{3.21}$$

Squaring this, we recover Fermi's Golden Rule. There is a slight subtlety in that the t integral leads to an amplitude proportional to $\delta(E_i - E_f)$. This appears to lead to a transition probability which is proportional to the square of a δ-function. We understand, however, that this is a result of taking the limits of integration to infinity carelessly: the square of the δ-function is actually a single δ-function multiplied by the difference between the initial and final times. Hence, this implies that the *transition rate* is:

$$\frac{dP}{dt} = |\langle f| \int_{-\infty}^{\infty} dt\, H_I(t) |i\rangle|^2 \tag{3.22}$$

with one δ-function dropped.

To get a sense of the meaning of the T symbol, it is instructive to consider the second-order term in the expansion of the exponential:

$$\frac{(-i)^2}{2!} \int_{t'}^{t} dt_1 \int_{t'}^{t} dt_2\, T\left(H_I(t_1) H_I(t_2)\right) \;=\; (-i)^2 \int_{t'}^{t} dt_1 \int_{t'}^{t_1} dt_2\, H_I(t_1) H_I(t_2) \tag{3.23}$$

3.4 Wick's Theorem

We would like to evaluate the terms of the perturbation series obtained by expanding Dyson's formula (3.15). To do this, we need to compute time-ordered products $T\{H_I\, H_I \ldots H_I\}$. This can be done efficiently if we can reduce the time-ordered products to normal-ordered products (which enjoy the relative simplicity of annihilating the vacuum).

To do this, we define the notion of the contraction of free fields (remember that, in the interaction picture, the operators are free and the states have complicated time-dependence), which we will denote by an overbrace:

$$\overbrace{A(t_1) B(t_2)} = T\left(A(t_1) B(t_2)\right) - :A(t_1) B(t_2): \tag{3.24}$$

Dividing A and B into their positive- and negative-frequency parts, $A^{(\pm)}$, $B^{(\pm)}$, we see that:

$$\overbrace{A(t_1) B(t_2)} = \left[A^{(-)}, B^{(+)}\right] \tag{3.25}$$

if $t_1 > t_2$ and

$$\overbrace{A(t_1) B(t_2)} = \left[B^{(-)}, A^{(+)}\right] \tag{3.26}$$

if $t_1 < t_2$. This is a c-number (i.e. it is an ordinary number which commutes with everything) since $[a, a^\dagger] = 1$. Hence, it is equal to its vacuum

expectation value:

$$\begin{aligned}\overbrace{A(t_1)B(t_2)} &= \langle 0|\,\overbrace{A(t_1)B(t_2)}\,|0\rangle \\ &= \langle 0|T\left(A(t_1)B(t_2)\right)|0\rangle \;-\; \langle 0|:A(t_1)B(t_2):|0\rangle \\ &= \langle 0|T\left(A(t_1)B(t_2)\right)|0\rangle \end{aligned} \tag{3.27}$$

The following theorem, due to Gian-Carlo Wick, uses the contraction to reduce time-ordered products to normal-ordered products:

$$\begin{aligned} T\{u_1\,u_2\,\ldots\,u_n\} = \quad & :u_1\,u_2\,\ldots\,u_n: \\ & + :\overbrace{u_1\,u_2}\,\ldots\,u_n: + \text{ other terms with one contraction} \\ & + :\overbrace{u_1\,u_2}\overbrace{u_3\,u_4}\,\ldots\,u_n: + \text{ other terms with two contractions} \\ & \quad\vdots \\ & + :\overbrace{u_1\,u_2}\,\ldots\,\overbrace{u_{n-1}u_n}: \\ & \quad + \text{ other such terms if } n \text{ is even} \\ & + :\overbrace{u_1\,u_2}\,\ldots\,\overbrace{u_{n-2}u_{n-1}}\,u_n: \\ & \quad + \text{ other such terms if } n \text{ is odd} \end{aligned} \tag{3.28}$$

The right-hand-side is normal-ordered. It contains all possible terms with all possible contractions appear, *each with coefficient* 1. The proof proceeds by induction. Let us call the right-hand-side $w(u_1\,u_2\,\ldots\,u_n)$. The equality of the left and right-hand sides is trivial for $n = 1, 2$. Suppose that it is true for time-ordered products of $n-1$ fields. Let us further suppose, without loss of generality, that t_1 is the latest time. Then,

$$\begin{aligned} T\{u_1\,u_2\,\ldots\,u_n\} &= u_1\,T\{u_2\,\ldots\,u_n\} \\ &= u_1\,w\left(u_2,\ldots,u_n\right) \\ &= u_1^{(+)}\,w\left(u_2,\ldots,u_n\right) \;+\; u_1^{(-)}\,w\left(u_2,\ldots,u_n\right) \\ &= u_1^{(+)}\,w\left(u_2,\ldots,u_n\right) \;+\; w\left(u_2,\ldots,u_n\right)\,u_1^{(-)} \;+\; \left[u_1^{(-)},w\right] \\ &= w\left(u_1,u_2,\ldots,u_n\right) \end{aligned} \tag{3.29}$$

The equality between the last two lines follows from the fact that the final expression is normal ordered and contains all possible contractions: the first two terms contain all contractions in which u_1 is not contracted while the third term contains all contractions in which u_1 is contracted.

A concise way of writing down Wick's theorem is the following:

$$T\{u_1\,u_2\,\ldots\,u_n\} =: e^{\frac{1}{2}\sum_{i,j=1}^{n}\overbrace{u_i u_j}\frac{\partial}{\partial u_i}\frac{\partial}{\partial u_j}}\,u_1\,u_2\,\ldots\,u_n: \tag{3.30}$$

3.5 The Phonon Propagator

As a result of Wick's theorem, the contraction of two phonon fields, $\overbrace{u_i u_j}$, is the basic building block of perturbation theory. Matrix elements of the time-evolution operator will be given by integrals of products of contractions. The contraction $\overbrace{u_i u_j}$ is also called the *phonon propagator*. In the problem set, you will compute the propagator in two different ways. First, you will calculate it directly from:

$$\langle T(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2))\rangle = \left[u_i{}^{(+)}, u_j{}^{(-)}\right] \tag{3.31}$$

You will also calculate it by noting that

$$\langle T\left(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right)\rangle = \theta(t_1-t_2)\left\langle u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right\rangle + \theta(t_2-t_1)\left\langle u_j(\vec{x}_2,t_2)u_i(\vec{x}_1,t_1)\right\rangle \tag{3.32}$$

and acting on this with (2.3) to obtain,

$$\begin{aligned}&\left(\rho\delta_{ik}\partial_t^2 - (\mu+\lambda)\,\partial_i\partial_k - \mu\delta_{ik}\partial_l\partial_l\right)\langle T\left(u_k(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right)\rangle \\ &= -i\delta\left(\vec{x}_1-\vec{x}_2\right)\,\delta\left(t_1-t_2\right)\,\delta_{ij}\end{aligned} \tag{3.33}$$

By Fourier transforming this equation, we find:

$$\langle T\left(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right)\rangle \;=\; \frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{i\epsilon_i^s\epsilon_j^s}{\omega^2-(\omega_p^s)^2} \tag{3.34}$$

Here, we have used $\epsilon_i^s\epsilon_j^s = \delta_{ij}$. However, the singularities at $\omega^2 = (\omega_p^s)^2$ are unresolved by this expression. As you will show in the problem set, the correct expression is:

$$\langle T\left(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right)\rangle \;=\; \frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_2)-\omega(t_1-t_2))}\,\frac{i\epsilon_i^s\epsilon_j^s}{\omega^2-(\omega_p^s)^2+i\delta} \tag{3.35}$$

Since $\epsilon_i^1 k_i = k$ while $\epsilon_i^{2,3}k_i = 0$,

$$\begin{aligned}\epsilon_i^1\epsilon_j^1 &= \frac{k_i k_j}{k^2} \\ \epsilon_i^2\epsilon_j^2 + \epsilon_i^3\epsilon_j^3 &= \delta_{ij} - \frac{k_i k_j}{k^2}\end{aligned} \tag{3.36}$$

Hence, using $\omega_p^1 = v_l p$, $\omega_p^{2,3} = v_s p$, we can rewrite the phonon propagator as:

$$\langle T\left(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2)\right)\rangle \;=\; \frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{i\,k_i k_j/k^2}{\omega^2-v_l^2p^2+i\delta}$$

$$+\frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{i\,(\delta_{ij}-k_ik_j/k^2)}{\omega^2-v_t^2p^2+i\delta} \tag{3.37}$$

For some purposes, it will be more convenient to consider a slightly different phonon field,

$$\varphi_i(\vec{r},t)=\int \frac{d^3\vec{p}}{(2\pi)^3}\sum_s \frac{1}{\sqrt{\rho}}\,\epsilon_i^s\left(a_{\vec{k},s}e^{i(\vec{k}\cdot\vec{r}-\omega_k^s t)}\,+\,a^\dagger_{\vec{k},s}e^{-i(\vec{k}\cdot\vec{r}-\omega_k^s t)}\right) \tag{3.38}$$

The difference with u_i is the missing $1/\sqrt{2\omega_k^s}$. This field has propagator:

$$\begin{aligned}\langle T\,(u_i(\vec{x}_1,t_1)u_j(\vec{x}_2,t_2))\rangle \;&=\; \frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{2i\omega_p^s\,\epsilon_i^s\epsilon_j^s}{\omega^2-(\omega_p^s)^2+i\delta}\\ &=\; \frac{1}{\rho}\int d^3\vec{p}d\omega\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\left(\frac{i\epsilon_i^s\epsilon_j^s}{\omega-\omega_p^s+i\delta}\,+\,\frac{i\epsilon_i^s\epsilon_j^s}{\omega+\omega_p^s-i\delta}\right)\end{aligned} \tag{3.39}$$

3.6 Perturbation Theory in the Interaction Picture

We are now in position to start looking at perturbation theory. Since transverse phonons are unaffected by the interaction (3.1), we only need to discuss longitudinal phonons. Consider the second-order contribution in our theory of phonons with a quartic anharmonicity (3.1),

$$\begin{aligned}U(-\infty,\infty)&=\frac{(-i)^2}{2!}\int_{-\infty}^{\infty}dt_1\int_{-\infty}^{\infty}dt_2\,T\,(H_I(t_1)H_I(t_2))\\ &=\frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1dt_1\,d^3\vec{x}_2dt_2\,T\left((\partial_k u_k(\vec{x}_1,t_1))^4\,(\partial_k u_k(\vec{x}_2,t_2))^4\right)\end{aligned} \tag{3.40}$$

When we apply Wick's theorem, we get such terms as:

$$\frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1dt_1\,d^3\vec{x}_2dt_2\;:\partial_k u_k\partial_k u_k\,\overbrace{\partial_k u_k\,\overbrace{\partial_k u_k(\vec{x}_1,t_1)\,\partial_k u_k}\,\partial_k u_k}\,\partial_k u_k\partial_k u_k(\vec{x}_2,t_2):\tag{3.41}$$

This term will contribute to such physical processes as the scattering between two longitudinal phonons. If we look at

$$\langle\vec{k}_3,l;\vec{k}_4,l;t=\infty|U(-\infty,\infty)|\vec{k}_1,l;\vec{k}_2,l;t=-\infty\rangle\;=$$

$$\ldots + \frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\ \overbrace{\partial_k u_k \overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k}\,\partial_k u_k(\vec{x}_2,t_2)}\times$$
$$\langle \vec{k}_3,l;\vec{k}_4,l;t=\infty| : \partial_k u_k \partial_k u_k(\vec{x}_1,t_1)\,\partial_k u_k \partial_k u_k(\vec{x}_2,t_2) : |\vec{k}_1,l;\vec{k}_2,l;t=-\infty\rangle$$
$$+\ldots \tag{3.42}$$

this will give a non-vanishing contribution since two of the uncontracted u_i's can annihilate the phonons in the initial state and the other two can create the phonons in the final state. Let's suppose that the incoming phonons are annihilated by $\partial_k u_k^{(-)}\partial_k u_k^{(-)}$ at $(\vec{x}_1,t_1)$ and the outgoing phonons are created by the $\partial_k u_k^{(+)}\partial_k u_k^{(+)}$ at $(\vec{x}_2,t_2)$. Since

$$\partial_k u_k^{(-)}\partial_k u_k^{(-)}(\vec{x}_1,t_1)|\vec{k}_1,l;\vec{k}_2,l;t=-\infty\rangle = -|\vec{k}_1|\,|\vec{k}_2|\left(e^{i(\vec{k}_1\cdot\vec{x}_1-\omega^l_{k_1}t_1)}\,e^{i(\vec{k}_2\cdot\vec{x}_1-\omega^l_{k_2}t_1)}\right)|0\rangle \tag{3.43}$$

we obtain a contribution to (3.42) of the form:

$$\frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\ \{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times$$
$$\left(e^{i((\vec{k}_1+\vec{k}_2)\cdot\vec{x}_1-(\omega^l_{k_1}+\omega^l_{k_2})t_1)}\right)\left(e^{-i((\vec{k}_3+\vec{k}_4)\cdot\vec{x}_2-(\omega^l_{k_3}+\omega^l_{k_4})t_2)}\right)\times$$
$$\overbrace{\partial_k u_k(\vec{x}_1,t_1)\overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k(\vec{x}_2,t_2)}\,\partial_k u_k(\vec{x}_2,t_2)}\}\tag{3.44}$$

Substituting the expression for $\overbrace{u_k(\vec{x}_1,t_1)u_k(\vec{x}_2,t_2)}$, we find:

$$\frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\ \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d\omega_2}{2\pi}\ \{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times$$
$$\left(e^{i((\vec{k}_1+\vec{k}_2-\vec{p}_1-\vec{p}_2)\cdot\vec{x}_1-(\omega^l_{k_1}+\omega^l_{k_2}-\omega_1-\omega_2)t_1)}\right)\left(e^{-i((\vec{k}_3+\vec{k}_4-\vec{p}_1-\vec{p}_2)\cdot\vec{x}_2-(\omega^l_{k_3}+\omega^l_{k_4}-\omega_1-\omega_2)t_2)}\right)\times$$
$$\frac{1}{\rho}|\vec{p}_1|^2\,\frac{i}{\omega_1^2-(\omega^l_{p_1})^2+i\delta}\,\frac{1}{\rho}|\vec{p}_2|^2\,\frac{i}{\omega_2^2-(\omega^l_{p_2})^2+i\delta}\}\tag{3.45}$$

The x and t integrals give δ functions which enforce momentum- and energy-conservation.

$$\frac{(-ig/4!)^2}{2!}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\frac{d\omega_2}{2\pi}\ \{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times$$
$$(2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{p}_1-\vec{p}_2)\,2\pi\delta(\omega^l_{k_1}+\omega^l_{k_2}-\omega_1-\omega_2)$$
$$(2\pi)^3\delta(\vec{k}_3+\vec{k}_4-\vec{p}_1-\vec{p}_2)\,2\pi\delta(\omega^l_{k_3}+\omega^l_{k_4}-\omega_1-\omega_2)$$
$$\frac{1}{\rho^2}|\vec{p}_1|^2\,\frac{i}{\omega_1^2-(\omega^l_{p_1})^2+i\delta}\,|\vec{p}_2|^2\,\frac{i}{\omega_2^2-(\omega^l_{p_2})^2+i\delta}\}\tag{3.46}$$

which, finally, gives us

$$\frac{(-ig/4!)^2}{2!}\frac{1}{\rho^{2'}}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\;\{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times \\ |\vec{p}_1|^2\,\frac{i}{\omega_1^2-(\omega^l_{p_1})^2+i\delta}\,|\vec{k}_1+\vec{k}_2-\vec{p}_1|^2\,\frac{i}{(\omega^l_{k_1}+\omega^l_{k_2}-\omega_1)^2-(\omega^l_{k_1+k_2-p_1})^2+i\delta} \\ (2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\,2\pi\delta(\omega^l_{k_1}+\omega^l_{k_2}-\omega^l_{k_3}-\omega^l_{k_4})\} \tag{3.47}$$

There are actually several ways in which an identical contribution can be obtained. By an identical contribution, we mean one in which there are two contractions of the form $\overbrace{u_k(\vec{x}_1,t_1)u_k(\vec{x}_2,t_2)}$; the incoming phonons are annihilated at the same point (which can be either $(\vec{x}_1,t_1)$ or $(\vec{x}_2,t_2)$ since these are dummy variables which are integrated over); and the outgoing phonons are created at the same point. The incoming phonons are annihilated by the u_i's at $(\vec{x}_1,t_1)$ and the outgoing phonons are annihilated by the u_i's at $(\vec{x}_2,t_2)$, which can be done in $(4\cdot 3)(4\cdot 3)$ ways. There are 2 ways in which we can choose how the remaining u_i's at $(\vec{x}_1,t_1)$ are contracted with the remaining u_i's at $(\vec{x}_2,t_2)$, giving us a multiplicity of $(4\cdot 3)(4\cdot 3)2$. It is now clear why we included a factor of $1/4!$ in our definition of g: the above multiplicity almost cancels the two factors of $1/4!$. Only a factor of $1/2$ remains. If we permute $(\vec{x}_1,t_1)$ and $(\vec{x}_2,t_2)$, then the incoming phonons are annihilated by the u_i's at $(\vec{x}_2,t_2)$ and the outgoing phonons are annihilated by the u_i's at $(\vec{x}_1,t_1)$. This gives an identical contribution, thereby cancelling the $1/2!$ which we get at second-order. Hence, the sum of all such contributions is:

$$\frac{(-ig)^2}{2}\frac{1}{\rho^2}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\;\{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times \\ |\vec{p}_1|^2\,\frac{i}{\omega_1^2-(\omega^l_{p_1})^2+i\delta}\,|\vec{k}_1+\vec{k}_2-\vec{p}_1|^2\,\frac{i}{(\omega^l_{k_1}+\omega^l_{k_2}-\omega_1)^2-(\omega^l_{k_1+k_2-p_1})^2+i\delta} \\ (2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\,2\pi\delta(\omega^l_{k_1}+\omega^l_{k_2}-\omega^l_{k_3}-\omega^l_{k_4})\} \tag{3.48}$$

There are, of course, other, distinct second-order contributions to the two phonon $\rightarrow$ two phonon transition amplitude which result, say, by contracting fields at the same point or by annihilating the incoming phonons at different points. Consider the latter contributions. Ther is a contirbution of the form:

$$\frac{(-ig)^2}{2}\frac{1}{\rho^2}\int\frac{d^3\vec{p}_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\;\{|\vec{k}_1|\,|\vec{k}_2||\vec{k}_3|\,|\vec{k}_4|\times \\ |\vec{p}_1|^2\,\frac{i}{\omega_1^2-(\omega^l_{p_1})^2+i\delta}\,|\vec{k}_1-\vec{k}_3-\vec{p}_1|^2\,\frac{i}{(\omega^l_{k_1}-\omega^l_{k_3}-\omega_1)^2-(\omega^l_{k_1-k_3-p_1})^2+i\delta}$$

$$(2\pi)^3\delta(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4)\,2\pi\delta(\omega^l_{k_1}+\omega^l_{k_2}-\omega^l_{k_3}-\omega^l_{k_4})\} \tag{3.49}$$

and one with $\vec{k}_3 \to \vec{k}_4$.

The cancellation which we obtained by permuting the different $(\vec{x}_i, t_i)$'s does not always occur. For instance, the following contraction at second-order makes a contribution to the amplitude for the vacuum at $t = -\infty$ to go into the vacuum at $t = \infty$:

$$\begin{aligned}
&\langle 0; t=\infty|U(-\infty,\infty)|0; t=-\infty\rangle \;= \\
&\ldots + \frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\;\overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k(\vec{x}_2,t_2)}\overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k(\vec{x}_2,t_2)}\times \\
&\quad \overbrace{\partial_k u_k \partial_k u_k}\,\overbrace{\partial_k u_k \partial_k u_k}\,\langle 0; t=\infty|0; t=-\infty\rangle \\
&+\ldots
\end{aligned} \tag{3.50}$$

We have written this term with contracted fields adjacent in order to avoid clutter. There are no distinct permutations, so there is nothing to cancel the 1/2!. In addition, there are only 4! ways to do the contractions, so there is an uncancelled factor of 1/4! as well, and hence, an overall factor of 1/2!4!. Consider, for a moment, how this works at n^{th} order. There will be a factor of $n!$. If this is incompletely cancelled by permutations of the $(\vec{x}_i, t_i)$'s, there will be a factor, $1/S$ (in the above, $S = 2$). In the next chapter, we will see that the *symmetry factor*, S, is related to the symmetries of Feynman diagrams. In addition, there will be factors arising from the incomplete cancellation of the $(1/4!)^n$. In the above, this additional factor is 1/4!.

Again, there are other second-order contributions to the vacuum-to-vacuum amplitude which result from contracting fields at the same point, but they will give a different contribution which is different in form from the one above. One such is the following:

$$\begin{aligned}
&\langle 0; t=\infty|U(-\infty,\infty)|0; t=-\infty\rangle \;= \\
&\ldots + \frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\;\overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k(\vec{x}_1,t_1)}\overbrace{\partial_k u_k(\vec{x}_2,t_2)\partial_k u_k(\vec{x}_2,t_2)}\times \\
&\quad \overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k}(\vec{x}_2,t_2)\;\overbrace{\partial_k u_k(\vec{x}_1,t_1)\partial_k u_k}(\vec{x}_2,t_2)\langle 0; t=\infty|0; t=-\infty\rangle \\
&+\ldots
\end{aligned} \tag{3.51}$$

There are $4 \cdot 3/2$ ways of choosing the two fields at $(\vec{x}_1, t_1)$ which are contracted and $4 \cdot 3/2$ ways of choosing the two fields at $(\vec{x}_2, t_2)$ which are contracted. Finally, there are 2 ways of contracting the remaining fields at $(\vec{x}_1, t_1)$ with those at $(\vec{x}_2, t_2)$. This multiplicity incompletely cancels the $1/2!(4!)^2$.

As another example, consider

$$\frac{(-ig/4!)^2}{2!}\int d^3\vec{x}_1 dt_1\, d^3\vec{x}_2 dt_2\ :\partial_k u_k\,\overbrace{\partial_k u_k\,\overbrace{\partial_k u_k\,\overbrace{\partial_k u_k(\vec{x}_1,t_1)\,\partial_k u_k}\,\partial_k u_k}\,\partial_k u_k\,\partial_k u_k}\,\partial_k u_k(\vec{x}_2,t_2)\,: \tag{3.52}$$

This contributes to the amplitude for processes in which both the initial and final states contain one longitudinal phonon. There are $4\cdot 4$ ways of choosing the u_i's which create the incoming phonon at $(\vec{x}_1,t_1)$ and annihilate the outgoing phonon at $(\vec{x}_2,t_2)$. There are 3! ways of contracting the remaining u_i's. Finally, $(\vec{x}_1,t_1)$ and $(\vec{x}_2,t_2)$ can be permuted. This gives an overall factor of $4\cdot 4\cdot 3!\cdot 2$, which incompletely cancels the $(1/2!)\cdot(1/4!)^2$, leaving $1/3!$.

CHAPTER 4

Feynman Diagrams and Green Functions

4.1 Feynman Diagrams

Feynman introduced a diagrammatic notation which will help us systematically enumerate all of the perturbative contributions which we generate using Wick's theorem. This diagrammatic notation will have the added benefit of having a simple physical interpretation which will guide our intuition about physical processes.

Suppose we want to construct a matrix element at n^{th} order in perturbation theory. We draw a diagram containing n vertices with 4 lines emanating from each vertex. Each such vertex represents a factor of $(\partial_k u_k)^4$. The lines emanating from the vertices can be connected. Each such connection represents a contraction. We will call such a line an *internal line*. The remaining (uncontracted) lines – *external lines* – represent incoming and outgoing phonons. We will adopt the convention that incoming phonon lines enter at the left of the diagram while outgoing phonon lines exit at the right of the diagram.

The first contribution which we considered in chapter 3 (3.47) can be represented as:

Given such a diagram – a Feynman diagram – you can immediately reconstruct the expression which it represents according to the following rules:

- Assign a directed momentum and energy to each line. For external

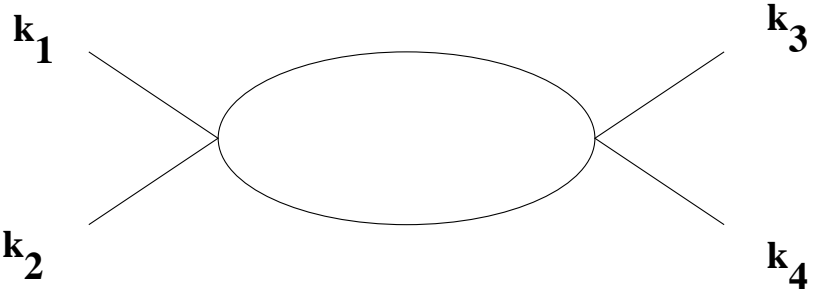


Figure 4.1: The diagram corresponding to (3.47).

lines, the momentum is directed into or out of the diagram for, respectively, incoming and outgoing phonons.

- For each external line with momentum $\vec{k}$, write $|\vec{k}|$.
- For each internal line with momentum and energy $\vec{p}$, ω write:

$$\frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\,|\vec{p}|^2\,\frac{i}{\omega^2 - v_l^2 p^2 + i\delta}$$

- For each vertex with momenta, energies $(\vec{p}_1,\omega_1),\ldots,(\vec{p}_4,\omega_4)$ directed into the vertex, write:

$$g\,(2\pi)^3\delta(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)\,2\pi\delta(\omega_1+\omega_2+\omega_3+\omega_4)$$

- Imagine labelling the vertices $1,2,\ldots,n$. Vertex i will be connected to vertices $j_1,\ldots,j_m$ $(m\leq 4)$ and to external momenta $p_1,\ldots,p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices i, i is still connected to vertices $j_1,\ldots,j_m$ $(m\leq 4)$ and to external momenta $p_1,\ldots,p_{4-m}$. If S is the number of permutations which leave the diagram invariant, we assign a factor $1/S$ to the diagram.
- If two vertices are connected by l lines, we assign a factor $1/l!$ to the diagram.

You can verify that by applying these rules to figure (4.1) we recover (3.47).

For the particular interaction we have chosen, we can ignore the transverse phonons – since they don't interact – and consider only the longitudinal phonons. If we were to consider a model in which both longitudinal

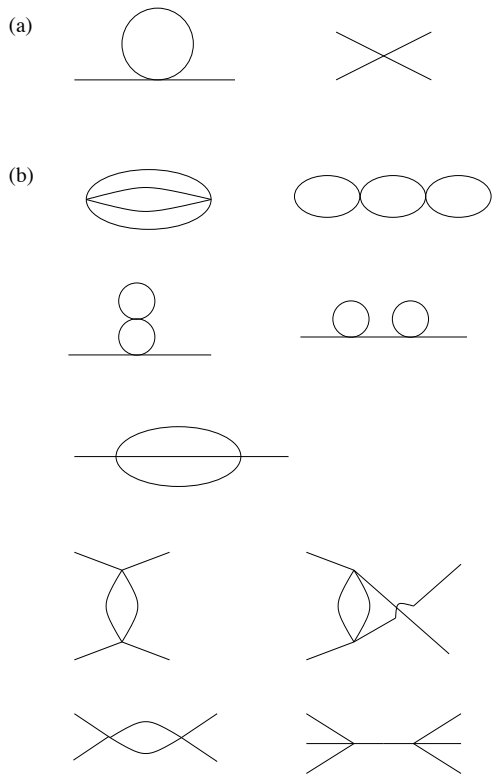


Figure 4.2: All *connected* Feynman diagrams for the theory (3.1) to $O(g^2)$. In (a), we have the diagrams of order g. In (b), we have the diagrams of order g^2.

and transverse phonons interact, our Feynman diagrams would have to have internal and external indices corresponding to the vector indices of the fields u_i, u_j, *etc.*, and our Feynman rules would have to tell us how to contract or route these indices.

In figure (4.2) we display all of the *connected* diagrams which appear to $O(g^2)$ in the theory given by (3.1). In the problem set, you will write down expressions for them.

The Feynman diagram representation for transition amplitudes suggests a beautiful visualization of perturbative processes. External lines correspond to 'real phonons' or simply phonons, while internal lines correspond to 'virtual phonons'. For the diagram of figure 4.1, we say that the incoming phonons with momenta $\vec{k}_1, \vec{k}_2$ interact at x_1, propagate as virtual phonons

with momenta $\vec{p}_1, \vec{k}_1 + \vec{k}_2 - \vec{p}_1$, and finally interact again at x_2, thereby scattering into the outgoing phonons with momenta $\vec{k}_3, \vec{k}_4$. For the first diagram of figure 4.2b, we say that the incoming phonons with momenta $\vec{k}_1, \vec{k}_2$ exchange a pair of virtual phonons, thereby scattering into $\vec{k}_3, \vec{k}_4$. External lines correspond to initial or final states with phonons of momentum, energy $(\vec{p}, \omega)$. These satisfy $\omega^2 = (\omega_p^l)^2$. Such a phonon is said to be 'on-shell'. Virtual phonons need not be 'on-shell'. Indeed, the phonon propagator diverges if a virtual phonon is on-shell, thereby signalling a resonance.

4.2 Loop Integrals

Suppose we have a Feynman diagram with E external lines, I internal lines, and V vertices. Suppose, further, that this diagram has L loops (e.g. the first diagram in figure 4.2 has one loop, while the third, fourth, and fifth have two loops. The second has no loops.). Then, let's imagine connecting all of the external lines at a single point so that the Feynman diagram defines a polyhedron with $E + I$ edges; $V + 1$ vertices - the extra vertex being the one at which the external lines are connected; and $L + E$ faces - with E faces formed as a result of connecting the external lines. According to Euler's fomula,

$$(\#\ faces) + (\#\ vertices) - (\#\ edges) = 2 \tag{4.1}$$

or,

$$L = I - V + 1 \tag{4.2}$$

The number of loops is given by the number of internal lines - i.e. the number of propagators, each coming with an integral - minus the number of vertices, each coming with momentum and energy-conserving δ-functions, plus 1 for the overall δ-functions satisfied by the external momenta, energies. In short, there are as many $(\vec{p}, \omega)$ pairs to be integrated over as there are loops. A diagram with no loops has as many δ-functions as integrals, so the integrals can all be evaluated trivially, and there are no remaining integrals to be evaluated. Such a diagram is said to be a *tree level* diagram. The tree-level diagrams are indicated in figure 4.3 These diagrams can be evaluated without doing any integrals. Note that most of these are not connected diagrams. In order to evaluate a one-loop diagram, we need to do one $\int d\omega d^3\vec{p}$ integral; to evaluate a two-loop diagram we need to do two such integrals, i.e. $\int\int d\omega_1 d^3\vec{p}_1\, d\omega_2 d^3\vec{p}_2$; and so on. The expansion in loops is actually an expansion in powers of Planck's constant, since, as you will show in the problem set each propagator comes with a factor of $\hbar$ and each vertex

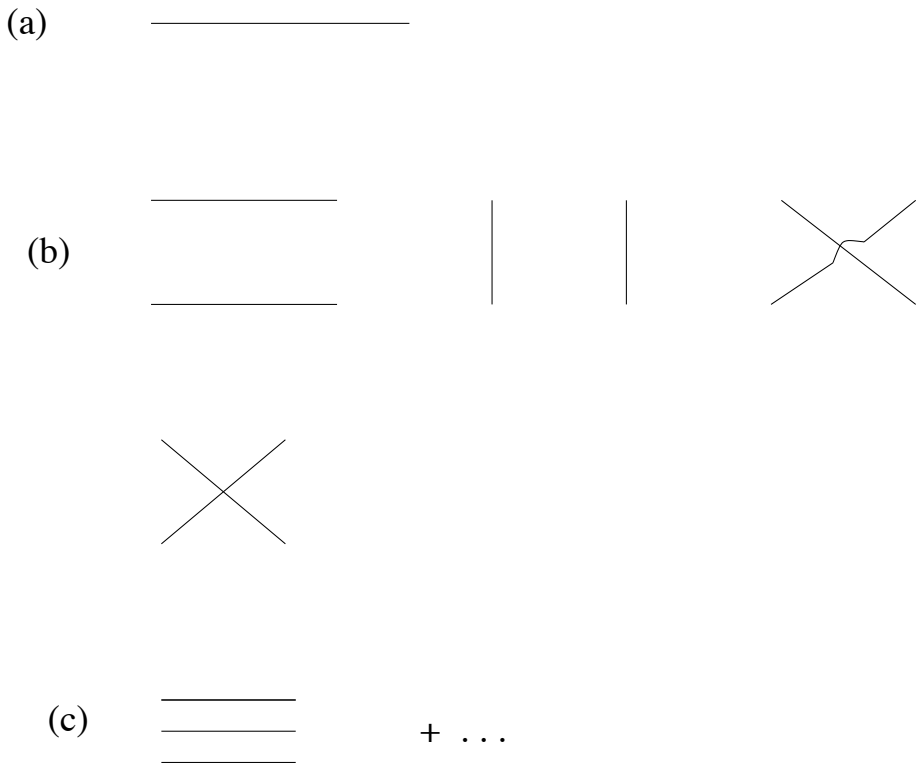


Figure 4.3: The tree-level Feynman diagrams of the theory (3.1).

comes with a factor of $1/\hbar$. An L-loop diagram comes with a coefficient of $\hbar^{L-1}$.

Turning now to the evaluation of multi-loop diagrams, we find the following trick (due to - you guessed it - Feynman) very useful. When we integrate the momenta in closed loops, we often encounter integrals of products of propagators. These are more easily evaluated if we combine the denominators of the propagators using the following formula:

$$\int_0^1 dx \frac{1}{[ax + b(1-x)]^2} = \frac{1}{ab} \tag{4.3}$$

For the more general case of the product of several propagators, which can occur in higher orders of perturbation theory, we will use the following formula:

$$\frac{\Gamma(\alpha)}{A^\alpha} = \int_0^\infty dt\, t^{\alpha-1} e^{-At} \tag{4.4}$$

Using this formula, we can write:

$$\frac{1}{\prod_j A_j^{\alpha_j}} = \prod_j \frac{1}{\Gamma(\alpha_j)} \int_0^\infty dt_j t_j^{\alpha_j - 1}\, e^{-A_j t_j} \left[\int_0^\infty ds\, \delta(s - \sum_j t_j) \right] \tag{4.5}$$

Here, the integral in brackets is equal to 1. Changing variables from t_i to x_i according to $t_j = s x_j$, we have:

$$\begin{aligned}
\frac{1}{\prod_j A_j^{\alpha_j}} &= \int_0^\infty ds \prod_j \frac{1}{\Gamma(\alpha_j)} \int_0^\infty dx_j\, s^{\alpha_j}\, x_j^{\alpha_j - 1}\, e^{-sA_j x_j}\, \frac{1}{s} \delta(1 - \sum_j x_j) \\
&= \prod_j \frac{1}{\Gamma(\alpha_j)} \int_0^1 dx_j\, x_j^{\alpha_j - 1} \int_0^\infty ds\, s^{\sum_j \alpha_j - 1}\, e^{-s\sum_j A_j x_j} \delta(1 - \sum_j x_j) \\
&= \frac{\Gamma\left(\sum_j \alpha_j\right)}{\prod_j \Gamma(\alpha_j)} \int_0^1 dx_1 \ldots dx_n\, \delta(1 - \sum_j x_j)\, \frac{\prod_j x^{\alpha_j - 1}}{\left(\sum_j x_j A_j\right)^{\sum \alpha_j}}
\end{aligned} \tag{4.6}$$

To see why these formulas are useful, consider the evaluation of diagram 4.1. We have

$$\frac{g^2}{2\rho} \int \frac{d\omega_1}{2\pi} \frac{d^3\vec{p}_1}{(2\pi)^3} \frac{|\vec{k}_1 + \vec{k}_2 - \vec{p}_1|^2}{(\epsilon_1 + \epsilon_2 - \omega_1)^2 - v_l^2 \left(\vec{k}_1 + \vec{k}_2 - \vec{p}_1\right)^2 + i\delta} \frac{|\vec{p}_1|^2}{\omega_1^2 - v_l^2 \vec{p}_1^2 + i\delta} \tag{4.7}$$

This can be brought into a more useful form using (4.3) with

$$\begin{aligned} a &= (\epsilon_1+\epsilon_2-\omega_1)^2 - v_l^2\Big(\vec{k}_1+\vec{k}_2-\vec{p}_1\Big)^2 + i\delta \\ b &= \omega_1^2 - v_l^2\vec{p}_1^{\,2} + i\delta \end{aligned} \tag{4.8}$$

Using (4.3), we can write

$$\begin{aligned} &\frac{|\vec{k}_1+\vec{k}_2-\vec{p}_1|^2}{(\epsilon_1+\epsilon_2-\omega_1)^2 - v_l^2\Big(\vec{k}_1+\vec{k}_2\Big)^2 + i\delta}\,\frac{|\vec{p}_1|^2}{\omega_1^2 - v_l^2\vec{p}_1^{\,2}+i\delta} \\ &= \int_0^1 dx\,\frac{|\vec{p}_1|^2|\vec{k}_1+\vec{k}_2-\vec{p}_1|^2}{\left[\left((\epsilon_1+\epsilon_2-\omega_1)^2 - v_l^2\Big(\vec{k}_1+\vec{k}_2-\vec{p}_1\Big)^2 + i\delta\right)x \;+\; \left(\omega_1^2 - v_l^2\vec{p}_1^{\,2}+i\delta\right)(1-x)\right]^2} \\ &= \int_0^1 dx\,\frac{|\vec{p}_1|^2|\vec{k}_1+\vec{k}_2-\vec{p}_1|^2}{\left[\omega_1^2 - v_l^2\vec{p}_1^{\,2} + x((\epsilon_1+\epsilon_2)^2 - v_l^2\vec{p}_1^{\,2}) - 2x\omega_1\,(\epsilon_1+\epsilon_2) + 2xv_l^2\vec{p}_1\cdot\Big(\vec{k}_1+\vec{k}_2-\vec{p}_1\Big)+i\delta\right]^2} \end{aligned}$$

If these integrals were from $-\infty$ to ∞, then we could shift the variables of integration without worrying. In condensed matter physics, these integrals are always cutoff, so we must be a little more careful; in our phonon theory, the momentum cutoff, Λ, is the inverse lattice spacing and the frequency cutoff, Λ_ω, is the Debye energy. However, so long as the external momenta and energies are much smaller than the cutoffs, i.e. $k_i \ll \Lambda, \omega_i \ll \Lambda_\omega$, we can shift the variables of integration and neglect the effect of this shift on the range of integration. Thus, we proceed by changing the variables of integration to $\omega = \omega_1 - x(\epsilon_1+\epsilon_2)$, $\vec{q} = \vec{p}_1 - x(\vec{k}_1+\vec{k}_2)$. Writing $a = x(1-x)((\epsilon_1+\epsilon_2)^2 - v_l^2(\vec{k}_1+\vec{k}_2)^2)$ we can write the loop integral as:

$$\begin{aligned} \int\frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{\vec{q}^{\,4}}{\left(\omega^2 - v_l^2\vec{q}^{\,2}+a\right)^2} &+ \int\frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{\vec{q}^{\,2}(\vec{k}_1+\vec{k}_2)^2\left(x^2+(1-x)^2+\frac{4}{3}x(1-x)\right)}{\left(\omega^2 - v_l^2\vec{q}^{\,2}+a\right)^2} \\ &+ \int\frac{d\omega}{2\pi}\frac{d^3q}{(2\pi)^3}\,\frac{x^2(1-x)^2(\vec{k}_1+\vec{k}_2)^4}{\left(\omega^2 - v_l^2\vec{q}^{\,2}+a\right)^2} \end{aligned} \tag{4.9}$$

Integrals of this form are often divergent. If we forget about the momentum and frequency cutoffs, then these integrals are ultraviolet divergent. If we're careful and we remember that these integrals are cutoff in the ultraviolet, then we will get finite (albeit cutoff-dependent) answers. On the other hand, these integrals are infrared divergent if $a = 0$ – i.e. if ϵ_i, $\vec{k}_i$ vanish. This is a real, physical effect: phonon Green functions *do* diverge when the

momenta, energies tend to zero. We will study the power-law forms of these divergences when we turn to the renormalization group in chapter 11.

It will sometimes be important to distinguish between the frequency cutoff, Λ_ω, and the momentum cutoff, Λ. Often, however, the distinction is unimportant and we can assume that there is a single cutoff, $\Lambda_\omega = v_l\Lambda$. In such a case, we can simplify (4.9) by using analytic continuation.

For either sign of $|\vec{q}|^2 - a$, the poles in ω are in the second and fourth quadrants. Hence, the contour of integration can be harmlessly rotated in an anti-counter-clockwise direction from the real ω axis to the imaginary ω axis. If we write $q_4 = -i\omega$ and $q^2 = q_4^2 + v_l^2\vec{q}^{\,2}$, then (4.9) is equal to

$$\frac{i}{v_l^7}\int\frac{d^4q}{(2\pi)^4}\,\frac{q^4}{\left(-q^2+a\right)^2} + \frac{i}{v_l^5}\int\frac{d^4q}{(2\pi)^4}\,\frac{\frac{3}{4}q^2(\vec{k}_1+\vec{k}_2)^2\left(x^2+(1-x)^2+\frac{4}{3}x(1-x)\right)}{\left(-q^2+a\right)^2} + \frac{i}{v_l^3}\int\frac{d^4q}{(2\pi)^4}\,\frac{x^2(1-x)^2(\vec{k}_1+\vec{k}_2)^4}{\left(-q^2+a\right)^2} \tag{4.10}$$

or

$$\frac{1}{v_l^7}I_{2,2}(a) + \frac{1}{v_l^5}\frac{3}{4}(\vec{k}_1+\vec{k}_2)^2\left(x^2+(1-x)^2+\frac{4}{3}x(1-x)\right)I_{2,1}(a) + \frac{1}{v_l^3}\left(x^2(1-x)^2(\vec{k}_1+\vec{k}_2)^4\right)I_{2,0}(a) \tag{4.11}$$

where the integrals which we need to study are:

$$I_{n,m}(a) = i\int\frac{d^4q}{(2\pi)^4}\,\frac{q^{2m}}{\left(-q^2+a\right)^n} \tag{4.12}$$

or, setting $z = q^2$, and $V(S^3) = 2\pi^2$,

$$\begin{aligned}
I_{n,m}(a) &= \frac{i}{16\pi^2}\int_0^{\Lambda^2}\frac{z^{m+1}\,dz}{(-z+a)^n} \\
&= \frac{(-1)^n}{(n-1)!}\,\frac{d^{n-1}}{da^{n-1}}\left(\frac{i}{16\pi^2}\int_0^{\Lambda^2}\frac{z^{m+1}\,dz}{(-z+a)}\right) \\
&= -\frac{(-1)^n}{(n-1)!}\,\frac{d^{n-1}}{da^{n-1}}\left(\frac{i}{16\pi^2}\int_{-a}^{\Lambda^2-a}\frac{(u+a)^{m+1}\,du}{u}\right) \\
&= -\frac{i}{16\pi^2}\frac{(-1)^n}{(n-1)!}\,\frac{d^{n-1}}{da^{n-1}}\left(\sum_{k=0}^{m+1}\binom{m+1}{k}u^{k-1}a^{m+1-k}\right) \\
&= -\frac{i}{16\pi^2}\frac{(-1)^n}{(n-1)!}\,\frac{d^{n-1}}{da^{n-1}}\left(a^{m+1}\ln u + \sum_{k=0}^{m+1}\frac{1}{k}\binom{m+1}{k}u^k a^{m+1-k}\right)_{-a}^{\Lambda^2-a}
\end{aligned}$$

$$= -\frac{i}{16\pi^2}\frac{(-1)^n}{(n-1)!}\frac{d^{n-1}}{da^{n-1}}(a^{m+1}\ln\left(\frac{\Lambda^2-a}{-a}\right) + \sum_{k=0}^{m+1}\frac{1}{k}\binom{m+1}{k}a^{m+1-k}\left[\left(\Lambda^2-a\right)^k-(-a)^k\right]) \tag{4.13}$$

Hence, we finally obtain:

$$\begin{aligned}
-\frac{i}{16\pi^2}\int_0^1 dx[\frac{1}{v_l^7}\left(x^2(1-x)^2(\vec{k}_1+\vec{k}_2)^4\right)\{-\frac{1}{2}\Lambda^4-3\Lambda^2x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)\\
-\Big(x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)\Big)^2\\
+3\Big(x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)\Big)^2\ln\left(\frac{\Lambda^2}{x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)}\right)\}\\
+\frac{1}{v_l^5}\frac{3}{4}(\vec{k}_1+\vec{k}_2)^2\left(x^2+(1-x)^2+\frac{4}{3}x(1-x)\right)\{\Lambda^2+x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)\\
-x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)\ln\left(\frac{\Lambda^2}{x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)}\right)\}\\
+\frac{1}{v_l^3}\left(x^2(1-x)^2(\vec{k}_1+\vec{k}_2)^4\right)\left\{\ln\left(\frac{\Lambda^2}{x(1-x)(v_l^2(\vec{k}_1+\vec{k}_2)^2-(\epsilon_1+\epsilon_2)^2)}\right)-1\right\}]
\end{aligned} \tag{4.14}$$

To summarize, we evaluate a Feynman diagram by the following steps:

- Use the Feynman rules to obtain a loop integral.
- Combine the denominators using Feynman's trick.
- Shift the variables of integration to make the denominator invariant under $\omega \to \omega$, $\vec{p} \to -\vec{p}$.
- Analytically continue to imaginary frequencies.
- Use rotational invariance to reduce the integral to an integral over a single variable for each $\omega, \vec{p}$.

4.3 Green Functions

In the preceding discussion, we have implicitly assumed that external phonon lines are 'on shell', i.e. they satisfy $\omega^2 = (\omega_p^l)^2$. It does, however, make sense to relax this requirement and allow even the external phonons to be "off-shell". One reason is that we may want to define a Feynman diagram – or

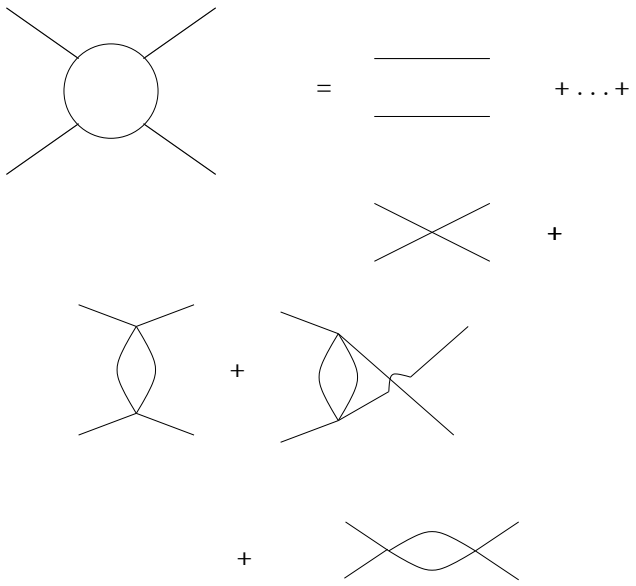


Figure 4.4: The definition of the 4 point Green function

a set of diagrams – which can be part of a larger diagram. In such a case, the lines which enter this part might not be on-shell.

Consider the diagram of figure 4.4a. The shaded circle represents all possible diagrams with 4 external legs. The first few are shown in figure 4.4b. We will call such an object

$$G(p_1, p_2, p_3, p_4) \tag{4.15}$$

(We will use p as a shorthand for $\vec{p}, \omega$.) $G(p_1, p_2, p_3, p_4)$ is defined to include the momentum conserving δ functions,

$$(2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)\, 2\pi\delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)$$

and a propagator

$$|\vec{p}_i|^2 \frac{i}{\omega_i^2 - v_l^2 p_i^2 + i\delta}$$

on each external leg.

We can define similar objects – called Green functions – for any number of external legs:

$$G(p_1, p_2, \ldots, p_n) \tag{4.16}$$

It is given by the sum of all diagrams with n external legs with (possibly off-shell) momenta and energies $p_1, p_2, \ldots, p_n$ with a propagator assigned to each external leg. We can Fourier transform the Green function to obtain the real-space *n-point Green function*, $G(x_1, \ldots, x_n)$.

While the notation $G(x_1, \ldots, x_n)$ is generically used for Green functions, the phonon two-point Green function is often denoted $D(x_1, x_2)$. However, we will reserve this notation for the two-point function of the other phonon field φ_i, which is more natural in some contexts.

The name 'Green function' is due to the fact that when the interaction is turned off, i.e. $g = 0$, the two-point Green function is a Green function of the differential operator

$$\rho\delta_{ik}\partial_t^2 - (\mu + \lambda)\,\partial_i\partial_k - \mu\delta_{ik}\partial_j\partial_j \tag{4.17}$$

This follows since the two-point Green function is just the derivative of the propagator:

$$\begin{aligned} G(x_1, x_2) &= \partial_i\partial_j \left\langle T\left(u_i(\vec{x}_1, t_1)u_j(\vec{x}_2, t_2)\right)\right\rangle \\ &= \partial_i\partial_j \frac{1}{\rho}\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{i(\vec{p}\cdot(\vec{x}_1 - \vec{x}_1) - \omega(t_1 - t_2))}\, \frac{i\epsilon_i^s\epsilon_j^s}{\omega^2 - (\omega_p^s)^2 + i\delta} \end{aligned} \tag{4.18}$$

It therefore satisfies

$$\left(\rho\delta_{ik}\partial_t^2 - (\mu + \lambda)\,\partial_i\partial_k - \mu\delta_{ik}\partial_j\partial_j\right) G(x_1, x_2) = i\delta\left(\vec{x}_1 - \vec{x}_2\right)\,\delta\left(t_1 - t_2\right) \tag{4.19}$$

as you showed in the first problem set.

4.4 The Generating Functional

Let's modify our Hamiltonian by adding a 'source term',

$$H \to H + \int d^3\vec{x}\, j(\vec{x}, t)\partial_k u_k(\vec{x}, t) \tag{4.20}$$

The source, j, is some arbitrary, prescribed function. We can think of $j(\vec{x}, t)$ as a knob which we can turn in order to set up compressional waves in the solid. By measuring the system at $(\vec{x}', t')$, we can study the propagation of sound waves.

Our interaction Hamiltonian is now $H_I + \int d^3\vec{x}\, j(\vec{x})\partial_k u_k(\vec{x})$, so our Feynman rules must be expanded to include a new vertex – which we will call a 'source vertex' –with only one line emerging from it. If this line has momentum, energy $\vec{p}$, ω, we assign $-i\,\tilde{j}(\vec{p}, \omega)$ to it ($\tilde{j}$ is the Fourier transform

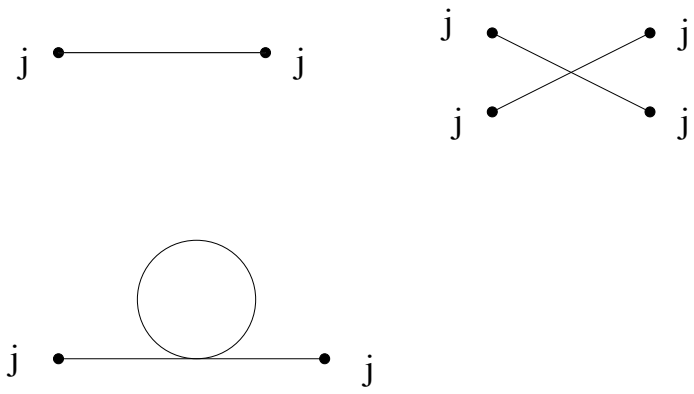


Figure 4.5: Some vacuum-to-vacuum diagrams in the presence of an external source.

of j). Let us now look at the vacuum-to-vacuum amplitude, which we will call $Z[j]$:

$$Z[j] = \langle 0|T\left\{e^{-i\int H_I + j\partial_k u_k}\right\}|0\rangle \tag{4.21}$$

This is given by the sum of all diagrams with no external legs. Several of these are shown in figure (4.5). We have denoted the new vertex by a large dot with a j next to it. To make life easy, let us shift the zero of energy by adding a constant to the Hamiltonian, $H \to H + E_0$ and choose E_0 so that:

$$Z[0] = 1 \tag{4.22}$$

Hence, the sum of all of the diagrams with no source vertices is 1. Consider a diagram with n source vertices. It will have an amplitude proportional to $\tilde{j}(\vec{p}_1, \omega_1) \ldots \tilde{j}(\vec{p}_n, \omega_n)$. Each $\tilde{j}(\vec{p}_i, \omega_i)$ creates a phonon with momentum $\vec{p}_i, \omega_i$. These n phonons enter the diagram along the external legs of the Green function $G(p_1, \ldots, p_n)$. We then have to integrate over all of the p_i's with a factor of $1/n!$ to avoid overcounting due to permutations of the p_i's. Hence,

$$Z[j] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^3\vec{p}_1\, d\omega_1 \ldots d^3\vec{p}_n\, d\omega_n\, j(p_1) \ldots j(p_n)\, G(p_1, \ldots, p_n) \tag{4.23}$$

We can Fourier transform this expression into real space:

$$Z[j] = 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^3\vec{x}_1\, dt_1 \ldots d^3\vec{x}_n\, dt_n\, j(x_1) \ldots j(x_n)\, G(x_1, \ldots, x_n) \tag{4.24}$$

We can understand the Green function in another way by considering the Hamiltonian with a source term,

$$\begin{aligned} H &\to H + j\partial_k u_k \\ &= H_0 + H' + j\partial_k u_k \\ &= (H_0 + H') + j\partial_k u_k \end{aligned} \tag{4.25}$$

We can now treat $H_0 + H'$ as our 'free' Hamiltonian and $j\partial_k u_k$ as our interaction Hamiltonian. Since $H_0 + H'$ is not actually free, we can't use Wick's theorem, but we we can still use Dyson's formula. The 'interaction' representation for this 'free' Hamiltonian is actually what we would call the Heisenberg representation for $j = 0$, so we will put an H superscript on all fields. Using Dyson's formula, we can express $Z[j]$ as:

$$\begin{aligned} Z[j] &= \langle 0|T\left\{e^{-i\int d^3\vec{x}\, dt\, j(\vec{x},t)\partial_k u_k^H(\vec{x},t)}\right\}|0\rangle \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^3\vec{x}_1\, dt_1 \ldots d^3\vec{x}_n\, dt_n\, j(x_1) \ldots j(x_n)\, \langle 0|T\left(\partial_k u_k^H(x_1) \ldots \partial_k u_k^H(x_n)\right)|0\rangle \end{aligned} \tag{4.26}$$

Comparing this with our earlier expression for $Z[j]$, we see that the Green function is given by:

$$G(x_1, x_2, \ldots, x_n) = \langle 0|T\left(\partial_k u_k^H(x_1) \ldots \partial_k u_k^H(x_n)\right)|0\rangle = \frac{\delta^n Z[j]}{\delta j(x_1) \ldots \delta j(x_n)} \tag{4.27}$$

In other words, the Green functions are the vacuum expectation values of the T-ordered product of a string of (Heisenberg picture) fields. These vacuum expectation values are the coefficients of the Taylor expansion of the vacuum-to-vacuum transition amplitude in the presence of an external source field. While our earlier definition – as a sum of Feynman diagrams – is convenient for perturbative calculation, the present definition as a vacuum expectation value is far more general since it is non-perturbative and can be compared with experiments. These vacuum expectation values are also called time-ordered *correlation functions*.

4.5 Connected Diagrams

There is a very useful theorem which states that

$$Z[j] = e^{W[j]} \tag{4.28}$$

where $Z[j]$ is the sum of vacuum-to-vacuum Feynman diagrams we defined above and $W[j]$ is the sum of *connected* vacuum-to-vacuum diagrams. To prove this theorem, observe that a given diagram which contributes to $Z[j]$ will, in general, be made up of several different connected diagrams and it will factor into the contributions from each of them. We can assemble the set of all vacuum-to-vacuum diagrams by putting together n_1 connected diagrams of type 1, n_2 connected diagrams of type 2,..., n_r connected diagrams of type r, etc. The contribution of such a diagram to $Z[j]$ will be the product of the contributions, C_r, of its connected components:

$$Z[j] = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \prod_{r=1}^{\infty} \frac{C_r^{n_r}}{n_r!} \tag{4.29}$$

The $n_r!$ in the denominator is the symmetry factor resulting from the permutations of the n_r identical connected components of type r. Commuting the sums and the product:

$$\begin{aligned} Z[j] &= \prod_{r=1}^{\infty} \left(\sum_{n_r=0}^{\infty} \frac{C_r^{n_r}}{n_r!} \right) \\ &= \prod_{r=1}^{\infty} e^{C_r} \\ &= e^{\sum_{r=1}^{\infty} C_r} \\ &= e^{W[j]} \end{aligned} \tag{4.30}$$

This theorem – sometimes called the linked cluster theorem – will be particularly useful when we construct a diagrammatic expansion for the partition function, $Z = tr(e^{-\beta H})$. In order to compute the free energy, $F = -T \ln Z$, we need only compute the connected diagrams.

The Taylor expansion of $W[j]$ is:

$$W[j] = W[0] + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int d^3\vec{x}_1\, dt_1 \ldots d^3\vec{x}_n\, dt_n\, j(x_1) \ldots j(x_n)\, G_c(x_1, \ldots, x_n) \tag{4.31}$$

where the G_c's are connected Green functions. The two-point connected Green function is given by:

$$\begin{aligned} G_c(x_1,x_2) &= \langle 0|T\left(\partial_k u_k(x_1)\partial_k u_k(x_2)\right)|0\rangle - \langle 0|\partial_k u_k(x_1)|0\rangle\langle 0|\partial_k u_k(x_2)|0\rangle \\ &= \langle 0|T\left(\left(\partial_k u_k(x_1) - \langle\partial_k u_k(x_1)\rangle\right)\left(\partial_k u_k(x_2)\right) - \langle\partial_k u_k(x_2)\rangle\right)|0\rangle \end{aligned} \tag{4.32}$$

This correlation function is often more useful since it measures fluctuations around mean values.

4.6 Spectral Representation of the Two-Point Green function

The spectral representation of the two-point Green function has the advantage of being intuitive yet well-suited for rigorous statements. It is obtained by inserting a complete set of states into

$$\begin{aligned} \langle T\left(\partial_i u_i(\vec{x},t)u_j(0,0)\right)\rangle &= \theta(t)\langle\partial_i u_i(\vec{x},t)\partial_j u_j(0,0)\rangle + \theta(-t)\langle\partial_j u_j(0,0)\partial_i u_i(\vec{x},t)\rangle \\ &= \theta(t)\sum_i \langle 0\left|\partial_i u_i(\vec{x},t)\right|i\rangle\langle i\left|\partial_j u_j(0,0)\right|0\rangle \\ &\quad + \theta(-t)\sum_i \langle 0\left|\partial_j u_j(0,0)\right|i\rangle\langle i\left|\partial_i u_i(\vec{x}_1,t_1)\right|0\rangle \end{aligned} \tag{4.33}$$

By translational invariance,

$$\langle 0\left|\partial_i u_i(\vec{x},t)\right|i\rangle = e^{i\vec{p}_i\cdot\vec{x}-i\omega_i t}\langle 0\left|\partial_i u_i(0,0)\right|i\rangle \tag{4.34}$$

where $\vec{p}_i$ and ω_i are the momentum and energy of the state $|i\rangle$.

Hence, we can write the Green function as

$$\begin{aligned} \langle T\left(\partial_i u_i(\vec{x},t)u_j(0,0)\right)\rangle &= \sum_i |\langle i\left|\partial_j u_j(0,0)\right|0\rangle|^2\left(\theta(t)e^{i(\vec{p}_i\cdot\vec{x}-\omega_i t)} + \theta(-t)e^{-i(\vec{p}_i\cdot\vec{x}-\omega_i t)}\right) \\ &= \int d^3\vec{P}dE\left[\sum_i |\langle i\left|\partial_j u_j(0,0)\right|0\rangle|^2\delta(\vec{P}-\vec{p_i})\,\delta(\omega_i - E)\right] \\ &\quad\times\left(\theta(t)e^{i(\vec{P}\cdot\vec{x}-Et)} + \theta(-t)e^{-i(\vec{P}\cdot\vec{x}-Et)}\right) \\ &\equiv \int d^3\vec{P}dE\,|P|^2 B(\vec{P},E)\left(\theta(t)e^{i(\vec{P}\cdot\vec{x}-Et} + \theta(-t)e^{-i(\vec{P}\cdot\vec{x}-Et}\right) \end{aligned} \tag{4.35}$$

Here, we have introduced the spectral function, $B(\vec{P},E)$, so that $|P|^2B(\vec{P},E)$ is given by the quantity in brackets. In order to take the Fourier transform, we have to add $i\delta$s to make the t integral convergent, so that we obtain:

$$G(\vec{x},\omega) = \int dt\int d^3\vec{P}dE\,|P|^2 B(\vec{P},E)\left(\theta(t)e^{i(\vec{P}\cdot\vec{x}+(\omega-E+i\delta)t)} + \theta(-t)e^{-i(\vec{P}\cdot\vec{x}-(\omega+E-i\delta)t)}\right)$$

$$= \int dt \int d^3\vec{P} dE \, |P|^2 B(\vec{P}, E) \left(\theta(t) e^{i(\omega - E)t - \delta t} \, e^{i\vec{P}\cdot\vec{x}} + \theta(-t) e^{i(\omega + E)t + \delta t} \, e^{-i\vec{P}\cdot\vec{x}} \right)$$
$$= \int d^3\vec{P} dE \, i|p|^2 B(\vec{p}, E) \left(\frac{e^{i\vec{P}\cdot\vec{x}}}{\omega - E + i\delta} - \frac{e^{-i\vec{P}\cdot\vec{x}}}{\omega + E - i\delta} \right) \tag{4.36}$$

or

$$G(\vec{p}, \omega) = \int dE \, i|p|^2 \left(\frac{B(\vec{p}, E)}{\omega - E + i\delta} - \frac{B(-\vec{p}, E)}{\omega + E - i\delta} \right) \tag{4.37}$$

For a parity-invariant system, $B(\vec{p}, E) = B(-\vec{p}, E)$, so

$$G(\vec{p}, \omega) = \int dE \, i|p|^2 \, \frac{2E \, B(\vec{p}, E)}{\omega^2 - E^2 + i\delta} \tag{4.38}$$

From its definition, $B(\vec{p}, E)$ is non-negative. If the phonons are non-interacting, $B(\vec{p}, E) = \delta(E^2 - \omega_p^2)$, and we recover the free-phonon two-point function.

We can split the sum over i into the vacuum state, the one-phonon states, and all other states. Now, let us assume that

$$\langle 0 \, | u_i(\vec{x}_1, t_1) | \, 0 \rangle = 0 \tag{4.39}$$

If it didn't, this would be the statement that there is some kind of static distortion present in the ground state. We could shift $u_i(\vec{x}_1, t_1)$ by $u_i(\vec{x}_1, t_1) \to u_i(\vec{x}_1, t_1) - \langle 0 | u_i(\vec{x}_1, t_1) | 0 \rangle$ and we would have the above for our new displacement field.

Consider a one-phonon state of momentum $\vec{p}$. Then, we will write:

$$|p|^2 \, Z = |\langle 0 \, | \partial_i u_i(0, 0) | \, \vec{p} \rangle|^2 \tag{4.40}$$

Rotational and Galilean invariance imply that the left-hand-side is independent of the direction of $\vec{p}$.

Then the spectral function can be broken into a piece, carrying weight Z, which looks like a non-interacting phonon, and the remaining 'incoherent' weight:

$$B(\vec{p}, E) = Z\delta(E^2 - \omega_p^2) + B_{\text{inc}}(\vec{p}, E) \tag{4.41}$$

The phonon propagates as a free phonon with probability Z and as a multi-phonon state of energy E with probability $B_{\text{inc}}(\vec{p}, E)$.

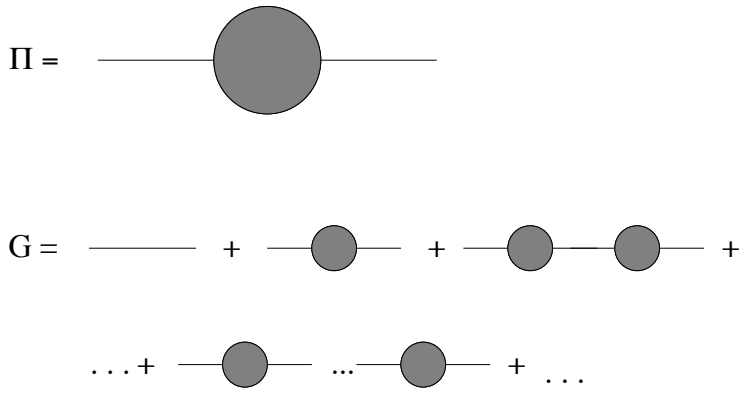


Figure 4.6: The relation between Π and G.

4.7 The Self-Energy and Irreducible Vertex

The two-point Green function $G(p_1, p_2)$ is given by the diagrams in figure (**??**). To zeroth order in g, it is simply the free phonon propagator. There is an $O(g)$ correction given by the diagram of figure (4.2) which leads to

$$G(p_1,p_2) = (2\pi)^3\delta(\vec{p}_1+\vec{p}_2)\,2\pi\delta(\omega_1+\omega_2)\,\Big(\,|\vec{p}_1|^2\,\frac{i}{\omega_1^2 - v_l^2 p_1^2 + i\delta} + \frac{g}{2}\left(|\vec{p}_1|^2\,\frac{i}{\omega_1^2 - v_l^2 p_1^2 + i\delta}\right)^2 \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{d\omega}{2\pi}\,|\vec{p}|^2\,\frac{i}{\omega^2 - v_l^2 p^2 + i\delta} + O(g^2)\Big) \tag{4.42}$$

For the two-point Green function, we can do better without doing much more work. Let us define the one-particle irreducible, or 1PI n-point Green function as the sum of all the Feynman graphs contributing to the n-point Green function which cannot be made disconnected by cutting a single internal line (this is a subset of the set of connected diagrams). For the 1PI n-point function, we do not include propagators on the external legs. The 1PI two-point Green function is given by $\Pi(p,\omega)/p^2$; $\Pi(p,\omega)$ is called the *self-energy* because the two-point Green function can be expressed in terms of it according to the graphical relation of figure 4.6. Summing this geometrical series, we have:

$$G(p_1,p_2) = (2\pi)^3\delta(\vec{p}_1+\vec{p}_2)\,2\pi\delta(\omega_1+\omega_2)\,|\vec{p}_1|^2\,\frac{i}{\omega_1^2 - v_l^2 p_1^2 - \Pi(\vec{p}_1,\omega_1) + i\delta} \tag{4.43}$$

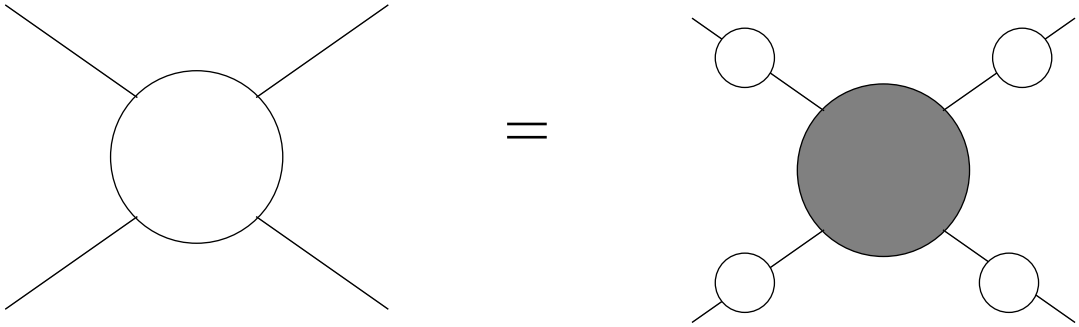


Figure 4.7: The relation between the regular and 1PI four point Green functions.

From our calculation above, we see that the self-energy is given by:

$$\Pi(p,\omega) = \frac{g}{2}\,|\vec{p}|^2 \int \frac{d^3\vec{q}}{(2\pi)^3}\,\frac{d\epsilon}{2\pi}\,|\vec{q}|^2\,\frac{i}{\epsilon^2 - v_l^2 q^2 + i\delta} \;+\; O(g^2) \tag{4.44}$$

In the problem set, you will show that $\mathrm{Im}\{\Pi(p,\omega)\}$ is related to the phonon *lifetime*.

The coherent weight in the phonon spectral function, Z, is given by:

$$Z^{-1} = 1 - \left(\frac{\partial}{\partial\omega^2}\mathrm{Re}(\Pi)\right)_{\omega=vp} \tag{4.45}$$

We can also define a 1PI 4-point Green function, $\Gamma(p_1,p_2,p_3,p_4)$. The full Green function $G(p_1,p_2,p_3,p_4)$ can be expressed in terms of $\Gamma(p_1,p_2,p_3,p_4)$ and the two-point Green function $G(p_1,p_2)$ according to the graphical relation of figure 4.7.

CHAPTER 5

Imaginary-Time Formalism

5.1 Finite-Temperature Imaginary-Time Green Functions

In the previous chapter, we found that the mathematical trick of analytically continuing to imaginary frequencies, $\omega \to i\omega$, facilitated the calculation of the integrals arising from Feynman diagrams. As we will demonstrate in this chapter, it is extremely convenient to work with imaginary-time from the outset. Such an imaginary-time formalism will have the advantage of having a natural extension to arbitrary temperature. It can also, in many cases, serve as a preliminary step in the calculation of retarded correlation functions which – as we will discuss in the next chapter – are the quantities most closely related to physical measurements.

We make the analytic continuation $it \to \tau$ and define the following object for $0 < \tau < \beta$:

$$\begin{aligned}
\mathcal{G}(\vec{x}-\vec{x}',\tau-\tau') &= \theta(\tau-\tau')Tr\left\{e^{-\beta H}\, e^{\tau H}\partial_k u_k(\vec{x})e^{-\tau H}\, e^{\tau' H}\partial_j u_j(\vec{x}')e^{-\tau' H}\right\} \\
&\quad +\theta(\tau'-\tau)Tr\left\{e^{-\beta H}\, e^{\tau' H}\partial_j u_j(\vec{x}')e^{-\tau' H}\, e^{\tau H}\partial_k u_k(\vec{x})e^{-\tau H}\right\} \\
&= \theta(\tau-\tau')Tr\left\{e^{-\beta H}\partial_k u_k(\vec{x},\tau)\partial_j u_j(\vec{x}',\tau')\right\} \\
&\quad +\theta(\tau'-\tau)Tr\left\{e^{-\beta H}\partial_j u_j(\vec{x}',\tau')\partial_k u_k(\vec{x},\tau)\right\} \\
&\equiv \left\langle T_\tau\left(\partial_k u_k(\vec{x},\tau)\partial_j u_j(\vec{x}',\tau')\right)\right\rangle \qquad (5.1)
\end{aligned}$$

We have passed from the Schrödinger representation in the first line to an imaginary-time Heisenberg representation in the third line. In the final line, we have defined the imaginary-time-ordering symbol, T_τ, by analogy with the real-time symbol, T: operators are arranged from right to left in order of increasing τ. In a similar way, we can define the imaginary-time-ordered product of strings of fields. If $\tau_1 > \tau_2 > \ldots > \tau_n$, then

$$\langle T_\tau \left(O_1 \ldots O_n\right)\rangle = Tr\left\{e^{-\beta H}\, O_1 \ldots O_n\right\} \tag{5.2}$$

Rather than take the expectation value in the gound state, we are averaging the expectation value over all energy eigenstates, $|n\rangle$, weighted by $e^{-\beta E_n}$. If we had used it rather than τ, we would have the finite-temperature time-ordered Green function. By working with τ, we will construct a Green function from which the retarded Green function can be constructed by analytic continuation. We will also exploit the formal analogy between the time-evolution operator $e^{-itH} \to e^{-\tau H}$ and the Boltzmann weight, $e^{-\beta H}$.

In analogy wth the real-time case, we write

$$U(\tau_2, \tau_1) = e^{-(\tau_2 - \tau_1)H} \tag{5.3}$$

We will add a source to the Hamiltonian and view the partition function, $Z[j] = Tr\{e^{-\beta H}\} = Tr\{U(\beta, 0)\}$, as the generating functional for imaginary-time Green functions. In the perturbative expansion of $U(\beta, 0)$, we will only encounter fields with imaginary-time arguments in the interval $[0, \beta]$.

There is a further condition which follows from the cyclic property of the trace. Since $0 < \tau, \tau' < \beta$, it follows that $-\beta < \tau - \tau' < \beta$. Now suppose that $\tau < \tau'$. Then,

$$\begin{aligned}
\mathcal{G}(\tau - \tau' < 0) &= Tr\left\{e^{-\beta H}\, e^{\tau' H} \partial_j u_j(\vec{x}') e^{-\tau' H}\, e^{\tau H} \partial_k u_k(\vec{x}) e^{-\tau H}\right\} \\
&= Tr\left\{e^{\tau H} \partial_k u_k(\vec{x}) e^{-\tau H}\, e^{-\beta H}\, e^{\tau' H} \partial_j u_j(\vec{x}') e^{-\tau' H}\right\} \\
&= Tr\left\{e^{-\beta H}\, e^{\beta H}\, e^{\tau H} \partial_k u_k(\vec{x}) e^{-\tau H}\, e^{-\beta H}\, e^{\tau' H} \partial_j u_j(\vec{x}') e^{-\tau' H}\right\} \\
&= \mathcal{G}(\tau - \tau' + \beta)
\end{aligned} \tag{5.4}$$

The first equality follows from the cyclic property of the trace. The final equality follows from the fact that $\tau - \tau' + \beta > 0$.

As a result of periodicity in imaginary-time, we can take the Fourier transform over the interval $[0, \beta]$:

$$\mathcal{G}(i\omega_n) = \int_0^\beta d\tau\, e^{i\omega_n \tau} \mathcal{G}(\tau) \tag{5.5}$$

where the *Matsubara frequencies* ω_n, are given by:

$$\omega_n = \frac{2n\pi}{\beta} \tag{5.6}$$

Inverting the Fourier transform, we have:

$$\mathcal{G}(\tau) = \frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n)\, e^{-i\omega_n\tau} \tag{5.7}$$

In the absence of interactions, we can evaluate the imaginary-time two-point Green function directly. Using the Planck distribution,

$$Tr\left\{e^{-\beta H_0} a_k^\dagger a_k\right\} = n_B(\omega_k) = \frac{1}{e^{\beta\omega_k}-1} \tag{5.8}$$

and substituting the mode expansion of u_k, we have:

$$\begin{aligned}
\mathcal{G}(\vec{x},\tau) &= \theta(\tau)\, Tr\left\{e^{-\beta H_0}\partial_k u_k(\vec{x},\tau)\partial_j u_j(0,0)\right\} \\
&\quad +\theta(-\tau)\, Tr\left\{e^{-\beta H_0}\partial_j u_j(0,0)\partial_k u_k(\vec{x},\tau)\right\} \\
&= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\left|\vec{k}\right|^2 [\theta(\tau)\left((n_B(\omega_k)+1)e^{i\vec{k}\cdot\vec{x}-\omega_k\tau} + n_B(\omega_k)e^{-i\vec{k}\cdot\vec{x}+\omega_k\tau}\right) \\
&\qquad +\theta(-\tau)\left(n_B(\omega_k)e^{i\vec{k}\cdot\vec{x}-\omega_k\tau} + (n_B(\omega_k)+1)e^{-i\vec{k}\cdot\vec{x}+\omega_k\tau}\right)
\end{aligned} \tag{5.9}$$

We can now compute the Fourier representation of the Green function:

$$\begin{aligned}
\mathcal{G}(\vec{p}, i\omega_n) &= \int d^3\vec{x} e^{i\vec{p}\cdot\vec{x}}\int_0^\beta d\tau e^{i\omega_n\tau}\mathcal{G}(\vec{x},\tau) \\
&= \frac{|\vec{p}|^2}{2\omega_k}\left(\frac{(n_B(\omega_p)+1)(e^{-\beta\omega_p}-1)}{i\omega_n-\omega_p} + \frac{n_B(\omega_p)(e^{\beta\omega_p}-1)}{i\omega_n+\omega_p}\right) \\
&= \frac{|\vec{p}|^2}{2\omega_k}\left(\frac{-1}{i\omega_n-\omega_p} + \frac{1}{i\omega_n+\omega_p}\right) \\
&= -|\vec{p}|^2\frac{1}{\omega_n^2+\omega_p^2} \\
&= -|\vec{p}|^2\frac{1}{\omega_n^2+v_l^2p^2}
\end{aligned} \tag{5.10}$$

In real-space, this is:

$$\mathcal{G}(\vec{x},\tau) = -\frac{1}{\beta}\sum_n \int \frac{d^3\vec{k}}{(2\pi)^3}\left|\vec{k}\right|^2 \frac{e^{-i\vec{k}\cdot\vec{x}-i\omega_n\tau}}{\omega_n^2+v_l^2k^2} \tag{5.11}$$

As we can see from the above derivation of the Green function, in the imaginary-time formalism, we have the decaying exponential $e^{-\omega\tau}$ rather than the oscillatory $e^{i\omega t}$, so we can dispense with the $i\delta$'s which are needed to ensure convergence in the real-time formulation. Indeed, from a mathematical point of view, imaginary-time is simpler than real-time; sadly, nature forces us to live in real-time.

5.2 Perturbation Theory in Imaginary Time

Following our real-time development, we consider a Hamiltonian of the form $H = H_0 + H_{\rm int}$, and go to the imaginary-time interaction representation. In this representation,

$$U(\tau_1, \tau_2) = T_\tau \left\{ e^{-\int_{\tau_2}^{\tau_1} d\tau H_{\rm int}(\tau)} \right\} \tag{5.12}$$

Hence, the imaginary-time Green function , which takes the Schrödinger picture form:

$$\begin{aligned} \mathcal{G}(\vec{x}, \tau - \tau') = \theta(\tau - \tau') Tr \left\{ e^{-\beta H}\, e^{\tau H} \partial_k u_k(\vec{x}) e^{-\tau H}\, e^{\tau' H} \partial_j u_j(0) e^{-\tau' H} \right\} \\ + \theta(\tau' - \tau) Tr \left\{ e^{-\beta H}\, e^{\tau' H}\, \partial_j u_j(0)\, e^{-\tau' H} e^{\tau H} \partial_k u_k(\vec{x}) e^{-\tau H} \right\} \end{aligned} \tag{5.13}$$

can be written in the interaction picture as:

$$\begin{aligned} \mathcal{G}(\vec{x}, \tau - \tau') &= \theta(\tau - \tau') Tr\{ e^{-\beta H_0}\, U(\beta, 0)\, U^{-1}(\tau, 0) \partial_k u_k(\vec{x}, \tau)\, U(\tau, 0) \times \\ &\qquad U^{-1}(\tau', 0) \partial_j u_j(0, \tau') U(\tau', 0) \} \\ &\quad + \theta(\tau' - \tau) Tr\{ e^{-\beta H_0}\, U(\beta, 0)\, U^{-1}(\tau', 0) \partial_j u_j(0, \tau')\, U(\tau', 0) \times \\ &\qquad U^{-1}(\tau, 0) \partial_k u_k(\vec{x}, \tau) U(\tau, 0) \} \\ &= \theta(\tau - \tau') Tr\{ e^{-\beta H_0}\, U(\beta, \tau)\, \partial_k u_k(\vec{x}, \tau)\, U(\tau, \tau')\, \partial_j u_j(0, \tau')\, U(\tau', 0) \} \\ &\quad + \theta(\tau' - \tau) Tr \left\{ e^{-\beta H_0}\, U(\beta, \tau')\, \partial_j u_j(0)\, U(\tau, \tau') \partial_k u_k(\vec{x}) U(\tau, 0) \right\} \end{aligned} \tag{5.14}$$

or, simply, as

$$\begin{aligned} \mathcal{G}(\vec{x}, \tau - \tau') &= Tr \left\{ e^{-\beta H_0} T_\tau \left(U(\beta, 0)\, \partial_k u_k(\vec{x}, \tau)\, \partial_j u_j(0, \tau') \right) \right\} \\ &\equiv \left\langle T_\tau \left(U(\beta, 0)\, \partial_k u_k(\vec{x}, \tau)\, \partial_j u_j(0, \tau') \right) \right\rangle \end{aligned} \tag{5.15}$$

As we noted earlier, only imaginary times $\tau \in [0, \beta]$ appear.

To evaluate this perturbatively, we expand $U(\beta, 0)$, as in the real-time case:

$$\mathcal{G}(\vec{x}, \tau - \tau') = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int_0^\beta \ldots \int_0^\beta d\tau_1 \ldots d\tau_n$$

$$\left\langle T_\tau \left(\partial_k u_k(\vec{x},\tau)\, \partial_j u_j(0,\tau')\, H_{\rm int}(\tau_1) \ldots H_{\rm int}(\tau_n) \right) \right\rangle \quad (5.16)$$

We can show that Wick's theorem holds for the imaginary-time-ordered product and use it to evaluate the above expectation values. Following our real-time development in this way, we can use Feynman diagrams to evaluate the perturbation series. The differences are that in imaginary-time,

- To each line, we associate a momentum, $\vec{p}$ and a Matsubara frequency, ω_n.
- For each external line with momentum $\vec{p}$, write $|\vec{p}|$.
- The propagator assigned to each internal line is:

$$-\frac{1}{\beta}\sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\, |\vec{p}|^2\, \frac{1}{\omega_n^2 + v_l^2 p^2}$$

- For each vertex with momenta, Matsubara frequencies $(\vec{p}_1,\omega_{n_1}),\ldots,(\vec{p}_4,\omega_{n_4})$ directed into the vertex, we write

$$g\,(2\pi)^3 \delta(\vec{p}_1+\vec{p}_2+\vec{p}_3+\vec{p}_4)\ \beta\,\delta_{n_1+n_2+n_3+n_4,0}$$

- Imagine labelling the vertices $1,2,\ldots,n$. Vertex i will be connected to vertices $j_1,\ldots,j_m$ $(m \le 4)$ and to external momenta $p_1,\ldots,p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices i, i is still connected to vertices $j_1,\ldots,j_m$ $(m \le 4)$ and to external momenta $p_1,\ldots,p_{4-m}$. If S is the number of permutations which leave the diagram invariant, we assign a factor $1/S$ to the diagram.
- If two vertices are connected by l lines, we assign a factor $1/l!$ to the diagram.

Using our result on connected Feynman diagrams from chapter 5, we see that the free energy, F, is given by

$$\begin{aligned} -\beta F &= \ln\left[Tr\left\{U(\beta,0)\right\}\right] \\ &= \sum \text{All connected diagrams with no external legs} \end{aligned} \quad (5.17)$$

5.3 Analytic Continuation to Real-Time Green Functions

In spite of their many charms, imaginary-time Green functions cannot be directly measured in experiments. Hence, we must contemplate real-time Green functions. In fact, it is useful to consider τ as a complex variable, and to analyze the properties of $\mathcal{G}(\tau)$ as τ varies through the complex plane. When τ lies on the real axis, we have the imaginary-time Green function:

$$\mathcal{G}(\vec{x}-\vec{x}',\tau-\tau') = \theta(\tau-\tau')Tr\left\{e^{-\beta H}\, e^{-\tau H}\partial_k u_k(\vec{x})e^{\tau H}\, e^{-\tau' H}\partial_j u_j(\vec{x}')e^{\tau' H}\right\} + \theta(\tau'-\tau)Tr\left\{e^{-\beta H}\, e^{-\tau' H}\partial_j u_j(\vec{x}')e^{\tau' H}\, e^{-\tau H}\partial_k u_k(\vec{x})e^{\tau H}\right\} \tag{5.18}$$

When τ is on the imaginary axis, $\tau = it$, we have the real-time Green function:

$$\mathcal{G}(\vec{x}-\vec{x}',t-t') = \theta(t-t')Tr\left\{e^{-\beta H}\, e^{-itH}\partial_k u_k(\vec{x})e^{itH}\, e^{-it' H}\partial_j u_j(\vec{x}')e^{it' H}\right\} + \theta(t'-t)Tr\left\{e^{-\beta H}\, e^{-it' H}\partial_j u_j(\vec{x}')e^{it' H}\, e^{-itH}\partial_k u_k(\vec{x})e^{itH}\right\} \tag{5.19}$$

For arbitrary complex τ, $\mathcal{G}(\tau)$ interpolates between these two. $\mathcal{G}(\tau)$ is not, however, an analytic function over the entire complex plane, as we can see from its spectral representation. We follow our earlier derivation of the spectral representation for the $T = 0$ real-time ordered Green function. The principal difference is that we now have $e^{-\beta E_n}|n\rangle$ rather than $|0\rangle$. Hence, by inserting a complete set of intermediate states, $|m\rangle\langle m|$, we have, in lieu of (4.35),

$$\mathcal{G}(\vec{x},\tau) = \int d^3\vec{p}\, d\omega [\sum_{n,m}\delta(\vec{p}-\vec{p}_m+\vec{p}_n)\delta(\omega-\omega_{nm})(\theta(\tau)e^{i\vec{p}\cdot\vec{x}-\omega\tau}\, e^{-\beta E_n} + \theta(-\tau))e^{-i\vec{p}\cdot\vec{x}+\omega\tau}\, e^{-\beta E_m})|\langle m\,|\partial_i u_i(0,0)|\, n\rangle|^2] \tag{5.20}$$

The Fourier transform,

$$\mathcal{G}(\vec{p},i\omega_j) = \int d^3\vec{x}\int_0^\beta d\tau\, \mathcal{G}(\vec{x},\tau)\, e^{i\omega_j\tau} \tag{5.21}$$

is given by:

$$\mathcal{G}(\vec{p},i\omega_j) = \int dE\, [\sum_{n,m}\left(e^{-\beta E_n} - e^{-\beta E_m}\right)\, |\langle m\,|\partial_i u_i(0,0)|\, n\rangle|^2$$

$$\times\ \delta(\vec{p}-\vec{p}_m+\vec{p}_n)\delta(E-E_m+E_n)]\,\frac{1}{E-i\omega_j} \tag{5.22}$$

Writing

$$p^2\,B(\vec{p},E)=\sum_{n,m}\left(e^{-\beta E_n}-e^{-\beta E_m}\right)\,|\langle m\,|\partial_i u_i(0,0)|\,n\rangle|^2\delta(\vec{p}-\vec{p}_m+\vec{p}_n)\,\delta(E-E_{mn}) \tag{5.23}$$

we have the spectral representation of $\mathcal{G}$:

$$\mathcal{G}(\vec{p},i\omega_n)=\int_{-\infty}^{\infty}dE\,\frac{p^2\,B(\vec{p},E)}{E-i\omega_j} \tag{5.24}$$

As usual, the spectral function $B(\vec{p},E)$ is real and positive.

$\mathcal{G}$ is not analytic as a result of the singularities in (11.53). Hence, it does not satisfy the Kramers-Kronig relations. However, the functions

$$\int_{-\infty}^{\infty}dE\,\frac{p^2\,B(\vec{p},E)}{E-\omega\pm i\delta} \tag{5.25}$$

are analytic functions of ω in the lower and upper-half planes respectively. Consequently, they do satisfy the Kramers-Kronig relations. As you will show in the problem set, these are the advanced and retarded correlation functions defined in the next section:

$$\begin{aligned}G_{\rm ret}(\vec{p},\omega)&=\int_{-\infty}^{\infty}dE\,\frac{p^2\,B(\vec{p},E)}{E-\omega-i\delta}\\ G_{\rm adv}(\vec{p},\omega)&=\int_{-\infty}^{\infty}dE\,\frac{p^2\,B(\vec{p},E)}{E-\omega+i\delta}\end{aligned} \tag{5.26}$$

Note that the spectral function is the difference between the retarded and advanced correlation functions.

$$G_{\rm ret}(\vec{p},\omega)-G_{\rm adv}(\vec{p},\omega)=2\pi i p^2\,B(\vec{p},\omega) \tag{5.27}$$

5.4 Retarded and Advanced Correlation Functions

In the previous chapter, we dealt with the time-ordered two-point correlation function,

$$\begin{aligned}G(\vec{x}_1,t_1;\vec{x}_2,t_2)=&\,\theta(t_1-t_2)\,\langle\partial_i u_i(\vec{x}_1,t_1)\partial_j u_j(\vec{x}_2,t_2)\rangle\\ &+\theta(t_2-t_1)\,\langle\partial_j u_j(\vec{x}_2,t_2)\partial_i u_i(\vec{x}_1,t_1)\rangle\end{aligned} \tag{5.28}$$

In this chapter, we have introduced the imaginary-time two-point correlation function:

$$\begin{aligned}\mathcal{G}(\vec{x}-\vec{x}',\tau-\tau') = \theta(\tau-\tau')Tr\left\{e^{-\beta H}\partial_k u_k(\vec{x},\tau)\partial_j u_j(\vec{x}',\tau')\right\} \\ + \theta(\tau'-\tau)Tr\left\{e^{-\beta H}\partial_j u_j(\vec{x}',\tau')\partial_k u_k(\vec{x},\tau)\right\}\end{aligned} \tag{5.29}$$

To this family of Green functions, we have now added the retarded and advanced correlation function. As we will see in the next chapter, the retarded correlation function is often more useful for comparison with experiments. At zero temperature, the retarded and advanced correlation functions are given by:

$$\begin{aligned}G_{\rm ret}(\vec{x}_1,t_1;\vec{x}_2,t_2) &= \theta(t_1-t_2)\left\langle 0\left|[\partial_i u_i(\vec{x}_1,t_1),\partial_j u_j(\vec{x}_2,t_2)]\right|0\right\rangle \\ G_{\rm adv}(\vec{x}_1,t_1;\vec{x}_2,t_2) &= \theta(t_2-t_1)\left\langle 0\left|[\partial_j u_j(\vec{x}_2,t_2),\partial_i u_i(\vec{x}_1,t_1)]\right|0\right\rangle\end{aligned} \tag{5.30}$$

At finite temperature, these are generalized to:

$$\begin{aligned}G_{\rm ret}(\vec{x}_1,t_1;\vec{x}_2,t_2) &= \theta(t_1-t_2)\,Tr\left\{e^{-\beta H}\left[\partial_i u_i(\vec{x}_1,t_1),\partial_j u_j(\vec{x}_2,t_2)\right]\right\} \\ G_{\rm adv}(\vec{x}_1,t_1;\vec{x}_2,t_2) &= \theta(t_2-t_1)\,Tr\left\{e^{-\beta H}\left[\partial_j u_j(\vec{x}_2,t_2),\partial_i u_i(\vec{x}_1,t_1)\right]\right\}\end{aligned} \tag{5.31}$$

For free phonons, the zero-temperature advanced and retarded correlation functions can be obtained by choosing the correct $i\delta$ prescription for the poles:

$$G_{\rm ret}(\vec{x}_1,t_1;\vec{x}_2,t_2) = \int d^3\vec{p}d\omega\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{ip^2}{(\omega+i\delta)^2-v_l^2p^2} \tag{5.32}$$

$$G_{\rm adv}(\vec{x}_1,t_1;\vec{x}_2,t_2) = \int d^3\vec{p}d\omega\, e^{i(\vec{p}\cdot(\vec{x}_1-\vec{x}_1)-\omega(t_1-t_2))}\,\frac{ip^2}{(\omega-i\delta)^2-v_l^2p^2} \tag{5.33}$$

For interacting phonons, the situation is not so simple. From (11.53), we see that for $i\omega_n$ in the upper-half-plane, we can obtain $\mathcal{G}$ from $G_{\rm ret}$ by taking $\omega \to i\omega_n$. From (11.53), we see that $\mathcal{G}(-i\omega_n) = \mathcal{G}^*(i\omega_n)$, from which we can obtain $\mathcal{G}$ for $i\omega_n$ in the lower-half-plane. In other words, we can always obtain $\mathcal{G}$ from $G_{\rm ret}$. What we would like to do, however, is obtain $G_{\rm ret}$ from $\mathcal{G}$. This analytic continuation from the Matsubara frequencies $i\omega_n$ to the entire upper-half-plane can often be done by simply taking $i\omega_n \to \omega + i\delta$. In the specific examples which we will look at, this procedure works. However, there is no general theorem stating that this can always be done.

In the next chapter, we will see why retarded correlation functions are intimately related to experimental measurements.

5.5 Evaluating Matsubara Sums

We can use contour integration and the fact that the poles of $n_B(\omega)$ are precisely the Matsubara frequencies, $\omega_n = 2n\pi/\beta$, to convert sums over Matsubara frequencies into integrals. As we will see, it is natural to rewrite these integrals in terms of advanced and retarded Green functions.

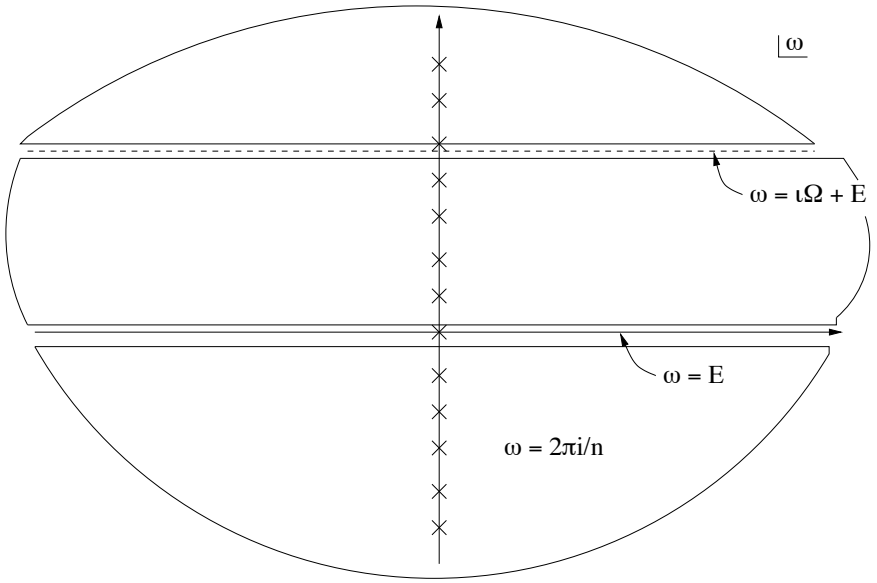


Figure 5.1: The contour of integration in (5.35).

Consider the sum

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m - i\omega_n, \vec{p}-\vec{q})\,\mathcal{G}(i\omega_n, \vec{q}) \tag{5.34}$$

This sum is equal to the following contour integral (see figure 5.1) since the integral avoids the singularities of the Green functions; consequently, it picks up only the poles of $n_B(\omega)$, thereby leading to the Matsubara sum.

$$\oint_C \frac{d\omega}{2\pi i}\, n_B(\omega)\,\mathcal{G}(i\Omega_m - \omega, \vec{p}-\vec{q})\,\mathcal{G}(\omega, \vec{q}) = \frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m - i\omega_n, \vec{p}-\vec{q})\,\mathcal{G}(i\omega_n, \vec{q}) \tag{5.35}$$

The singularities of the Green functions occur when ω or $i\Omega_m - \omega$ are real, as may be seen from the spectral representation (11.53). The only non-vanishing segments of the contour integral are those which run on either

side of the lines $\omega = E$ (the first term on the right-hand-side below) or $\omega = i\Omega_m - E$ (the second term) where E is real:

$$\oint_C \frac{d\omega}{2\pi i}\, n_B(\omega)\, \mathcal{G}(\omega + i\Omega_m)\, \mathcal{G}(\omega) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, n_B(E)\, \mathcal{G}(i\Omega_m - E)\, (\mathcal{G}(E + i\delta) - \mathcal{G}(E - i\delta)) + \frac{1}{2\pi i}\int_{\infty}^{-\infty} dE\, n_B(i\Omega_m - E)\, (\mathcal{G}(E + i\delta) - \mathcal{G}(E - i\delta))\, \mathcal{G}(i\Omega_m - E)$$

Note the reverse limits of integration in the second integral; they arise from the fact that E and ω are oppositely directed.

If we assume that the analytic continuation is straightforward (as it often is), then we can use (11.56) to write this as:

$$\oint_C \frac{d\omega}{2\pi i}\, n_B(\omega)\, \mathcal{G}(\omega + i\Omega_m)\, \mathcal{G}(\omega) = \int_{-\infty}^{\infty} dE\, n_B(E)\, \mathcal{G}(i\Omega_m - E, \vec{p} - \vec{q}) q^2\, B(E, \vec{q}) - \int_{-\infty}^{\infty} dE\, n_B(i\Omega_m - E)\, \mathcal{G}(i\Omega_m - E, \vec{q})(p - q)^2\, B(E, \vec{p} - \vec{q})$$

Since $n_B(i\Omega_m - E) = -(1 + n_B(E))$ for Matsubara frequencies $i\Omega_n$, we finally have:

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\omega_n + i\Omega_m)\, \mathcal{G}(i\omega_n) = \int_{-\infty}^{\infty} dE\, n_B(E)\, \mathcal{G}(i\Omega_m - E, \vec{p} - \vec{q})\, q^2\, B(E, \vec{q}) + \int_{-\infty}^{\infty} dE\, (n_B(E) + 1)\, \mathcal{G}(i\Omega_m - E, \vec{q})\, (p - q)^2\, B(E, \vec{p} - \vec{q}) \tag{5.36}$$

If we also continue $i\Omega_m \to \Omega + i\delta$, then we have:

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m - i\omega_n, \vec{p} - \vec{q})\, \mathcal{G}(i\omega_n, \vec{q}) = \int_{-\infty}^{\infty} dE\, n_B(E)\, G_{\rm ret}(\Omega - E, \vec{p} - \vec{q})\, q^2\, B(E, \vec{q}) + \int_{-\infty}^{\infty} dE\, (n_B(E) + 1)\, G_{\rm ret}(\Omega_m - E, \vec{q})\, (p - q)^2\, B(E, \vec{p} - \vec{q}) \tag{5.37}$$

It is important that we did the analytic continuation of the external frequency, Ω, at the end of the calculation, rather than during some intermediate step. This type of contour integral trick can be used rather generally to bring Matsubara sums to a convenient form, as you will see in the problem set.

5.6 The Schwinger-Keldysh Contour

The formalism which we have thus far constructed was designed to determine the transition amplitudes from some given intial state at $t = -\infty$ to some final state at $t = \infty$. In the previous chapter, we were able to relate these amplitudes to the amplitude for a system to remain in its vacuum state in the presence of an external source $j(x,t)$ and, hence, to correlation functions in the vacuum state. We may, however, wish to consider a situation in which the system is in a given intial state – say the vacuum state or the state at thermal equilibrium at inverse temperature β – but *we make no assumptions about the final state*. We may then wish to compute the correlation functions of some observable such as $\partial_i u_i(x,t)$ at time t.

In order to do this, we imagine evolving the system from some initial state $|i\rangle$ at $t = -\infty$ to $t = \infty$ and then back to $t = -\infty$. The evolution operator for such a process is:

$$U(\infty, -\infty)\, U(\infty, -\infty) \tag{5.38}$$

Clearly, this is simply equal to 1 and[2]

$$\langle i|U(-\infty, \infty)\, U(\infty, -\infty)|i\rangle = 1 \tag{5.39}$$

Suppose, however, that we switch on a source field, $j(x,t)$, only during the forward propagation in time. Then

$$\langle i|U(-\infty, \infty)\, U_j(\infty, -\infty)|i\rangle \neq 1 \tag{5.40}$$

and, by differentiating with respect to $j(x,t)$, we can obtain correlation functions.

If we wish to work with a system at zero-temperature which is in its ground state at $t = -\infty$, then we define the generating functional:

$$Z[j] = \langle 0|U(-\infty, \infty)\, U_j(\infty, -\infty)|0\rangle \tag{5.41}$$

At finite temperature, we have

$$Z[j] = Tr\left\{e^{-\beta H} U(-\infty, \infty)\, U_j(\infty, -\infty)\right\} \tag{5.42}$$

or

$$Z[j] = Tr\left\{U(-\infty - i\beta, -\infty)\, U(-\infty, \infty)\, U_j(\infty, -\infty)\right\} \tag{5.43}$$

i.e. we evolve the system from $t = -\infty$ to $t = \infty$ then back to $t = -\infty$ and thence to $t = -\infty - i\beta$.

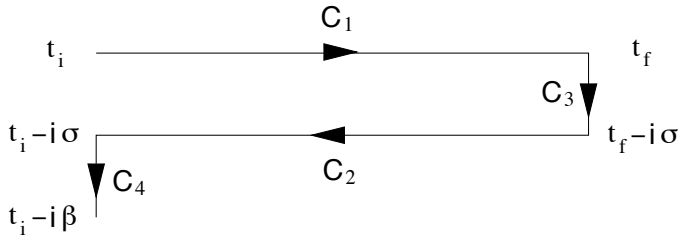


Figure 5.2: Real-time contour separated into four parts that factorize into separate contributions: $C_1 \cup C_2$ and $C_3 \cup C_4$.

This generating functional allows us to calculate non-equilibrium correlation functions: $j(x,t)$ can drive the system out of equilibrium since we make no assumptions about the final state of the system. The price which must be paid is the doubling of the number of the fields in the theory; the second copy of each field propagates backwards in time.

The Keldysh contour which we just described is just one example of a possible contour along which the time-evolution is performed. There is a more general class of contours, C, which go from $-\infty$ to ∞; from ∞ to $\infty - i\sigma$; from $\infty - i\sigma$ to $-\infty - i\sigma$; and thence to $\infty - i\beta$. We make the choice $\sigma = \beta/2$ for which the propagator takes a particularly simple form; however, this is a matter of taste. All choices of σ share the advantage of being real-time formulations and thereby obviating the need for potentially ill-defined analytical continuations.

There is an important factorization property, which we won't prove here, for the contributions from each piece of the contour to the functional integral Z:

$$Z = Z^{C_1 \cup C_2}\, Z^{C_3 \cup C_4}\ . \tag{5.44}$$

where

$$\begin{aligned} Z &= Tr\left\{U(-\infty - i\beta, -\infty - i\sigma)\, U(-\infty - i\sigma, \infty - i\sigma)\, U(\infty - i\sigma, \infty)\, U_j(\infty, -\infty)\right\} \\ &= Tr\left\{T_c e^{-i\int_C dt\, H_{\rm int}(t)}\right\} \end{aligned} \tag{5.45}$$

Only C_1 and C_2 are important in obtaining correlation functions. Using Dyson's formula for $U(t_f, t_i)$, we can expand $Z = Z^{C_1 \cup C_2}$ perturbatively as we did in the equilibrium zero-temperature and imaginary-time formalisms. We can use Wick's theorem to evaluate T_c-ordered products. To construct the resulting perturbation theory, it is useful to denote the fields on the

upper (C_1) and the lower (C_2) pieces of the countour by

$$u_i^1(t) = u_i(t), \qquad u_i^2(t) = u_i(t - i\sigma), \qquad (t = \text{real}) \tag{5.46}$$

The Feynman rules are similar to those of our equilibrium zero-temperature theory with the major difference being each vertex is labelled by an index $a = 1, 2$ and the amplitude is assigned a factor of -1 for each $a = 2$ vertex. The vertex resulting from the source field $j(x, t)$ is assigned $a = 1$. The propagator between two vertices with indices a and b is given by:

$$-i\Delta^{ab}(t - t', x - x') =< T_c[\partial_i u_i^a(t, x)\partial_j u_j^b(t', x')] > \tag{5.47}$$

where T_c denotes ordering of fields according their position along the contour of Fig. 5.2. For the Keldysh contour, the diagonal elements of the propagator are the real-time zero-temperature time- and anti-time ordered propagators. The off-diagonal element contains all of the information about the occupation numbers of states; in thermal equilibrium the occupation numbers are given by $n_B(\omega)$, but they can be more general. For our choice, $\sigma = \beta/2$, the dynamical information contained in the zero-temperature propagator and the information about occupation numbers can be untangled by the parametrization:

$$i\Delta(\omega, k) = u(\omega)\; i\Delta_0(\omega, k)\; u^\dagger(\omega)\;, \tag{5.48}$$

where

$$i\Delta_0(\omega, k) = \begin{pmatrix} iG_0(\omega, k) & 0 \\ 0 & -iG_0^*(\omega, k) \end{pmatrix}\;, \tag{5.49}$$

with $iG_0(\omega, k)$ the usual time-ordered propagator

$$iG_0(t - t', x - x') = \langle T[\partial_i u_i(t, x)\partial_j u_j(t', x')]\rangle \tag{5.50}$$

and $-iG_0^*(\omega, k)$, consequently, the anti-time-ordered one

$$-iG_0^*(t - t', x - x') = (\langle T[\partial_i u_i(t, x)\partial_i u_i(t', x')]\rangle)^* = \langle \bar{T}[\partial_i u_i(t', x')\partial_i u_i(t, x)]\rangle \tag{5.51}$$

The matrix u contains the information about the temperature. This matrix is given by

$$u(\omega) = \begin{pmatrix} \cosh\Delta\theta_\omega & \sinh\Delta\theta_\omega \\ \sinh\Delta\theta_\omega & \cosh\Delta\theta_\omega \end{pmatrix}\;, \quad \text{where} \;\; \Delta\theta_\omega = \theta_\omega^T - \theta_\omega^{T=0},$$
$$\text{and} \qquad \cosh^2\theta_\omega^T = \frac{1}{1 - e^{-\omega/T}}\;. \tag{5.52}$$

Notice that at zero temperature $u = \mathbf{1}$.

To summarize,

- To each line, we associate a momentum, $\vec{p}$ and a frequency, ω.
- To each vertex we assign an index $a = 1, 2$. External lines are assigned $a = 1$ at their free end.
- For each external line with momentum $\vec{p}$, write $|\vec{p}|$.
- The propagator assigned to an internal line carrying momentum $\vec{p}$ and frequency ω which connectes vertices labelled by indices a and b is:

$$\Delta_0^{ab}(\vec{p}, \omega)$$

- For each vertex carrying index a with momenta, frequencies $(\vec{p}_1, \omega_1), \ldots, (\vec{p}_4, \omega_4)$ directed into the vertex, we write

$$(3 - 2a)\, g\, (2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)\, \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4)$$

- Imagine labelling the vertices $1, 2, \ldots, n$. Vertex i will be connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices i, i is still connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$. If S is the number of permutations which leave the diagram invariant, we assign a factor $1/S$ to the diagram.
- If two vertices are connected by l lines, we assign a factor $1/l!$ to the diagram.

In equilibrium, the Schwinger-Keldysh formalism gives results which are identical to those of the Matsubara formalism. Out of equilibrium, however, the two formalisms give different results; only the Schwinger-Keldysh is correct.

CHAPTER 6

Measurements and Correlation Functions

6.1 A Toy Model

We will now take a break from our development of techniques for calculating correlation functions to relate retarded correlation functions to experimental measurements. We will also discuss those properties of retarded correlation functions which follow from causality, symmetries, and conservation laws.

Let us take a look at a toy model to see why retarded correlation functions are useful objects which are simply related to experimentally measurable quantities. Consider a single damped harmonic oscillator, with equation of motion

$$\frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = F^{\text{ext}}(t) \tag{6.1}$$

We define the retarded response function, $\chi_{\text{ret}}(t-t')$ by

$$x(t) = \int_{-\infty}^{\infty} dt' \, \chi_{\text{ret}}(t-t') F^{\text{ext}}(t') \tag{6.2}$$

By causality, $\chi_{\text{ret}}(t-t')$ must vanish for $t-t' < 0$. Substituting the definition of $\chi_{\text{ret}}(t-t')$, (6.2) into the equation of motion (6.1), we see that $\chi_{\text{ret}}(t-t')$ satisfies:

$$\frac{d^2}{dt^2}\chi_{\text{ret}}(t-t') + \gamma \frac{d}{dt}\chi_{\text{ret}}(t-t') + \omega_0^2 \chi_{\text{ret}}(t-t') = \delta(t-t') \tag{6.3}$$

Thus, $\chi_{\rm ret}(t-t')$ is the Green function of the differential operator on the left-hand-side of (6.1) subject to the boundary condition that $\chi_{\rm ret}(t-t')=0$ for $t-t'<0$.

We can define the Fourier transform of χ:

$$\chi(\omega)=\int_{-\infty}^{\infty}dte^{i\omega t}\chi(t)=\int_{0}^{\infty}dte^{i\omega t}\chi(t) \tag{6.4}$$

Since $\chi(t)$ vanishes for $t<0$, the integral is well-defined for ω anywhere in the upper-half-plane. Therefore, $\chi(\omega)$ is an analytic function in the upper-half-plane.

Substituting the Fourier representation of $\chi(t)$ in the equation of motion, we find $\chi(\omega)$:

$$\chi(\omega)=\frac{1}{\omega_0^2-\omega^2-i\gamma\omega} \tag{6.5}$$

We can break $\chi(\omega)$ into its real and imaginary parts:

$$\chi(\omega)=\chi'(\omega)+i\chi''(\omega) \tag{6.6}$$

From (6.5), we have in our toy model:

$$\begin{aligned}\chi'(\omega)&=\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}\\ \chi''(\omega)&=\frac{\gamma\omega}{(\omega_0^2-\omega^2)^2+(\gamma\omega)^2}\end{aligned} \tag{6.7}$$

From the above definition,

$$\begin{aligned}\chi''(\omega)&=\mathrm{Im}\left\{\int_{-\infty}^{\infty}dte^{i\omega t}\chi(t)\right\}\\ &=\int_{-\infty}^{\infty}dt\,\frac{1}{2i}\left(e^{i\omega t}-e^{-i\omega t}\right)\chi(t)\\ &=\int_{-\infty}^{\infty}dte^{i\omega t}\,\frac{1}{2i}\left(\chi(t)-\chi(-t)\right)\end{aligned} \tag{6.8}$$

Similarly,

$$\chi'(\omega)=\int_{-\infty}^{\infty}dte^{i\omega t}\,\frac{1}{2}\left(\chi(t)+\chi(-t)\right) \tag{6.9}$$

Thus, $\chi''(\omega)$ is the Fourier transform of the part of $\chi(\omega)$ which is not invariant under $t\to-t$ while $\chi'(\omega)$ is the Fourier transform of the part of $\chi(\omega)$ which is invariant under $t\to-t$. In other words, $\chi''(\omega)$ knows about

the arrow of time which is implicit in the condition $\chi(t-t')$ vanishes for $t-t'<0$. $\chi'(\omega)$, on the other hand, does not.

This is reflected in the fact that $\chi''(\omega)$ determines the dissipative response. To see this, suppose that we apply a force, $F(t)$. The work done by this force is the energy which is tranferred to the system – i.e. the dissipation:

$$\begin{aligned}\frac{dW}{dt} &= F(t)\,\frac{dx}{dt}\\ &= F(t)\,\frac{d}{dt}\int_{-\infty}^{\infty} dt'\,\chi_{\rm ret}(t-t')F(t')\\ &= F(t)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,e^{i\omega t}\,i\omega\chi(\omega)\,F(\omega)\\ &= \int\frac{d\omega}{2\pi}\,\frac{d\omega'}{2\pi}\,e^{i(\omega+\omega')t}\,i\omega\chi(\omega)\,F(\omega)F(\omega') \qquad (6.10)\end{aligned}$$

If we assume $F(t) = F_0\cos\Omega_0 t$ and compute the zero-frequency part of dW/dt (rather than the part which oscillates at $2\omega_0$), we find:

$$\begin{aligned}\frac{dW}{dt}(\omega=0) &= \frac{1}{4}\,F_0^2\,i\,(\Omega_0\chi(\Omega_0)-\Omega_0\chi(-\Omega_0))\\ &= \frac{1}{2}\,F_0^2\,\Omega_0\chi''(\Omega_0) \qquad (6.11)\end{aligned}$$

The essential reason that χ' doesn't enter the dissipation is that the time-reversal symmetry of χ' implies that the energy gain and loss due to χ' are the same. χ' is often called the *reactive* part of the susceptibility while χ'' is the dissipative or absorptive part.

For our toy model, the energy dissipation,

$$\frac{dW}{dt} = \frac{1}{2}\,F_0^2\,\Omega_0\,\frac{\gamma\Omega_0}{(\omega_0^2-\Omega_0^2)^2+(\gamma\Omega_0)^2} \qquad (6.12)$$

is maximum at $\Omega_0=\pm\omega_0$, i.e. on resonance. Consider the resonance $\Omega_0\approx\omega_0$. Approximating $\Omega_0+\omega_0\approx 2\omega_0$, we have a Lorentzian lineshape:

$$\frac{dW}{dt} = \frac{1}{2}\,F_0^2\,\frac{\gamma}{4(\omega_0-\Omega_0)^2+\gamma^2} \qquad (6.13)$$

with full-width γ at half-maximum.

As a result of the analyticity of χ in the upper-half-plane – i.e. as a result of causality – we have the Kramers-Kronig relations. The analyticity of $\chi(z)$ in the upper-half-plane implies that:

$$\oint_C\frac{dz}{\pi i}\,\frac{\chi(z)}{z-\omega} = 0 \qquad (6.14)$$

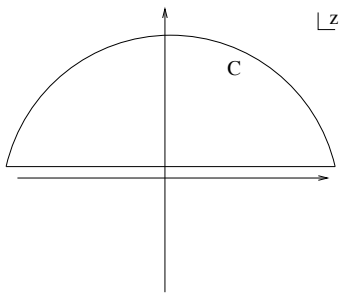


Figure 6.1: The countour of integration taken for the Kramers-Kronig relation.

for the contour of figure 6.1 The integral of the semicircular part of the contour vanishes, so we have:

$$\int_{-\infty}^{\infty} \frac{d\omega'}{\pi i} \frac{\chi(\omega')}{\omega' + i\epsilon - \omega} = 0 \tag{6.15}$$

Using the relation:

$$\frac{1}{\omega' + i\epsilon - \omega} = \mathcal{P}\left(\frac{1}{\omega' - \omega}\right) - i\pi\,\delta(\omega' - \omega) \tag{6.16}$$

we have

$$\chi(\omega) = -i\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\chi(\omega')}{\omega' - \omega} \tag{6.17}$$

Comparing real and imaginary parts, we have the Kramers-Kronig relations:

$$\begin{aligned} \chi'(\omega) &= \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\chi''(\omega')}{\omega' - \omega} \\ \chi''(\omega) &= -\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\chi'(\omega')}{\omega' - \omega} \end{aligned} \tag{6.18}$$

6.2 General Formulation

We can cull the essential features of the toy model and apply them to a many-body system such as our phonon theory. It is fairly clear that this can be done for a free field theory which is, after all, just a set of harmonic oscillators. As we will see momentarily, this can be done – at least at a formal level – for any theory.

Suppose our external probe couples to our system through a term in the Hamiltonian of the form:

$$H_{\rm probe} = \int d^3\vec{x}\, \phi(\vec{x},t)\, f(\vec{x},t) \tag{6.19}$$

$f(\vec{x},t)$ is our external probe, which we control, and $\phi(\vec{x},t)$ is some quantum field describing the system. In our phonon theory, we could take $\phi(\vec{x},t) = \partial_k u_k(\vec{x},t)$ in which case our probe compresses the solid. Alternatively, we could take $\phi(\vec{x},t) = u_3(\vec{x},t)$, causing displacements along the 3-direction. We will work – as in the last chapter when we defined Green functions – in an interaction representation in which $H_{\rm probe}$ is the interaction Hamiltonian and the rest of the Hamiltonian is the 'free' Hamiltonian. Let us suppose that we now measure the field $\eta(\vec{x},t)$, which may or may not be the same as $\phi(\vec{x},t)$. Its expectation value is given by:

$$\langle \eta(\vec{x},t)\rangle = \left\langle 0 \left| U_I^{-1}(t,-\infty)\, \eta(\vec{x},t)\, U_I(t,-\infty) \right| 0 \right\rangle \tag{6.20}$$

where

$$\begin{aligned} U_I(t,-\infty) &= T\left\{ e^{-i\int_{-\infty}^{t} dt'\, H_{\rm probe}(t')} \right\} \\ &= 1 - i\int_{-\infty}^{t} dt'\, H_{\rm probe}(t') + \ldots \end{aligned} \tag{6.21}$$

If we keep only terms up to first-order in $H_{\rm probe}$, then we have:

$$\begin{aligned} \langle \eta(\vec{x},t)\rangle &= \left\langle 0 \left| \left(1 + i\int_{-\infty}^{t} dt'\, H_{\rm probe}(t')\right)\, \eta(\vec{x},t)\, \left(1 - i\int_{-\infty}^{t} dt'\, H_{\rm probe}(t')\right) \right| 0 \right\rangle \\ &= \langle 0\,|\,\eta(\vec{x},t)\,|\,0\rangle_0 + \left\langle 0 \left| i\int_{-\infty}^{t} dt'\, \left[H_{\rm probe}(t'), \eta(\vec{x},t)\right] \right| 0 \right\rangle \\ &= \langle 0\,|\,\eta(\vec{x},t)\,|\,0\rangle_0 + i\int d^3\vec{x}' \int_{-\infty}^{t} dt'\, f(\vec{x}',t') \langle 0 \,|\, \left[\phi(\vec{x}',t'), \eta(\vec{x},t)\right] | 0 \rangle \end{aligned} \tag{6.22}$$

We have added a subscript 0 to emphasize that these are interaction picture states – i.e. these are expectation values in the absence of the probe. Let us assume, as is usually the case, that

$$\langle 0\,|\,\eta(\vec{x},t)\,|\,0\rangle_0 = 0 \tag{6.23}$$

Then,

$$\langle \eta(\vec{x},t)\rangle = i\int d^3\vec{x}' \int_{-\infty}^{t} dt'\, f(\vec{x}',t') \langle 0 \,|\, \left[\phi(\vec{x}',t'), \eta(\vec{x},t)\right] | 0 \rangle \tag{6.24}$$

The commutator on the right-hand-side, $\langle 0|\ [\phi(\vec{x}',t'),\eta(\vec{x},t)]|0\rangle$, is an example of a *response function*.

Let us specialize to the case $\eta(\vec{x},t)=\phi(\vec{x}',t')$. Then, we write:

$$\langle\eta(\vec{x},t)\rangle = -i\int d^3\vec{x}'\int_{-\infty}^{t} dt'\, f(\vec{x}',t')\,\chi(\vec{x},\vec{x};t,t') \tag{6.25}$$

If the Hamiltonian is space- and time-translationally in the absence of $H_{\rm probe}$, then we can write:

$$\chi(\vec{x},\vec{x}';t,t') = \chi(\vec{x}-\vec{x}',t-t') \tag{6.26}$$

We can also extend the dt' integral to ∞

$$\langle\eta(\vec{x},t)\rangle = -i\int d^3\vec{x}'\int_{-\infty}^{\infty} dt'\, f(\vec{x}',t')\,\chi(\vec{x}-\vec{x}',t-t') \tag{6.27}$$

if we define

$$\chi(\vec{x}-\vec{x}',t-t') \equiv i\theta(t-t')\left\langle 0\,\middle|\,\left[\phi(\vec{x},t),\phi(\vec{x}',t')\right]\,\middle|\,0\right\rangle \tag{6.28}$$

As in our toy model, we can define the Fourier transform with respect to time, $\chi(\vec{x}-\vec{x}',\omega)$ and its real and imaginary parts,

$$\chi(\vec{x}-\vec{x}',\omega) = \chi'(\vec{x}-\vec{x}',\omega) + i\chi''(\vec{x}-\vec{x}',\omega) \tag{6.29}$$

Following the steps of (6.8), we see that $\chi''(\vec{x}-\vec{x}',\omega)$ is the Fourier transform of the commutator without a θ-function:

$$\chi''(\vec{x}-\vec{x}',t-t') = \int\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')}\chi''(\vec{x}-\vec{x}',\omega) \tag{6.30}$$

where

$$\chi''(\vec{x}-\vec{x}',t-t') = \left\langle 0\,\middle|\,\left[\phi(\vec{x},t),\phi(\vec{x}',t')\right]\,\middle|\,0\right\rangle \tag{6.31}$$

As in our toy model, χ'', satisfies the antisymmetry properties:

$$\begin{aligned}\chi''(\vec{x}-\vec{x}',t-t') &= -\chi''(\vec{x}'-\vec{x},t'-t)\\ \chi''(\vec{x}'-\vec{x},\omega) &= -\chi''(\vec{x}'-\vec{x},-\omega)\end{aligned} \tag{6.32}$$

These properties follow from the fact that χ'' is a commutator.

Following the steps which we took in our toy model, the dissipation rate under the influence of a periodic probe, $f(\vec{x},t)=f_\omega(\vec{x})\cos\omega t$ is given by

$$\frac{dw}{dt} = \int d^3\vec{x}\,d^3\vec{x}'\,\omega\chi''(\vec{x}'-\vec{x},\omega)f_\omega(\vec{x})f_\omega(\vec{x}') \tag{6.33}$$

We can also follow the derivation for our toy model to show that $\chi(\vec{x}' - \vec{x}, \omega)$ satisfies the Kramers-Kronig relations (6.18). The Kramers-Kronig relations can be approached from a different angle by constructing a spectral representation for the response function (6.28). Following our earlier derivations of spectral representations, we have:

$$\chi(\vec{q}, \omega) = \int_{-\infty}^{\infty} dE \, \frac{\rho(\vec{q}, E)}{E - \omega - i\delta} \tag{6.34}$$

where

$$\rho(\vec{q}, E) = \sum_m |\langle m \, |\phi(0,0)| \, 0\rangle|^2 \delta(\vec{q} - \vec{p}_m) \, \delta(E - \omega_m) \tag{6.35}$$

If we construct a spectral representation for $\chi''(\vec{q}, \omega)$ – which can be done trivially since there are no θ-functions – or simply take the imaginary part of (6.34), we see that:

$$\chi''(\vec{q}, \omega) = \pi \, \rho(\vec{q}, \omega) \tag{6.36}$$

In other words, $\chi''(\vec{q}, \omega)$ is the spectral function for $\chi(\vec{q}, \omega)$:

$$\chi(\vec{q}, \omega) = \int_{-\infty}^{\infty} \frac{dE}{\pi} \, \frac{\chi''(\vec{q}, E)}{E - \omega - i\delta} \tag{6.37}$$

which is the Kramers-Kronig relation. According to (6.37) $\chi(\vec{q}, \omega)$ is singular whenever $\chi''(\vec{q}, \omega)$ is non-vanishing. These are the regions of phase space where there are states of the system with which the external probe can resonate, thereby causing dissipation.

6.3 The Fluctuation-Dissipation Theorem

Consider the correlation function:

$$S_{\eta\phi}(\vec{x}, t) = Tr \left(e^{-\beta H} \, \eta(\vec{x}, t) \, \phi(0,0) \right) \tag{6.38}$$

The response function can be expressed in terms of the correlation function:

$$\chi_{\eta\phi}(\vec{x}, t) = \theta(t) \, \left(S_{\eta\phi}(\vec{x}, t) - S_{\phi\eta}(-\vec{x}, -t) \right) \tag{6.39}$$

and its dissipative part is simply

$$\chi_{\eta\phi}{}''(\vec{x}, t) = \left(S_{\eta\phi}(\vec{x}, t) - S_{\phi\eta}(-\vec{x}, -t) \right) \tag{6.40}$$

or

$$\chi_{\eta\phi}{}''(\vec{x}, \omega) = \left(S_{\eta\phi}(\vec{x}, \omega) - S_{\phi\eta}(-\vec{x}, -\omega) \right) \tag{6.41}$$

By the cyclic property of the trace,

$$\begin{aligned} S_{\eta\phi}(\vec{x},t) &= Tr\left(e^{-\beta H}\,\eta(\vec{x},t)\,\phi(0,0)\right) \\ &= Tr\left(\phi(0,0)e^{-\beta H}\,\eta(\vec{x},t)\right) \\ &= Tr\left(e^{-\beta H}e^{\beta H}\phi(0,0)e^{-\beta H}\,\eta(\vec{x},t)\right) \\ &= Tr\left(e^{-\beta H}\phi(0,-i\beta)\,\eta(\vec{x},t)\right) \\ &= S_{\phi\eta}(-\vec{x},-t-i\beta) \end{aligned} \tag{6.42}$$

Hence,

$$S_{\eta\phi}(\vec{x},\omega) = e^{\beta\omega}S_{\phi\eta}(-\vec{x},-\omega) \tag{6.43}$$

Thus, we finally have:

$$\chi_{\eta\phi}{}''(\vec{x},\omega) = \left(1-e^{-\beta\omega}\right)\,S_{\eta\phi}(\vec{x},\omega) \tag{6.44}$$

Since the right-hand-side is a measure of the dissipation and the left-hand-side is a measure of the fluctuation, (6.44) is called the fluctuation-dissipation theorem. As we will see shortly, neutron scattering experiments measure $S_{\rho\rho}(q,\omega)$, and the fluctuation-dissipation theorem relates this to a quantity which we can attempt to calculate using the imaginary-time formalism: the imaginary part of a retarded correlation function, $\chi_{\rho\rho}{}''(q,\omega)$.

6.4 Perturbative Example

Let us consider the case in which $\phi = \eta = \partial_t u_1$. $\partial_t u_1$ is the current in the x_1-direction carried by the ions in the solid, so we are driving the solid in the x_1-direction and measuring the subsequent flow of the positive ions in this direction. Then $\chi_{\phi\phi}$ is given by the retarded correlation function of $\partial_t u_1$ with itself. Let us make the simplifying assumption that $v_l = v_t$. According to the $i\delta$ prescription of (5.32), for free phonons this is:

$$\chi_{\phi\phi}(\omega,\vec{q}) = \frac{\omega^2}{(\omega+i\delta)^2 - v_l^2 q^2} \tag{6.45}$$

Hence,

$$\chi''_{\phi\phi} = \omega^2\, sgn(\omega)\delta(\omega^2 - v_l^2 q^2) \tag{6.46}$$

In other words, there will only be dissipation if the compressional force has a component with $\omega^2 = v_l^2 q^2$. Thus a measurement of this response function is a direct measurment of v_l, i.e. of the phonon spectrum.

If the phonons are interacting,

$$\chi_{\phi\phi}(\omega,\vec{q}) = \frac{\omega^2}{(\omega+i\delta)^2 - v_l^2 q^2 + \Pi_{ret}(\omega,q)} \tag{6.47}$$

and the δ-function is broadened:

$$\chi''_{\phi\phi} = \frac{\omega^2 \,\mathrm{Im}\,\{\Pi_{ret}(\omega,q)\}}{\left(\omega^2 - v_l^2 q^2 + \mathrm{Re}\,\{\Pi_{ret}(\omega,q)\}\right)^2 + \left(\mathrm{Im}\,\{\Pi_{ret}(\omega,q)\}\right)^2} \tag{6.48}$$

Consider a perturbative computation of $\Pi_{ret}(\omega,q)$. At $O(g)$, there is diagram (a) of figure **??**. This diagram gives a purely real contribution

$$\Pi(i\omega_n,q) = -\frac{g}{2}\,q^2\,\frac{1}{\beta}\sum_n \int \frac{d^3\vec{p}}{(2\pi)^3}\,|\vec{p}|^2\,\frac{1}{\omega_n^2 + v_l^2 p^2} \;+\; O(g^2) \tag{6.49}$$

In the problem set, you will compute the Matsubara sum. At zero temperature, we find:

$$\begin{aligned}\Pi(i\omega_n,q) &= \frac{g}{2}\,q^2 \int \frac{d^3\vec{p}}{(2\pi)^3}\,\frac{p}{v} \;+\; O(g^2) \\ &= (\text{const.})\,g\,q^2\,\Lambda^4/v \;+\; O(g^2) \\ &= (\delta v_l^2) q^2 \end{aligned} \tag{6.50}$$

where $(\delta v_l^2) = (\text{const.})\,g\,\Lambda^4/v$. The analytic continuation to the retarded Green function is trivial.

The first contribution to the imaginary part of the self-energy comes from diagram (b) of **??**. It is given by

$$\begin{aligned}\mathrm{Im}\,\{\Pi(i\omega_n,q)\} = \mathrm{Im}\{q^2\,\frac{g^2}{6}\,\frac{1}{\beta^2}\sum_{n_1,n_2}\int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\,\mathcal{G}(i\omega_n - i\omega_{n_1} - i\omega_{n_2}, q-p_1-p_2) \\ \mathcal{G}(i\omega_{n_1},p_1)\,\mathcal{G}(i\omega_{n_2},p_2)\}_{i\omega_n\to\omega+i\delta} + O(g^3)\end{aligned} \tag{6.51}$$

Evaluating this integral is tedious, but we can make some simple observations. When we convert the two Matsubara sums to integrals, we will convert the Green functions to spectral functions; taking the imaginary part will convert the final one to a spectral function:

$$\begin{aligned}\mathrm{Im}\,\{\Pi(i\omega_n,q)\} &= \mathrm{Im}\{q^2\,\frac{g^2}{6}\int d\omega_1 \int d\omega_2 \int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\,\mathcal{G}\,B\,B\}_{i\omega_n\to\omega+i\delta} + O(g^3) \\ &= \mathrm{Im}\{q^2\,\frac{g^2}{6}\int d\omega_1 \int d\omega_2 \int \frac{d^3\vec{p}_1}{(2\pi)^3}\,\frac{d^3\vec{p}_2}{(2\pi)^3}\,B\,B\,B\}_{i\omega_n\to\omega+i\delta}\end{aligned}$$

$$+O(g^3) \tag{6.52}$$

By taking the imaginary part, we have put the three internal phonon lines on-shell. There is no phase space for the δ-functions to be satisfied if $\omega = 0$ (since $\vec{p}_1$ and $\vec{p}_2$ will have to be collinear), so the integral is propotional to ω. Hence, we have:

$$\mathrm{Im}\left\{\Pi(i\omega_n, q)\right\} = Dq^2\omega + O(q^4\omega) + O(g^3) \tag{6.53}$$

where $D = (\text{const.})g^2\Lambda^7$. Keeping only the first term, we can now write the spectral function as:

$$\chi''_{\phi\phi} = \frac{\omega^2\left(Dq^2\omega\right)}{\left(\omega^2 - \tilde{v}_l^2 q^2\right)^2 + \left(Dq^2\omega\right)^2} \tag{6.54}$$

where $\tilde{v}_l^2 = v_l^2 + \delta v_l^2$.

This is the form of the response function which we expect when g is small and $\Pi_{ret}(\omega, q)$ can be calculated perturbatively. In such a case, the corrections due to $\Pi_{ret}(\omega, q)$ are small and lead to a small *damping* of a *propagating mode*. In general, however, the calculation of the response function, χ, is a difficult problem. Nevertheless, we can often say something about χ since some of its general features follow from conservation laws and symmetries. The resulting equations satisfied by physical quantities (including response functions) are *hydrodynamic equations*.

6.5 Hydrodynamic Examples

Let us consider as an example some particles dissolved in a fluid. The density, ρ, and current, $\vec{J}$ of these particles will satisfy a conservation law:

$$\frac{\partial}{\partial t}\rho + \vec{\nabla}\cdot\vec{J} = 0 \tag{6.55}$$

Unlike in the case of a propagating mode, ρ and $\vec{J}^M$ will satisfy a *constitutive relation*:

$$\vec{J} = -D\vec{\nabla}\rho + \vec{f}_{\text{ext}} \tag{6.56}$$

where D is the diffusion constant and B_{ext} is an external force acting on the particles (such as gravity). Ideally, we would like to compute D (perturbatively or by going beyond perturbation theory), but in many cases we must

leave it as a phenomenological parameter. As a result of the constitutive relation, ρ satisfies the diffusion equation:

$$\frac{\partial}{\partial t}\rho - D\,\nabla^2\rho = \vec{\nabla}\cdot\vec{f}_{\text{ext}} \tag{6.57}$$

Hence,

$$\chi_{\rho\rho}(\omega,q) = \frac{iq}{-i\omega + Dq^2} \tag{6.58}$$

and

$$\chi''_{\rho\rho}(\omega,q) = \frac{Dq^3}{\omega^2 + (Dq^2)^2} \tag{6.59}$$

Thus, $\chi''_{\rho\rho}/q$ is a Lorentzian centered at $\omega = 0$ with width Dq^2. Similarly,

$$\chi_{JJ}(\omega,q) = \frac{i\omega}{-i\omega + Dq^2} \tag{6.60}$$

and

$$\chi''_{JJ}(\omega,q) = \frac{Dq^2\omega}{\omega^2 + (Dq^2)^2} \tag{6.61}$$

Note that this is precisely the same as (6.54) for $\tilde{v} = 0$.

In this example, we have seen how the low q, ω behavior of response functions of conserved quantities and their associated currents can be determined by a knowledge of the *hydrodynamic modes* of the system. In general, there will be one hydrodynamic mode for each conservation law satisfied by the system. The conservation law, together with a constitutive relation, leads to hydrodynamic equations satisfied by the conserved quantity and its current. These equations, in turn, determine the correlation functions. Note that such constraints usually only hold for conserved quantities; correlation functions of arbitrary fields are typically unconstrained.

Observe that (6.61) is of precisely the same form as (6.54) above, but with $\tilde{v}_l = 0$. In this section and the last, we have seen how response functions which are calculated perturbatively (as we imagined doing in the first example) are often of the same form as those which are deduced from the *hydrodynamic* – or long-wavelength, low-frequency – equations which they satisfy. Hydrodynamic laws hold in the long-wavelength, low-frequency limit in which local equilibrium is maintained so that constitutive relations such as (6.56) hold. Linear response theory holds in the limit of small f_{ext}. When both of these conditions are satisfied, we can sometimes perturbatively calculate response functions which satisfy the constraints imposed by

hydrodynamic relations. In chapter 7, we will see examples of this in the context of spin systems.

In a solid, we might be interested in the response functions for the energy density and the mass density. It turns out that these quantities are coupled so that the "normal modes" of the solid are a combination of the energy and mass. One of thse normal modes diffuses while the other is a (damped) propagating mode. Consequently, $\chi_{\rho\rho}$, $\chi_{\rho E}$, and χ_{EE} are given by linear combinations of functions of the form of (6.54) and (6.59).

6.6 Kubo Formulae

Transport measurements fit naturally into the paradigm of linear response theory: a weak external probe – such as a potential or temperature gradient – is applied and the resulting currents are measured. Transport coefficients relate the resulting currents to the applied gradients. These coefficients – or the corresponding response functions – may be derived by following the steps of section 2.

We have already encountered one example of a transport coefficient which can be obtained from a response function, namely the diffusion constant, D, which can be obtained from (6.54) or (6.61) by:

$$D = \lim_{\omega\to 0}\lim_{q\to 0}\left(\frac{\omega}{q^2}\,\chi''_{JJ}(q,\omega)\right) \tag{6.62}$$

To see why transport properties should, in general, be related to such limits of response functions, let us derive the corresponding relation, or Kubo formula, for the electrical conductivity of a system. Let $\vec{j}$ denote the current in our condensed matter system when the external vector and scalar potentials, $\vec{A}$ and φ, are zero. Let ρ be the charge density. Then, when we turn on the electromagnetic field, the current is given by $\vec{J} = \vec{j} - \frac{ne^2}{m}\vec{A}$. Meanwhile, $H_{\rm probe}$ is given by:

$$H_{\rm probe} = \int d^3\vec{x}\ \left(-\rho(\vec{x},t)\varphi(\vec{x},t) + \vec{j}(\vec{x},t)\cdot\vec{A}(\vec{x},t) + \frac{e}{m}\rho(\vec{x},t)A^2(\vec{x},t)\right) \tag{6.63}$$

Following our derivation of the response function in section 2, we have

$$\begin{aligned}\left\langle \vec{J}(\vec{x},t)\right\rangle &= \left\langle 0\left|\left(1+i\int_{-\infty}^{t}dt'\,H_{\rm probe}(t')\right)\,\vec{J}(\vec{x},t)\,\left(1-i\int_{-\infty}^{t}dt'\,H_{\rm probe}(t')\right)\right|0\right\rangle \\ &= \left\langle 0\left|\,\vec{J}(\vec{x},t)\,\right|0\right\rangle_0 + \left\langle 0\left|\,i\int_{-\infty}^{t}dt'\left[H_{\rm probe}(t'),\vec{J}(\vec{x},t)\right]\right|0\right\rangle\end{aligned} \tag{6.64}$$

Using

$$\begin{aligned}\left\langle 0 \left| \vec{J} \right| 0 \right\rangle_0 &= \left\langle 0 \left| \vec{j} \right| 0 \right\rangle_0 - \frac{ne^2}{m}\vec{A} \\ &= -\frac{ne^2}{m}\vec{A}\end{aligned} \tag{6.65}$$

and the expression (6.63) for H_{probe} and keeping only terms linear in $\vec{A}$, we have:

$$\begin{aligned}\langle J_i(\vec{x},t)\rangle = i &\int d^3\vec{x}' \int_{-\infty}^{t} dt' \ \left[j_j(\vec{x}',t'), j_i(\vec{x},t)\right] A_i(\vec{x}',t') \\ - i &\int d^3\vec{x}' \int_{-\infty}^{t} dt' \ \left[j_i(\vec{x}',t'), \rho(\vec{x},t)\right] \varphi(\vec{x}',t') \\ - &\frac{ne^2}{m} A_i(\vec{x},t)\end{aligned} \tag{6.66}$$

We are free to choose any gauge we want, so let's take $\varphi = 0$ gauge. Then,

$$\langle J_i(\vec{x},t)\rangle = i \int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \, \theta(t-t') \ \left[j_j(\vec{x}',t'), j_i(\vec{x},t)\right] A_i(\vec{x}',t') \ - \ \frac{ne^2}{m} A_i(\vec{x},t) \tag{6.67}$$

In this gauge, $\vec{E} = d\vec{A}/dt$, so we naively have:

$$\langle J_i(\vec{x},t)\rangle = \int d^3\vec{x}' \int_{-\infty}^{\infty} dt' \, \sigma_{ij}(\vec{x}-\vec{x}', t-t') \, \vec{E}(\vec{x}',t') \tag{6.68}$$

or

$$\langle J_i(\vec{q},\omega)\rangle = \sigma_{ij}(\vec{q},\omega) \, E_j(\vec{q},\omega) \tag{6.69}$$

with

$$\sigma_{ij}(\vec{q},\omega) = \frac{1}{\omega} \int_0^{\infty} dt \, e^{-i\omega t} \ [j_j(-\vec{q},0), j_i(\vec{q},t)] - \frac{1}{i\omega} \, \frac{ne^2}{m} \tag{6.70}$$

In terms of the response function,

$$\sigma_{ij}(\vec{q},\omega) = \frac{1}{i\omega} \, \chi_{jj}(\vec{q},\omega) - \frac{1}{i\omega} \, \frac{ne^2}{m} \tag{6.71}$$

The first term on the right-hand-side leads to the real part of the conductivity:

$$\operatorname{Re} \sigma_{ij}(\vec{q},\omega) = \frac{1}{\omega} \, \chi''_{jj}(\vec{q},\omega) \tag{6.72}$$

while the second term – if it's not cancelled by the imaginary part of the first term – leads to superconductivity.

The DC conductivity is obtained by taking the $q \to 0$ limit first, to set up a spatially uniform current, and then taking the DC limit, $\omega \to 0$.

$$\sigma_{ij}^{DC} = \lim_{\omega \to 0} \lim_{q \to 0} \left(\frac{1}{\omega} \chi''_{jj}(\vec{q}, \omega) \right) \tag{6.73}$$

If we take $\omega \to 0$, then we'll get a static, inhomogenous charge distribution and the $q \to 0$ limit won't tell us anything about the conductivity.

The above formulas are almost right. The problem with them is that they give the response to the *applied* electric field. In fact, we want the response to the *total* electric field. Using Maxwell's equations and our linear response result for $\vec{J}$, we can compute the total field and thereby find the correction to (6.73). This issue is most relevant in the context of interacting *electrons*, so we will defer a thorough discussion of it to that chapter.

6.7 Inelastic Scattering Experiments

Another way of experimentally probing a condensed matter system involves scattering a neutron off the system and studying the energy and angular dependence of the resulting cross-section. This is typically (but not exclusively) done with neutrons rather than photons – for which the requisite energy resolution has not yet been achieved – or electrons – which have the complication of a form factor arising from the long-range Coulomb interactions.

Let us assume that our system is in thermal equilibrium at inverse temperature β and that neutrons interact with our system via the Hamiltonian H'. Suppose that neutrons of momentum $\vec{k}_i$, and energy ω_i are scattered by our system. The differential cross-section for the neutrons to be scattered into a solid angle $d\Omega$ centered about $\vec{k}_f$ and into the energy range between $\omega_f \pm d\omega$ is:

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \sum_{m,n} \frac{k_f}{k_i} \left(\frac{M}{2\pi} \right)^2 \left| \left\langle \vec{k}_f; m \left| H' \right| \vec{k}_i; n \right\rangle \right|^2 e^{-\beta E_n} \, \delta\left(\omega + E_n - E_m\right) \tag{6.74}$$

where $\omega = \omega_i - \omega_f$ and n and m label the initial and final states of our system.

For simplicity, let us assume that there are simple δ-function interactions

between the neutrons and the particles in our system:

$$H' = V\sum_j \delta(\mathbf{x} - \mathbf{X}_j) = V\,\rho(x) \tag{6.75}$$

Then

$$\begin{aligned}\left\langle \vec{k}_f; m \left| H' \right| \vec{k}_i; n \right\rangle &= V \int d^3\vec{x}\, e^{i\vec{q}\cdot\vec{x}}\, \langle m \left|\rho(x)\right| n\rangle \\ &= V \langle m \left|\rho(\vec{q})\right| n\rangle \end{aligned} \tag{6.76}$$

If we use the Fourier representation of the δ-function,

$$\delta(\omega + E_n - E_m) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i(\omega + E_n - E_m)t} \tag{6.77}$$

and pass from the Schrödinger to the Heisenberg representation,

$$\begin{aligned} e^{i(E_n - E_m)t} \langle m \left|\rho(\vec{q})\right| n\rangle &= \left\langle n \left| e^{iHt}\rho(\vec{q})e^{-iHt} \right| m \right\rangle \\ &= \langle n \left|\rho(\vec{q},t)\right| m\rangle \end{aligned} \tag{6.78}$$

then we can rewrite (6.74) as

$$\begin{aligned}\frac{d^2\sigma}{d\Omega\, d\omega} &= \sum_m \frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 \left|\left\langle \vec{k}_f; m \left| H' \right| \vec{k}_i; n \right\rangle\right|^2 e^{-\beta E_n}\, \delta\left(\omega + E_n - E_m\right) \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} dt\, e^{i\omega t} \sum_m \frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 e^{-\beta E_n} \langle n \left|\rho(\vec{q},t)\right| m\rangle\, \langle m \left|\rho(-\vec{q},0)\right| n\rangle \end{aligned} \tag{6.79}$$

We can now use $|m\rangle\langle m| = 1$ and write this as

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \frac{1}{2\pi}\frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 \int_{-\infty}^{\infty} dt\, e^{i\omega t}\, Tr\left\{e^{-\beta H}\rho(\vec{q},t)\rho(-\vec{q},0)\right\} \tag{6.80}$$

If we define the *dynamic structure factor*, $S(\vec{q},\omega) \equiv S_{\rho\rho}(\vec{q},\omega)$:

$$S(\vec{q},\omega)\,\delta(\omega + \omega') = Tr\left\{e^{-\beta H}\rho(\vec{q},\omega)\rho(-\vec{q},\omega')\right\} \tag{6.81}$$

then we can write the inelastic scattering cross-section as:

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \frac{1}{2\pi}\frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 S(\vec{q},\omega) \tag{6.82}$$

According to the fluctuation-dissipation theorem, this can be written

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \frac{1}{2\pi}\frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 \left(\frac{1}{1 - e^{-\beta\omega}}\right) \chi''(\vec{q},\omega) \tag{6.83}$$

In our elastic theory of a solid,

$$\rho(\vec{x}) = \sum_i \delta\left(\vec{x} - \vec{R}_i - \vec{u}(\vec{R}_i)\right) \tag{6.84}$$

$$\rho(\vec{q}) = \sum_i e^{i\vec{q}\cdot(\vec{R}_i - \vec{u}(\vec{R}_i))} \tag{6.85}$$

Let us assume that the displacements of the ions are small and expand the exponential,

$$\begin{aligned}\rho(\vec{q}) &= \sum_i e^{i\vec{q}\cdot\vec{R}_i}\left(1 - i\vec{q}\cdot\vec{u}(\vec{R}_i)\right) \\ &\approx \sum_Q \delta(\vec{q} - \vec{Q}) \; - \; i\vec{q}\cdot\vec{u}(\vec{q})\end{aligned} \tag{6.86}$$

where $\vec{Q}$ is the set of reciprocal lattice vectors. By dropping the higher-order terms in the expansion of the exponential, we are neglecting multi-phonon emission processes. Hence, the scattering cross-section is given by the sum of the contributions of the Bragg peaks together with the contributions of one-phonon emission processes:

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \frac{1}{2\pi}\frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 \left(\left[\sum_Q \delta(\vec{q} - \vec{Q})\delta(\omega)\right] \; + \; \left[Tr\left\{e^{-\beta H}\, \vec{q}\cdot\vec{u}(\vec{q},\omega)\, \vec{q}\cdot\vec{u}(-\vec{q},-\omega)\right\}\right]\right) \tag{6.87}$$

Recognizing our longitudinal phonon Green function on the right-hand-side and using the fluctuation-dissipation theorem, we can write this as:

$$\frac{d^2\sigma}{d\Omega\, d\omega} = \frac{1}{2\pi}\frac{k_f}{k_i}\left(\frac{M}{2\pi}\right)^2 \left(\left[\sum_Q \delta(\vec{q} - \vec{Q})\delta(\omega)\right] \; + \; \left[\left(\frac{1}{1 - e^{-\beta\omega}}\right) \mathrm{Im}\left\{G_{\mathrm{ret}}(q,\omega)\right\}\right]\right) \tag{6.88}$$

Hence, the quantity which our imaginary-time perturbation theory is designed to compute – $G_{\mathrm{ret}}(q,\omega)$ is precisely the quantity which is measured in inelastic scattering experiments. If we assume a self-energy as we did in (6.54), then there will be Lorentzian peaks at $\omega = \pm\tilde{v}_l q$ of width Dq^2.

6.8 Neutron Scattering by Spin Systems-xxx

6.9 NMR Relaxation Rate

In nuclear magnetic resonance, or NMR, experiments, a material is placed in a constant magnetic field. As a result of this magnetic field, there is

an energy splitting ω_0 between the up-spin excited state and the down-spin ground state of the nuclei (let's assume spin-1/2 nuclei). If an up-spin state were an energy eigenstate, then electromagnetic radiation at frequency ω_0 would be perfectly resonant with the nuclear spins; the absorption cross section would have a δ-function at ω_0. As a result of the interaction between nuclear spins and the other excitations in the system (electrons, phonons, magnons), the up-spin state has a finite lifetime, T_1. The width of the resonance is, therefore, $1/T_1$. A measurement of T_1 is an important probe of the spin-carrying excitations of a system.

The interaction Hamiltonian for the coupling between a nuclear spin and the other excitations is:

$$H_{\rm int} = \int \frac{d^2q}{(2\pi)^2}\, A(q)\, \left[I_+ S_-(q) + I_- S_+(q)\right] \tag{6.89}$$

$A(q)$ is the hyperfine coupling between the the nuclear spin $\vec{I}$ and the spin density $\vec{S}(q)$ due to the excitations of the system. The lifetime of the up-spin state is given by:

$$\frac{1}{T_1} = \sum_{m,n} \left|\langle \downarrow; m \left| H' \right| \uparrow; n\rangle\right|^2 e^{-\beta E_n}\, \delta\left(\omega_0 + E_n - E_m\right) \tag{6.90}$$

Following the steps which we used in the derivation of the scattering cross-section, we rewrite this as:

$$\begin{aligned}
\frac{1}{T_1} &= \int \frac{d^2q}{(2\pi)^2}\, A(q)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i(\omega_0 + E_n - E_m)t} \sum_{m,n} \langle n \left| S_-(q) \right| m\rangle \langle m \left| S_+(q) \right| n\rangle\, e^{-\beta E_n} \\
&= \int \frac{d^2q}{(2\pi)^2}\, A(q)\, \frac{1}{2\pi} \int_{-\infty}^{\infty} dt\, e^{i\omega_0 t} \sum_{m,n} \langle n \left| S_-(-q,0) \right| m\rangle \langle m \left| S_+(q,t) \right| n\rangle\, e^{-\beta E_n} \\
&= \int \frac{d^2q}{(2\pi)^2}\, A(q) \sum_{m,n} \langle n \left| S_-(-q,-\omega_0) \right| m\rangle \langle m \left| S_+(q,\omega_0) \right| n\rangle\, e^{-\beta E_n} \\
&= \int \frac{d^2q}{(2\pi)^2}\, A(q) \sum_{n} \langle n \left| S_-(-q,-\omega_0)\, S_+(q,\omega_0) \right| n\rangle\, e^{-\beta E_n} \\
&= \int \frac{d^2q}{(2\pi)^2}\, A(q)\, Tr\left\{ e^{-\beta H} S_-(-\vec{q},-\omega_0) S_+(\vec{q},\omega_0) \right\}
\end{aligned} \tag{6.91}$$

or, using the fluctuation-dissipation theorem,

$$\frac{1}{T_1} = \int \frac{d^2q}{(2\pi)^2}\, A(q)\, \frac{\chi''_{+-}(q,\omega_0)}{1 - e^{-\beta\omega_0}} \tag{6.92}$$

ω_0 is usually a very small frequency, compared to the natural frequency scales of electrons, spin waves, etc., so we can take $\omega_0 \to 0$:

$$\frac{1}{T_1 T} = \int \frac{d^2q}{(2\pi)^2} A(q) \lim_{\omega \to 0} \frac{1}{\omega} \chi''_{+-}(q, \omega) \tag{6.93}$$

CHAPTER 7

Functional Integrals

7.1 Gaussian Integrals

We will now shift gears and develop a formalism which will give us a fresh perspective on many-body theory and its associated approximation methods. This formalism – functional integration – will also reveal the underlying similarity and relationship between quantum and classical statistical mechanics.

Consider the Gaussian integral,

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2}ax^2} = \left(\frac{2\pi}{a}\right)^{1/2} \tag{7.1}$$

This integral is well-defined for any complex a so long as $\mathrm{Re}\{a\} > 0$. We can generalize this to integration over n variables,

$$\int d^n\vec{x} e^{-\frac{1}{2}x_i A_{ij} x_j} = (2\pi)^{n/2} \left(\mathrm{det}A\right)^{-1/2} \tag{7.2}$$

and even to integration over complex variables z_i with $d^n\vec{z}d^n\vec{z}^* \equiv d^n(\mathrm{Re}\vec{z})d^n(\mathrm{Im}\vec{z})$,

$$\int d^n\vec{z}d^n\vec{z}^* e^{-{z_i}^* A_{ij} z_j} = (4\pi)^n \left(\mathrm{det}A\right)^{-1} \tag{7.3}$$

so long as A_{ij} is a symmetric matrix with $\mathrm{Re}\{A_{ij}\} > 0$.

By completing the square,

$$\frac{1}{2}\,x_i A_{ij} x_j + b_i x_i = \frac{1}{2}\left(x_i - x_i^0\right) A_{ij}\left(x_j - x_j^0\right) - \frac{1}{2}\,b_i \left(A^{-1}\right)_{ij} b_j \tag{7.4}$$

where $x_i^0 = (A^{-1})_{ij} b_j$, we can do the integral of the exponential of a quadratic form:

$$\int d^n\vec{x} e^{-\frac{1}{2}x_i A_{ij} x_j + b_i x_i} = (2\pi)^{n/2}\,(\det A)^{-1/2}\, e^{\frac{1}{2}\,b_i \left(A^{-1}\right)_{ij} b_j} \tag{7.5}$$

By differentiating with respect to b_i, we can also do the integrals of polynomials multiplying Gaussians:

$$\int d^n\vec{x}\, P\left(x_1, \ldots, x_n\right)\, e^{-\frac{1}{2}x_i A_{ij} x_j + b_i x_i} = P\left(\frac{\partial}{\partial b_1}, \ldots, \frac{\partial}{\partial b_n}\right)\left((2\pi)^{n/2}\,(\det A)^{-1/2}\, e^{\frac{1}{2}\,b_i \left(A^{-1}\right)_{ij} b_j}\right) \tag{7.6}$$

A non-Gaussian integral can often be approximated by a Gaussian integral using the *saddle-point approximation*:

$$\begin{aligned}\int d^n\vec{x}\, e^{-\lambda f(x_i)} &\approx \int d^n\vec{x}\, e^{-\lambda f\left(x_i^0\right) - \lambda\left(\partial_i \partial_j f\right)_{x_i = x_i^0}\left(x_i - x_i^0\right)\left(x_j - x_j^0\right)} \\ &= (2\pi)^{n/2}\, e^{-\lambda f\left(x_i^0\right)}\left(det\left(\lambda \partial_i \partial_j f\right)_{x_i = x_i^0}\right)^{-\frac{1}{2}}\end{aligned} \tag{7.7}$$

Where x_i^0 is a stationary point of $f(x_i)$, i.e. $\partial_j f(x_i^0) = 0$ for all j. This approximation is good in the $\lambda \to \infty$ limit where the minimum of $f(x_i)$ dominates the integral.

Nothing that we have done so far depended on having n finite. If we blithely allow n to be infinite (ignoring the protests of our mathematician friends), we have the Gaussian functional integral. In the next section, we will do this by making the replacement $i \to t$, $x_i \to x(t)$, and

$$\int d^n\vec{x}\, e^{-\frac{1}{2}x_i A_{ij} x_j} \to \int Dx(t)\, e^{-\int_{t_i}^{t_f} dt\, \frac{1}{2} x(t)\left(-\frac{d^2}{dt^2}\right) x(t)} \tag{7.8}$$

As we will see, the generating functional, Z, can be expressed in this way. Such an expression for the generating functional will facilitate many formal manipulations such as changes of variables and symmetry transformations. It will also guide our intuition about quantum mechanical processes and emphasize the connections with classical statistical mechanics.

7.2 The Feynman Path Integral

In this section, we will - following Feynman - give an argument relating the matrix elements of the evolution operator, U, of a free particle to a Gaussian functional integral. This derivation can be made more or less rigorous, but shouldn't be taken overly seriously. We could just as well write down the functional integral without any further ado and justify it by the fact that it gives the same result as canonical quantization - ultimately, this is its real justification. Here, our reason for discussing Feynman's derivation lies its heuristic value and its intuitive appeal.

Suppose that we have a particle in one dimension moving in a potential $V(x)$. Then

$$H = \frac{p^2}{2m} + V(x) \tag{7.9}$$

Then the imaginary-time evolution operator is given by

$$U(t_f, t_i) = e^{-(\tau_f - \tau_i)H} \tag{7.10}$$

We would like to compute

$$\langle x_f \left| U(\tau_f, \tau_i) \right| x_i \rangle \tag{7.11}$$

In order to do this, we will write (the Trotter product formula)

$$e^{-(\tau_f - \tau_i)H} \to \left(e^{-\delta\tau H} \right)^{N_\tau} \tag{7.12}$$

in the limit $\delta\tau = (\tau_f - \tau_i)/N_\tau \to 0$. We will now take the desired matrix element (7.11),

$$\left\langle x_f \left| e^{-\delta\tau H} \dots e^{-\delta\tau H} \right| x_i \right\rangle \tag{7.13}$$

and insert a resolution of the identity,

$$\int dx |x\rangle\langle x| = 1 \tag{7.14}$$

between each factor of $e^{-\delta\tau H}$:

$$e^{-\delta\tau H} = \int \dots \int dx_1 \dots dx_{N_\tau - 1} \left\langle x_f \left| e^{-\delta\tau H} \right| x_{N_\tau - 1} \right\rangle \dots \left\langle x_1 \left| e^{-\delta\tau H} \right| x_i \right\rangle \tag{7.15}$$

When we repeat this derivation for a quantum mechanical spin in the next chapter, we will insert a resolution of the identity in terms of an overcomplete set of states. As a result, the integration measure will be non- trivial.

We now make an approximation which is accurate to $O(\delta\tau^2)$. In the $\delta\tau \to 0$ limit, we will have an exact (though formal for arbitrary $V(x)$) expression for the desired matrix element. Observe that

$$e^{-\delta\tau(\frac{p^2}{2m}+V(x))} = e^{-\delta\tau(\frac{p^2}{2m})}\, e^{-\delta\tau(V(x))} \; + \; O(\delta\tau^2) \tag{7.16}$$

Hence, we can write

$$\left\langle x_n \left| e^{-\delta\tau H} \right| x_{n-1} \right\rangle = \left\langle x_n \left| e^{-\delta\tau(\frac{p^2}{2m})} \right| x_{n-1} \right\rangle e^{-\delta\tau(V(x_{n-1}))} + \, O(\delta\tau^2) \tag{7.17}$$

We now insert a complete set of momentum eigenstates into the right-hand-side of this expression:

$$\begin{aligned} \left\langle x_n \left| e^{-\delta\tau\,\frac{p^2}{2m}} \right| x_{n-1} \right\rangle e^{-\delta\tau\, V(x_{n-1})} &= \int dp_n \, \langle x_n | p_n \rangle \left\langle p_n \left| e^{-\delta\tau\,\frac{p^2}{2m}} \right| x_{n-1} \right\rangle e^{-\delta\tau\, V(x_{n-1})} \\ &= \int dp_n e^{ip_n(x_n - x_{n-1})} e^{-\delta\tau\,\frac{p_n^2}{2m}} e^{-\delta\tau\, V(x_{n-1})} \end{aligned} \tag{7.18}$$

Note that the second line is an ordinary integral of c-numbers. Doing the Gaussian p_n integral, we have

$$e^{-\delta\tau\left(\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\delta\tau}\right)^2 + V(x_{n-1})\right)} \tag{7.19}$$

Hence,

$$\langle x_f \, | U(\tau_f, \tau_i) | \, x_i \rangle = \int \ldots \int dx_1 \ldots dx_{N_\tau - 1} \, e^{-\sum_n \delta\tau(\frac{m}{2}\left(\frac{x_n - x_{n-1}}{\delta\tau}\right)^2 + V(x_{n-1}))} \tag{7.20}$$

In the $\delta\tau \to 0$ limit, we write

$$\begin{aligned} n\delta\tau &\to \tau \\ x_n &\to x(\tau) \\ \sum_n \delta\tau &\to \int d\tau \end{aligned} \tag{7.21}$$

(but we make no assumption that $x(\tau)$ is differentiable or even continuous) and

$$\langle x_f \, | U(\tau_f, \tau_i) | \, x_i \rangle = \int \mathcal{D}x(\tau) \, e^{-S_E} \tag{7.22}$$

where S_E is the imaginary-time – or 'Euclidean' – action:

$$S_E[x(\tau)] = \int_{t_i}^{t_f} d\tau \, \left(\frac{1}{2} \, m \left(\frac{dx}{d\tau} \right)^2 + V(x) \right) \tag{7.23}$$

The path integral representation suggests a beautiful interpretation of the quantum-mechanical transition amplitude: *the particle takes all possible trajectories with each trajectory $x(\tau)$ contributing $e^{-S_E[x(\tau)]}$ to the amplitude.*

If the particle is free or is in a harmonic oscillator potential, $V(x) = m\omega^2 x^2/2$, this is a Gaussian functional integral:

$$\int \mathcal{D}x(\tau)\, e^{-\int_{t_i}^{t_f} d\tau\, \frac{1}{2}\, m\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}\, m\omega^2 x^2} = N\, det\left(-\frac{d^2}{dt^2} + \omega^2\right) \tag{7.24}$$

where the determinant is taken over the space of functions satisfying $x(t_i) = x_i$, $x(t_f) = x_f$ and N is a 'normalization constant' into which we have absorbed factors of m, π, etc.

For a more general potential, the path integral can be defined perturbatively using (7.6)

$$\int \mathcal{D}x(\tau)\, e^{-\int_{t_i}^{t_f} d\tau\, \frac{1}{2}\, m\left(\frac{dx}{d\tau}\right)^2 + V(x)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\int_{t_i}^{t_f} dt' V\left(\frac{\partial}{\partial b(t')}\right)\right)^n \int \mathcal{D}x(\tau)\, e^{-\int_{t_i}^{t_f} d\tau \left(\frac{1}{2}\, m\left(\frac{dx}{d\tau}\right)^2 + b(t)x(t)\right)} \tag{7.25}$$

Time-ordered expectation values can be simply handled by the path-integral formalism:

$$\langle x_f \left| T\left(x(\tau_1)\ldots x(\tau_n)\, U(\tau_f,\tau_i)\right)\right| x_i\rangle = \int \mathcal{D}x(\tau)\, x(\tau_1)\ldots x(\tau_n)\, e^{-\int_{t_i}^{t_f} d\tau \left(\frac{1}{2}\, m\left(\frac{dx}{d\tau}\right)^2 + V(x)\right)} \tag{7.26}$$

As you will show in the problem set, this follows because the T-symbol puts the operators in precisely the right order so that they can act on the appropriate resolutions of the identity and become c-numbers.

7.3 The Functional Integral in Many-Body Theory

Instead of following the steps of the previous section to derive the functional integral for a field theory, we will simply demonstrate that the generating functional, $Z[j]$, is given by[3]:

$$Z[j] = N \int \mathcal{D}u\, e^{-S_E[j]} \tag{7.27}$$

where $S_E[j]$ is the imaginary-time – or Euclidean – action in the presence of an external source field $j(\vec{x},\tau)$ and N is a normalization factor.

To show that this is true, we will first show that it is true for a free field, i.e. $g = 0$ in our phonon Lagrangian. The generating functional, $Z[j]$ is given by the exponential of the generating functional for connected Green functions, $W[j]$, which, in turn, is given by a single diagram:

$$Z_0[j] = e^{W_0[j]} = e^{\frac{1}{2}\int j(x)G_0(x-y)j(y)}\, Z[0] \tag{7.28}$$

The functional integral, on the other hand, is given by

$$N\int \mathcal{D}u\, e^{-S_E[j]} = N\left[det\left(-\delta_{ij}\partial_\tau^2 - (\mu+\lambda)\partial_i\partial_j - \mu\delta_{ij}\partial_k\partial_k\right)\right]^{-\frac{1}{2}} e^{\frac{1}{2}\int j(x)G_0(x-y)j(y)} \tag{7.29}$$

which follows because

$$G_0 = \partial_l\partial_l\left(-\delta_{ij}\partial_\tau^2 - (\mu+\lambda)\partial_i\partial_j - \mu\delta_{ij}\partial_k\partial_k\right)^{-1} \tag{7.30}$$

If we choose N to cancel the determinant, we have the desired result.

Now consider the interacting case, $S = S_0 + \int \mathcal{L}_{int}(\partial_k u_k)$. Then, following (7.6), we can write the functional integral as:

$$N\int \mathcal{D}u\, e^{-S_0[j]-\int \mathcal{L}_{int}(\partial_k u_k)} = \frac{N}{N_0}\, e^{-\int \mathcal{L}_{int}\left(\frac{\delta}{\delta j}\right)}\, e^{W_0[j]} \tag{7.31}$$

According to Dyson's formula, we have precisely the same thing for the generating functional. Hence (7.27) is true even for an interacting theory. By straightforward extension, we can show that the same relation holds at finite-temperature, where imaginary-time integrals run from 0 to β.

An important result which follows from this discussion is that the propagator is simply the inverse of the differential operator in the quadratic term in the action. The inverse is almost always most easily taken in momentum space.

With the functional integral representation of $Z[j]$ in hand, we can give simple proofs of Wick's theorem and the Feynman rules. To do this, it's helpful to use the following identity (which we state for finite-dimensional vector spaces)

$$F\left(-i\frac{\partial}{\partial \mathbf{x}}\right) G(\mathbf{x}) = \left(G\left(-i\frac{\partial}{\partial \mathbf{y}}\right) F(\mathbf{y})\, e^{i\mathbf{x}\cdot\mathbf{y}}\right)_{\mathbf{y}=0} \tag{7.32}$$

This identity can be proven by expanding F and G in plane waves.

$$\begin{aligned} F(\mathbf{x}) &= e^{i\mathbf{a}\cdot\mathbf{x}} \\ G(\mathbf{y}) &= e^{i\mathbf{b}\cdot\mathbf{y}} \end{aligned} \tag{7.33}$$

The left-hand-side of (7.32) is

$$e^{i\mathbf{a}\cdot\partial/\partial\mathbf{x}}e^{i\mathbf{b}\cdot\mathbf{x}} = e^{i\mathbf{b}\cdot(\mathbf{x}+\mathbf{a})} \tag{7.34}$$

while the right-hand-side is:

$$e^{i\mathbf{b}\cdot\partial/\partial\mathbf{y}}e^{i(\mathbf{x}+\mathbf{a})\cdot\mathbf{y}} = e^{i(\mathbf{x}+\mathbf{a})\cdot(\mathbf{y}+\mathbf{b})} \tag{7.35}$$

from which (7.32) follows.

Using this, we can compute the time-ordered product of a string of fields in a free field theory:

$$\begin{aligned}\langle 0|T\left(\partial_k u_k(x_1)\dots\partial_k u_k(x_n)\right)|0\rangle &= \frac{\delta^n Z[j]}{\delta j(x_1)\dots\delta j(x_n)} \\ &= \left[\frac{\delta}{\delta j(x_1)}\cdots\frac{\delta}{\delta j(x_1)}\,e^{\frac{1}{2}\int j(x)G_0(x-y)j(y)}\right]_{j=0} \\ &= \left[e^{\frac{1}{2}\int G_0(x-y)\frac{\delta}{\delta(\partial_i u_i(x))}\frac{\delta}{\delta(\partial_j u_j(y))}}\,\partial_i u_i(x_1)\dots\partial_i u_i(x_n)\right]_{u_i=0}\end{aligned} \tag{7.36}$$

This is Wick's theorem in the form in which we rewrote it at the end of section 4.4.

In the same way, we can derive the Feynman rules for an interacting theory. Using our expression (7.31),

$$\begin{aligned}Z[j] &= N\,e^{-\int\mathcal{L}_{int}\left(\frac{\delta}{\delta j}\right)}\,e^{W_0[j]} \\ &= N\left[e^{-\int\mathcal{L}_{int}\left(\frac{\delta}{\delta j}\right)}\,e^{\frac{1}{2}\int j(x)G_0(x-y)j(y)}\right]\end{aligned} \tag{7.37}$$

Using our identity, (7.32) we can rewrite this as:

$$Z[j] = N\,e^{\frac{1}{2}\int G_0(x-y)\frac{\delta}{\delta\partial_k u_k(x)}\frac{\delta}{\delta\partial_j u_j(y)}}\,e^{-\int\mathcal{L}_{int}(\partial_k u_k)+j\partial_k u_k} \tag{7.38}$$

which is a compact expression of the Feynman rules for the generating functional.

7.4 Saddle Point Approximation, Loop Expansion

As we pointed out in chapter 5, the loop expansion is an expansion in powers of $\hbar$. Using the functional integral, we can obtain another perspective on the expansion in powers of $\hbar$. Restoring the $\hbar$, the functional integral is

$$Z[j] = N\int\mathcal{D}u\,e^{-\frac{1}{\hbar}\left(S_E[u]+\int j\partial_k u_k\right)} \tag{7.39}$$

At saddle-point level, this is given by:

$$Z[j] = N\, e^{-\frac{1}{\hbar}\left(S_E[u^c]+\int j\partial_k u^c_k\right)} \tag{7.40}$$

where u^c is a classical solution of the equation of motion:

$$K_{ij}(x-y)u_j(y) - \frac{g}{3!}\partial_i(\partial_k u_k)^3 = -\partial_i j(x) \tag{7.41}$$

and

$$K_{ij}(x-x') = \delta(x-x')\left(\delta_{ij}\partial_\tau^2 + (\mu+\lambda)\partial_i\partial_j + \mu\delta_{ij}\partial_k\partial_k\right) \tag{7.42}$$

If we use the classical solution which is obtained by starting with the $g=0$ solution:

$$u_i(x) = -K^{-1}_{ij}(x-y)\partial_j j(y) \tag{7.43}$$

and solve this iteratively:

$$u_i(x) = -K^{-1}_{ij}(x-y)\partial_j j(y) + \frac{g}{3!}K^{-1}_{il}(x-x')\partial_l\left(\partial_k K^{-1}_{kj}(x'-x'')\partial_j j(x'')\right)^3 + \ldots \tag{7.44}$$

we obtain:

$$\begin{aligned} W[j] = \ln Z[j] = \frac{1}{\hbar}\Big(&\frac{1}{2}\partial_i j(x)K^{-1}_{ij}(x-y)\partial_j j(y) \\ &-\frac{1}{2}\partial_i j(x)\frac{g}{3!}K^{-1}_{il}(x-x')\partial_l\left(\partial_k K^{-1}_{kj}(x'-x'')\partial_j j(x'')\right)^3 + \ldots\Big) \end{aligned} \tag{7.45}$$

This is the contribution to the generating functional for connected diagrams coming from tree-level diagrams. In the terms not shown, for each additional vertex carrying a g we have one extra internal line – i.e. a propagator K^{-1}_{ij} which is not attached to a $\partial_i j$.

The Gaussian fluctuations about the saddle point contribute to the functional integral a factor of

$$det\left(-K_{ij} + \frac{g}{2}\partial_i\partial_j(\partial_k u^c_k)^2\right) = e^{Tr\ln\left(-K_{ij}+\frac{g}{2}\partial_i\partial_j\left(\partial_k u^c_k\right)^2\right)} \tag{7.46}$$

This gives the following contribution to the generating functional of connected diagrams:

$$\begin{aligned} Tr\ln\left(-K_{ij} + \frac{g}{2}\partial_i\partial_j(\partial_k u^c_k)^2\right) &= Tr\ln\left(-K_{ij}\right) + Tr\ln\left(1 - K^{-1}_{ji}\frac{g}{2}\partial_i\partial_j(\partial_k u^c_k)^2\right) \\ &= Tr\ln\left(-K_{ij}\right) - Tr\left(K^{-1}_{ji}\frac{g}{2}\partial_i\partial_j(\partial_k u^c_k)^2\right) + \end{aligned}$$

$$\frac{1}{2}Tr\left(K^{-1}_{nm}\frac{g}{2}\partial_m\partial_l(\partial_k u^c_k)^2 K^{-1}_{li}\frac{g}{2}\partial_i\partial_j(\partial_k u^c_k)^2\right)+\dots \tag{7.47}$$

or, writing out the traces,

$$\begin{aligned} W_{1-\text{loop}}[J] = &\int d^3x dt \ln\left(-K_{ii}\right)(0) \quad - \\ &\int d^3x dt\ \left(K^{-1}_{ji}(0)\frac{g}{2}\partial_i\partial_j(\partial_k u^c_k(x,t))^2\right) \ + \\ &\frac{1}{2}\int d^3x dt d^3x' dt' (K^{-1}_{jm}(x'-x,t'-t)\frac{g}{2}\partial_m\partial_l(\partial_k u^c_k(x,t))^2 \times \\ &K^{-1}_{li}(x-x',t-t')\frac{g}{2}\partial_i\partial_j\big(\partial_k u^c_k(x',t')\big)^2) + \dots \end{aligned} \tag{7.48}$$

The first term is independent of $j(x,t)$ and can be absorbed in the normalization. The rest of the series gives the connected one-loops contributions. The second term is the loop obtained by connecting a point to itself with a propagator; the u^c_k's attach all possible tree diagrams to this loop. The third term is the loop obtained by connecting two points by two propagators and again attaching all possible tree diagrams. The next term (not written) is the loop obtained by connecting three points with three propagators, and so on.

Hence, the tree-level diagrams give the $O(1/\hbar)$ contribution to $W[J]$ while the one-loop diagrams give the $O(1)$ contribution. To see that the L-loop diagrams give the $O(\hbar^{L-1})$ contribution to $W[J]$, observe that each propagator comes with a factor of $\hbar$ (since it is the inverse of the quadratic part of the action) while each vertex comes with a factor of $1/\hbar$ (from the perturbative expansion of $e^{S/\hbar}$). Hence a diagram with I internal lines and V vertices is $O(\hbar^{I-V})$. According to the graphical argument we gave in chapter 5, $I-V=L-1$, which proves the claim.

Note that we have chosen a particular saddle-point, u^c_i, namely the one which can be obtained by solving the classical equations perturbatively about $g=0$. In principle, we must, of course, sum the contributions from all saddle-points of the functional integral.

7.5 The Functional Integral in Statistical Mechanics

7.5.1 The Ising Model and φ^4 Theory

The functional integral representation of the generating functional of a quantum mechanical many-body system bears a strong resemblance to the parti-

tion function of a classical statistical mechanical system. Indeed the formal similarity between the two allows us to use the same language and calculational techniques to analyze both. To see the correspondance, let's consider the Ising model,

$$H = -\frac{1}{2}\sum_{i,j} J_{ij}\sigma_i\sigma_j \tag{7.49}$$

where the spins $\sigma_i = \pm 1$ lie on a lattice. The classical partition function is:

$$\mathcal{Z} = \sum_{\sigma_i} e^{\frac{1}{2T}\sum_{i,j} J_{ij}\sigma_i\sigma_j} \tag{7.50}$$

We can introduce auxiliary variables, φ_i, to rewrite this as:

$$\mathcal{Z} = N\sum_{\sigma_i}\int \mathcal{D}\varphi_i\, e^{\sum_i \varphi_i\sigma_i}\, e^{-\frac{T}{2}\sum_{i,j}\left(J^{-1}\right)_{ij}\varphi_i\varphi_j} \tag{7.51}$$

The sum over the σ_i's can be done, giving:

$$\mathcal{Z} = N\sum_{\sigma_i}\int \mathcal{D}\varphi_i\; e^{-\frac{T}{2}\sum_{i,j}\left(J^{-1}\right)_{ij}\varphi_i\varphi_j + \sum_i \ln\cosh\varphi_i} \tag{7.52}$$

If $J_{ij} = J(i-j)$, then the first term in the exponential can be brought to a more convenient form by Fourier transforming:

$$\sum_{i,j}\left(J^{-1}\right)_{ij}\varphi_i\varphi_j = \int \frac{d^dq}{(2\pi)^d}\,\frac{1}{J(q)}\varphi(q)\varphi(-q) \tag{7.53}$$

The momenta are cutoff at large q by the inverse lattice spacing. If we are interested in small q, this cutoff is unimportant, and we can write

$$J(q) = J_0 - \frac{1}{2}J_2q^2 + O(q^4) \tag{7.54}$$

so long as $J(i-j)$ falls off sufficiently rapidly. Hence, we can write

$$\begin{aligned}\int \frac{d^dq}{(2\pi)^d}\,\frac{1}{J(q)}\varphi(q)\varphi(-q) &= \int \frac{d^dq}{(2\pi)^d}\left(\frac{1}{J_0}\varphi(q)\varphi(-q) + \frac{J_2}{2J_0^2}q^2\varphi(q)\varphi(-q) + O(q^4)\right)\\ &= \int d^dx\left(\frac{J_2}{2J_0^2}(\nabla\varphi)^2 + \frac{1}{J_0}\varphi^2 + O\left((\nabla\varphi)^4\right)\right)\end{aligned} \tag{7.55}$$

In the second line, we have gone back to real space and taken the continuum limit, $\varphi_i \to \varphi(x)$. Expanding the other term,

$$\ln\cosh\varphi_i = \ln 2 + \frac{1}{2}\varphi^2 - \frac{1}{12}\varphi^4 + O\left(\varphi^6\right) \tag{7.56}$$

If we neglect the $O((\nabla\varphi)^4)$ and higher gradient terms – which seems reasonable in the small q limit – as well as powers of φ higher than the quartic term – which is not obviously the right thing to do, but later will be shown to be reasonable – then we can write the partition function as the following functional integral.

$$\mathcal{Z} = N \int \mathcal{D}\varphi \, e^{-\int d^d x \left(\frac{1}{2}K(\nabla\varphi)^2 + \frac{1}{2}r\varphi^2 + \frac{1}{4!}u\varphi^4\right)} \tag{7.57}$$

where $K = TJ_2/2J_0^2$, $r = T/J_0 - 1$, $u = 2$. Hence, the imaginary-time functional integral which we have introduced for the generating functional of quantum-mechanical correlation functions is analogous to the classical partition function. The weighted sum over all possible classical histories is in direct analogy with the sum over all classical configurations. The similarity between the functional integral (7.57) and the functional integral for our theory of interacting phonons allows us to immediately deduce its Feynman rules:

- Assign a directed momentum to each line. For external lines, the momentum is directed into the diagram.

- For each internal line with momentum $\vec{q}$ write:

$$-\int \frac{d^d\vec{p}}{(2\pi)^d} \frac{1}{Kp^2 + r}$$

- For each vertex with momenta $\vec{p}_1, \ldots, \vec{p}_4$ directed into the vertex, write:

$$u\,(2\pi)^d \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4)$$

- Imagine labelling the vertices $1, 2, \ldots, n$. Vertex i will be connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices i, i is still connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$. If S is the number of permutations which leave the diagram invariant, we assign a factor $1/S$ to the diagram.

- If two vertices are connected by l lines, we assign a factor $1/l!$ to the diagram.

In classical equilibrium statistical mechanics, time plays no role, so there are momenta but no frequencies (unlike in quantum statistical mechanics where, as we have seen, statics and dynamics are intertwined). So long as the system is rotationally invariant, the theory will have Euclidean invariance in d spatial dimensions, and therefore be formally the same as the imaginary-time description of a quantum system in $d-1$-spatial dimensions (and one time dimension).[4] The classical analog of a quantum system at finite-temperature is a classical system which has a finite extent, β, in one direction. Much of what we have to say about quantum-mechanical many-body systems can be applied to classical systems with little modification. Of course, classical statistical mechanics should be contained within quantum statistical mechanics in the small β limit.

7.5.2 Mean-Field Theory and the Saddle-Point Approximation

We can apply the saddle-point – or 0-loop – approximation to our functional integral (7.57). The classical equation of motion is:

$$-K\nabla^2\varphi(x) + r\varphi(x) + \frac{u}{6}\varphi^3(x) = 0 \tag{7.58}$$

Let's look for spatially uniform solutions $\varphi(x) = \varphi$. If $r > 0$, there is only

$$\varphi = 0 \tag{7.59}$$

However, for $r < 0$ there are also the solutions

$$\varphi = \pm\sqrt{\frac{-6r}{u}} \tag{7.60}$$

These latter solutions have larger saddle-point contribution: $\exp(3r^2/2u)$ compared to 1 and are therefore more important.

$r = 0$ occurs at $T = T_c = J_0$. For $T > T_c$, there is only one saddle-point solution, $\varphi = 0$. According to the saddle-point approximation, at $T = T_c$, a phase transition occurs, and for $T < T_c$ there is *spontaneous magnetization* in the system, $\varphi \sim \pm\sqrt{T_c - T}$. Of course, we shouldn't stop with the saddle-point approximation but should include higher-loop processes. In the problem set, you will find the Ginzburg criterion which determines whether higher-loop processes invalidate the saddle-point analysis.

The saddle-point analysis reproduces and is completely equivalent to the standard mean-field-theory. To make this more obvious, let's use the full

potential (7.56) rather than the one truncated at quartic order:

$$\mathcal{Z} = N \int \mathcal{D}\varphi \, e^{-\int d^d x \left(\frac{1}{2} K (\nabla \varphi)^2 + \frac{T}{2J_0} \varphi^2 - \ln \cosh \varphi \right)} \tag{7.61}$$

The saddle-point equation is:

$$\frac{T}{J_0} \varphi = \tanh \varphi \tag{7.62}$$

This is the usual self-consistency condition of mean-field theory which predicts $T_c = J_0$.

For purposes of comparison, let's recapitulate mean-field theory. We replace the effective field which each spins sees as a result of its interaction with its neighbors,

$$H = -\frac{1}{2} \sum_{i,j} J_{ij} \sigma_i \sigma_j \tag{7.63}$$

by a mean-field, h:

$$H = -\sum_i h \sigma_i \tag{7.64}$$

with h given by

$$h = \sum_i J_{ij} \langle \sigma_i \rangle = J_0 \langle \sigma_i \rangle \tag{7.65}$$

In this field, the partition function is just $2 \cosh \frac{h}{T}$ and

$$\langle \sigma \rangle = \tanh \frac{h}{T} \tag{7.66}$$

Using the self-consistency condition, this is:

$$\langle \sigma \rangle = \tanh \frac{J_0 \langle \sigma \rangle}{T} \tag{7.67}$$

which is the saddle-point condition above.

7.6 The Transfer Matrix**

Part III

Goldstone Modes and Spontaneous Symmetry Breaking

CHAPTER 8

Spin Systems and Magnons

8.1 Coherent-State Path Integral for a Single Spin

Let us follow Feynman's derivation of the functional integral to formulate a functional integral for a quantum-mechanical spin. A quantum mechanical spin of magnitude s has a $2s+1$-dimensional Hilbert space of states. One basis is:

$$S^z|s^z\rangle = s^z|s^z\rangle\,, \qquad s^z = -s, -s+1, \ldots, s-1, s \tag{8.1}$$

For the functional integral representation, it is more convenient to use the overcomplete coherent state basis $|\vec{\Omega}\rangle$:

$$\vec{S}\cdot\vec{\Omega}|\vec{\Omega}\rangle = s|\vec{\Omega}\rangle \tag{8.2}$$

where $\vec{\Omega}$ is a unit vector. For $s = 1/2$, we can write this basis in terms of the spinor z^α, $\alpha = \pm 1/2$:

$$|\vec{\Omega}\rangle = z^\alpha|\alpha\rangle \tag{8.3}$$

where:

$$\vec{\Omega} = z^{*\alpha}\,\vec{\sigma}_{\alpha\beta}\,z^\beta \tag{8.4}$$

in terms of the spherical angles θ, ϕ of $\vec{\Omega}$

$$z = \begin{pmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} \\ e^{i\phi/2}\sin\frac{\theta}{2} \end{pmatrix} \tag{8.5}$$

and $z^{*\alpha}z^{\alpha} = 1$. Of course, there is arbitrariness in our choice of the overall phase of z^{α}, but so long as we choose a phase and stick with it, there is no problem. Therefore, the states $|\vec{\Omega}\rangle$ and $|\vec{\Omega}'\rangle$ have overlap:

$$\langle\vec{\Omega}'|\vec{\Omega}\rangle = z'^{*\alpha}z^{\alpha} \tag{8.6}$$

To obtain larger s, we can simply symmetrize $2s$ spin$-1/2$'s:

$$|\vec{\Omega}\rangle = z^{\alpha_1}\ldots z^{\alpha_{2s}}|\alpha_1,\ldots,\alpha_{2s}\rangle \tag{8.7}$$

with

$$\langle\vec{\Omega}'|\vec{\Omega}\rangle = \left(z'^{*\alpha}z^{\alpha}\right)^{2s} \tag{8.8}$$

In terms of the spherical angles,

$$\begin{aligned}\langle\hat{\mathbf{z}}|\vec{\Omega}\rangle &= \left(\frac{1}{2}\left(1+\cos\theta\right)e^{-i\phi}\right)^{s} \\ &= \left(\frac{1}{2}\left(1+\hat{\mathbf{z}}\cdot\vec{\Omega}\right)e^{-i\phi}\right)^{s}\end{aligned} \tag{8.9}$$

Hence the general relation is:

$$\langle\vec{\Omega}'|\vec{\Omega}\rangle = \left(\frac{1}{2}\left(1+\vec{\Omega}'\cdot\vec{\Omega}\right)e^{-i\phi}\right)^{s} \tag{8.10}$$

where ϕ is the phase of $z'^{*\alpha}z^{\alpha}$. In this basis, the resolution of the identity is given by:

$$I = \frac{2s+1}{4\pi}\int d^2\vec{\Omega}\,|\vec{\Omega}\rangle\langle\vec{\Omega}| \tag{8.11}$$

as may be seen by taking its matrix elements between states $\langle s|$ and $|s-n\rangle$. The usefulness of this basis lies in the following property:

$$\langle\vec{\Omega}|f(\vec{S})|\vec{\Omega}\rangle = f(s\vec{\Omega}) \tag{8.12}$$

To see this, use (8.7) to write this as:

$$\begin{aligned}\langle\vec{\Omega}|f(\vec{S})|\vec{\Omega}\rangle &= z^{*\beta_1}\ldots z^{*\beta_{2s}}\langle\beta_1,\ldots,\beta_{2s}|f\left(\frac{\vec{\sigma}_1}{2}+\ldots+\frac{\vec{\sigma}_{2s}}{2}\right)z^{\alpha_1}\ldots z^{\alpha_{2s}}|\alpha_1,\ldots,\alpha_{2s}\rangle \\ &= f(s\vec{\Omega})\end{aligned} \tag{8.13}$$

where we have used $\vec{\Omega} = z^{*\alpha}\,\vec{\sigma}_{\alpha\beta}\,z^{\beta}$ in the second line.

Let us construct the functional integral representation for the partition function of a single spin. Following our derivation of the path integral in chapter 8, we write the imaginary-time evolution operator as

$$e^{-\beta H(\vec{S})} = \left(e^{-\Delta\tau H(\vec{S})}\right)^N \tag{8.14}$$

where $N\Delta\tau = \beta$. Then we can write the partition function as:

$$Tr\left\{\left(e^{-\Delta\tau H(\vec{S})}\right)^N\right\} = \int \prod_{i=1}^{N} \frac{2s+1}{4\pi} d^2\vec{\Omega}_i \left\langle \vec{\Omega}_{i+1} \left| e^{-\Delta\tau H(\vec{S})} \right| \vec{\Omega}_i \right\rangle \tag{8.15}$$

Taking $\Delta\tau \to 0$, we have

$$\left\langle \vec{\Omega}_{i+1} \left| e^{-\Delta\tau H(\vec{S})} \right| \vec{\Omega}_i \right\rangle \approx e^{-\Delta\tau H(s\vec{\Omega}_i) - s\,\frac{1}{4}(\Delta\tau)^2\left(\frac{d\Omega}{d\tau}\right)^2 + s\left(z_i^{*\alpha} z_{i+1}^{\alpha} - z_{i+1}^{*\alpha} z_i^{\alpha}\right)} \tag{8.16}$$

The second term in the exponent was obtained by making the approximation

$$\left(1 + \vec{\Omega}' \cdot \vec{\Omega}\right)^s = \left(2 - \frac{1}{2}\left(\vec{\Omega} - \vec{\Omega}'\right)^2\right)^s \approx 2^s \left(1 - \frac{1}{4}(\Delta\tau)^2 \left(\frac{d\Omega}{d\tau}\right)^2\right)^s \approx 2^s\, e^{-\frac{s}{4}(\Delta\tau)^2\left(\frac{d\Omega}{d\tau}\right)^2} \tag{8.17}$$

while the third term follows from

$$\begin{aligned} e^{s\phi} &\approx e^{s\sin\phi} \\ &= e^{s\left(z_i^{*\alpha} z_{i+1}^{\alpha} - z_{i+1}^{*\alpha} z_i^{\alpha}\right)} \end{aligned} \tag{8.18}$$

Hence, we have

$$Tr\left\{\left(e^{-\Delta\tau H(\vec{S})}\right)^N\right\} = N \lim_{\Delta\tau\to 0} \int \prod_{i=1}^{N} \frac{2s+1}{4\pi} d^2\vec{\Omega}_i\, e^{\sum_i\left(-\Delta\tau H(s\vec{\Omega}_i) - \frac{s}{4}(\Delta\tau)^2\left(\frac{d\Omega}{d\tau}\right)^2 + 2s\Delta\tau z^{*\alpha}\frac{dz^{\alpha}}{d\tau}\right)} \tag{8.19}$$

The $(\Delta\tau)^2$ term can be dropped and we can write the partition function as

$$Tr\left\{e^{-\beta H(\vec{S})}\right\} = N \int D\vec{\Omega}(\tau)\, e^{-S_{\rm spin}[\vec{\Omega}]} \tag{8.20}$$

where

$$S_{\rm spin}[\vec{\Omega}] = \int d\tau \left(H(s\vec{\Omega}) - 2sz^{*\alpha}\frac{dz^{\alpha}}{d\tau}\right) \tag{8.21}$$

In terms of the spherical angles, the second term can be written

$$2s\int z^{*\alpha}\frac{dz^{\alpha}}{d\tau} = i\,s\int d\tau \frac{d\phi}{d\tau}\,\cos\theta(\tau)$$

$$\begin{aligned} &= i\,s \int d\phi\, \cos\theta(\phi) \\ &= i\,s \int d\phi \left(1 - \int_{\cos\theta(\phi)}^{1} d\,(\cos\theta)\right) \\ &= i\,s\,(2\pi - A) \end{aligned} \tag{8.22}$$

where A is the area of the region of the sphere enclosed by the curve traced out by $\vec{\Omega}(\tau)$. Actually, the curve traced out by $\vec{\Omega}(\tau)$ divides the surface of the sphere into two pieces, so there is some ambiguity in the definition of A. However, these two areas add up to 4π and $e^{4\pi i s} = 1$, so we can take either choice of A.

We can check that we get the correct equation of motion from this Lagrangian. As you will show in the problem set,

$$\delta\left(z^{*\alpha}\frac{dz^{\alpha}}{d\tau}\right) = \delta z^{*\alpha}\,\frac{dz^{\alpha}}{d\tau} - \delta z^{\alpha}\,\frac{dz^{*\alpha}}{d\tau} \tag{8.23}$$

while

$$\delta\vec{\Omega} = \delta z^{*\alpha}\,\vec{\sigma}_{\alpha\beta} z^{\beta} + z^{*\alpha}\,\vec{\sigma}_{\alpha\beta}\delta z^{\beta} \tag{8.24}$$

so that

$$\delta\left(z^{*\alpha}\frac{dz^{\alpha}}{d\tau}\right) = \delta\vec{\Omega}\cdot\vec{\Omega}\times\frac{d\vec{\Omega}}{d\tau} \tag{8.25}$$

Hence, the equation of motion following from our Lagrangian is:

$$i\,s\,\vec{\Omega}\times\frac{d\vec{\Omega}}{d\tau} = \frac{\partial H}{\partial\vec{\Omega}} \tag{8.26}$$

Since $\vec{\Omega}$ is a unit vector, $\vec{\Omega}$ and $d\vec{\Omega}/d\tau$ are perpendicular and the equation of motion can be written

$$i\,s\,\frac{d\vec{\Omega}}{d\tau} = \vec{\Omega}\times\frac{\partial H}{\partial\vec{\Omega}} \tag{8.27}$$

which is what we expect from $d\vec{\Omega}/d\tau = [\vec{\Omega}, H]$. If the spin is in a magnetic field, but is otherwise free, $H = \mu\,\vec{B}\cdot\vec{S}$, then the spin precesses about the magnetic field:

$$i\,\frac{d\vec{\Omega}}{d\tau} = \mu\,\vec{\Omega}\times\vec{B} \tag{8.28}$$

The time-derivative term,

$$S = s\int d\tau\,\left(-i\,\cos\theta\,\frac{d\phi}{d\tau}\right) \tag{8.29}$$

can be written in the alternative form:

$$S = s \int d\tau \left(-i \vec{A}\left(\vec{\Omega}\right) \cdot \frac{d\vec{\Omega}}{d\tau} \right) \tag{8.30}$$

where $\vec{A}(\vec{\Omega})$ satisfies

$$\vec{\nabla}_\Omega \times \vec{A}\left(\vec{\Omega}\right) = s\vec{\Omega} \tag{8.31}$$

This clearly leads to the same equation of motion and leads to the following interpretation of a spin: the spin can be modelled by a charged particle moving on the surface of a sphere with a magnetic monopole of magnetic charge s located at the origin. By Stokes' theorem, the action is given by the magnetic flux through the area enclosed by the orbit.

It can also be written in the form

$$S = s \int_0^\beta d\tau \int_0^1 dr \left(-i \frac{d\vec{\Omega}}{dr} \cdot \vec{\Omega} \times \frac{d\vec{\Omega}}{d\tau} \right) \tag{8.32}$$

where $\vec{\Omega}(r,\tau)$ interpolates between the north pole, $\hat{\mathbf{z}}$ and $\vec{\Omega}(\tau)$: $\vec{\Omega}(0,\tau) = \hat{\mathbf{z}}$, $\vec{\Omega}(1,\tau) = \vec{\Omega}(\tau)$. $\vec{\Omega}(r,\tau)$ thereby covers the region of the sphere enclosed by $\vec{\Omega}(\tau)$. To see that the action (8.32) gives the area of this region, observe that the integrand is equal to the Jacobian of the map from the (r,τ) plane to the surface of the sphere. Hence, the action depends only on $\vec{\Omega}(1,\tau) = \vec{\Omega}(\tau)$ and is independent of the particular interpolating function $\vec{\Omega}(r,\tau)$. To see that the same equation of motion results from this form of the action, imagine varying the upper limit of integration:

$$S = s \int_0^\beta d\tau \int_0^{r'} dr \left(-i \frac{d\vec{\Omega}}{dr} \cdot \vec{\Omega} \times \frac{d\vec{\Omega}}{d\tau} \right) \tag{8.33}$$

so that $\delta\vec{\Omega}(\tau) = \vec{\Omega}(r',\tau) - \vec{\Omega}(1,\tau)$. Then, writing $\delta r = r' - 1$, we have $\delta\vec{\Omega}(\tau) = \delta r(d\Omega/dr)$ and:

$$\begin{aligned} \delta S &= s \int_0^\beta d\tau \delta r \left(-i \frac{d\vec{\Omega}}{dr} \cdot \vec{\Omega} \times \frac{d\vec{\Omega}}{d\tau} \right) \\ &= s \int_0^\beta d\tau \left(-i\, \delta\vec{\Omega} \cdot \vec{\Omega} \times \frac{d\vec{\Omega}}{d\tau} \right) \end{aligned} \tag{8.34}$$

from which the correct equation of motion follows. Terms of this form are often called Wess-Zumino terms.

8.2 Ferromagnets

8.2.1 Spin Waves

Suppose that we have the ferromagnetic Heisenberg Hamiltonian:

$$H = -J\sum_{i,j}\vec{S}_i \cdot \vec{S}_i \tag{8.35}$$

with $J > 0$ and the sum restricted to nearest neighbors. The equation of motion is:

$$i\,s\,\frac{d\vec{\Omega}_i}{d\tau} = -Js^2\sum_j \vec{\Omega}_i \times \vec{\Omega}_j \tag{8.36}$$

The ground state of the Heisenberg ferromagnet is one in which the $SU(2)$ spin-rotational symmetry is *spontaneously broken*. The system chooses a direction along which to order. Let us call this direction $\vec{\Omega}_0$. Linearizing about the uniform ground state, $\vec{\Omega}_i = \vec{\Omega}_0 + \delta\vec{\Omega}_i$, we have:

$$i\,s\,\frac{d\delta\vec{\Omega}_i}{d\tau} = -Js^2\sum_j \vec{\Omega}_0 \times \left(\delta\vec{\Omega}_j - \delta\vec{\Omega}_i\right) \tag{8.37}$$

Substituting a plane-wave solution, $\delta\vec{\Omega} = \vec{\epsilon}\,e^{i\vec{q}\cdot\vec{R}_i}$, we have the dispersion relation of spin-wave theory:

$$E(\vec{q}) = Js\left(z - \sum_{i=1}^{z} e^{i\vec{q}\cdot\vec{\delta}_i}\right) \tag{8.38}$$

where the sum is over the z nearest neighbors. For a d-dimensional hypercubic lattice, the cordination number, z, is $2d$. For a square lattice of spacing a in d-dimensions, this gives:

$$E(\vec{q}) = 2Js\left(d - \sum_{i=1}^{d}\cos q_i a\right) \tag{8.39}$$

In the small $\vec{q}$ limit, the spin-waves have quadratic dispersion:

$$E(\vec{q}) = 2Jsa^2q^2 \tag{8.40}$$

8.2.2 Ferromagnetic Magnons

The small $\vec{q}$ behavior can be obtained directly from the continuum limit of the Lagrangian. Since $S_i^2 = s(s+1)$,

$$\vec{S}_i \cdot \vec{S}_i = \frac{1}{2}\Big(\vec{S}_i - \vec{S}_j\Big)^2 + \text{ const.} \tag{8.41}$$

the action takes the following form in the continuum limit:

$$S = s \int d^d\vec{x}\, d\tau \left(-i\, \vec{A}\left(\vec{\Omega}\right) \cdot \frac{d\vec{\Omega}}{d\tau} + \frac{1}{2} D \Big(\nabla \vec{\Omega}\Big)^2 \right) \tag{8.42}$$

where $D = 2Jsa^2$.

As in the previous section, we linearize the Lagrangian about an ordered state which, without loss of generality, we take to be $\vec{\Omega} = \hat{\mathbf{z}}$. We write

$$\vec{\Omega} = (m_x, m_y, \sqrt{1 - m_x^2 - m_y^2}) \tag{8.43}$$

and assume that m_x, m_y are small so that we can neglect all terms in the action higher than quadratic. We can now write

$$A(m_x, m_y) = \frac{s}{2}\,(-m_y, m_x, 0) \tag{8.44}$$

since

$$\nabla_m \times A(m_x, m_y) = s(0,0,1) = s\Omega \tag{8.45}$$

Hence, we can write the action as:

$$S = s \int d^d\vec{x}\, d\tau \left(\frac{1}{2}\, i\epsilon_{ji} m_i \frac{\partial m_j}{\partial \tau} + \frac{1}{2} D (\nabla m_i)^2 \right) \tag{8.46}$$

Introducing the fields $m_\pm = m_x \pm i m_y$, we can write:

$$S = s \int d^d\vec{x}\, d\tau \left(\frac{1}{2}\, m_+ \frac{\partial m_-}{\partial \tau} \;+\; \frac{1}{2} D\, \vec{\nabla} m_+ \cdot \vec{\nabla} m_- \right) \tag{8.47}$$

In chapter 8, we learned that the propagator is simply the inverse of the differential operator in the quadratic part of the action. The inverse can be taken trivially in momentum space:

$$\langle T_\tau\, (m_+(\vec{x},\tau) m_-(0,0))\rangle \;=\; \frac{2}{s}\, \frac{1}{\beta} \sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\, \frac{e^{i(\vec{p}\cdot\vec{x} - \omega_n \tau)}}{i\omega_n - Dp^2} \tag{8.48}$$

Alternatively, we can expand $m_\pm$ in normal modes:

$$\begin{aligned} m_-(x,\tau) &= \int \frac{d^d\vec{k}}{(2\pi)^{d/2}}\, a^\dagger_{\vec{k}}\, e^{-Dk^2\tau + i\vec{k}\cdot\vec{x}} \\ m_+(x,\tau) &= \int \frac{d^d\vec{k}}{(2\pi)^{d/2}}\, a_{\vec{k}}\, e^{Dk^2\tau - i\vec{k}\cdot\vec{x}} \end{aligned} \tag{8.49}$$

and compute the propagator directly as we did for phonons. $a^\dagger_{\vec{k}}$ and $a_{\vec{k}}$ are called *magnon* creation and annihilation operators. Magnons are the quantum particles which correspond to spin waves in analogy with the correspondence between phonons and sound waves or photons and electromagnetic waves. In the ground state, all of the spins point up. To create a magnon, we flip one spin down with $a^\dagger$; to annihilate it, we flip the spin back up.

Using the propagator (8.48), we can compute the magnetization as a function of temperature. To lowest order in $m_\pm$,

$$\Omega_z = \sqrt{1 - m_+ m_-} \approx 1 - \frac{1}{2} m_+ m_- \tag{8.50}$$

Hence,

$$\begin{aligned} \langle \Omega_z(x,\tau)\rangle &= 1 - \frac{1}{2}\langle m_+(x,\tau) m_-(x,\tau)\rangle \\ &= 1 - \frac{1}{s}\frac{1}{\beta}\sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\, \frac{1}{i\omega_n - Dp^2} \\ &= 1 - \frac{1}{s}\int \frac{d^d\vec{p}}{(2\pi)^d}\int \frac{d\omega}{2\pi i}\, n_B(\omega)\left(\frac{1}{\omega + i\delta - Dp^2} - \frac{1}{\omega - i\delta - Dp^2}\right) \\ &= 1 - \frac{1}{s}\int \frac{d^d\vec{p}}{(2\pi)^d}\int d\omega\, n_B(\omega)\delta(\omega - Dp^2) \\ &= 1 - \frac{1}{s}\int \frac{d^d\vec{p}}{(2\pi)^d}\, n_B\left(Dp^2\right) \\ &= 1 - \frac{1}{2s}\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(d/2)}\left(\frac{T}{D}\right)^{d/2}\int_0^\infty \frac{x^{\frac{d}{2}-1}\,dx}{e^x - 1} \end{aligned} \tag{8.51}$$

In the third line, we have converted the sum over Matsubara frequencies to an integral, as usual, obtaining a contribution only from the real axis.

Hence in $d = 3$, the magnetization decreases as $M_z(0) - M_z(T) \sim T^{\frac{3}{2}}$. In $d \leq 2$, however, the integral is divergent so we cannot trust the approximation (8.50). In fact, this divergence is a sign that the magnetization vanishes for any finite temperature in $d \leq 2$. Note that at $T = 0$, the exact

ground state of the ferromagnetic Heisenberg Hamiltonian is fully polarized for arbitrary d. For $d > 2$, the magnetization decreases continuously from its full value as the temperature is increased. For $d \leq 2$, M_z discontinuously jumps to zero as the temperature is raised above zero.

Thus far, we have neglected anharmonic terms in the magnon Lagrangian. By including these terms, we would have interactions between the magnons. Magnon-magnon interactions affect, for instance, the magnetization as a function of temperature. We will not discuss these interactions here, but we will discuss the analogous interactions in the next section in the context of antiferromagnetism.

8.2.3 A Ferromagnet in a Magnetic Field

Suppose we place our ferromagnet in a magnetic field, $\vec{B}$. At zero temperature, the magnetization will line up along the direction of the field. Let us suppose that the field is in the $\hat{\mathbf{z}}$ direction. Then the action is:

$$S = s \int d^d\vec{x}\, d\tau \left(-i\, \vec{A}\left(\vec{\Omega}\right) \cdot \frac{d\vec{\Omega}}{d\tau} + \frac{1}{2} D \left(\nabla\vec{\Omega}\right)^2 + \mu\, s\, B\, \Omega_z \right) \tag{8.52}$$

where μ is the gyromagnetic ratio. If we expand about the ordered state, $\vec{\Omega} = \Omega_z\, \hat{\mathbf{z}}$, then we have the quadratic action:

$$S = s \int d^d\vec{x}\, d\tau \left(\frac{1}{2}\, m_+ \frac{\partial m_-}{\partial \tau} \;+\; D\, \vec{\nabla} m_+ \cdot \vec{\nabla} m_- \;-\; \frac{1}{2}\, \mu\, s\, B\, m_+ m_- \right) \tag{8.53}$$

The propagator is now

$$\langle T_\tau \left(m_+(\vec{x},\tau) m_-(0,0) \right) \rangle \;=\; \frac{2}{s}\, \frac{1}{\beta} \sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\, \frac{e^{i(\vec{p}\cdot\vec{x} - \omega_n \tau)}}{i\omega_n - Dp^2 - \mu s B/2} \tag{8.54}$$

As a result of the magnetic field, there is a minumum energy cost μB to flip a spin. If we repeat the calculation of the magnetization as a function of temperature, we find:

$$\begin{aligned} \langle \Omega_z(x,\tau) \rangle &= 1 - \frac{1}{2} \langle m_+(x,\tau) m_-(x,\tau) \rangle \\ &= 1 - \frac{1}{s} \int \frac{d^d\vec{p}}{(2\pi)^d}\, n_B \left(Dp^2 + \mu s B/2 \right) \end{aligned} \tag{8.55}$$

Unlike in the $B = 0$ case, this integral is not infrared divergent: the magnetic field stabilizes the ferromagnetic state against thermal fluctuations.

8.3 Antiferromagnets

8.3.1 The Non-Linear σ-Model

Let us now consider the antiferromagnetic case,

$$H = J\sum_{i,j}\vec{S}_i \cdot \vec{S}_i \tag{8.56}$$

with $J > 0$ and the sum restricted to nearest neighbors. We expand about the state in which the spins are staggered – the Néel state:

$$\vec{\Omega}_i = (-1)^i\vec{n}(x_i) + \frac{1}{s}\,\vec{l}(x_i) \tag{8.57}$$

$\vec{l}$ is the $q = 0$ part of the spin field while $\vec{n}$ is the $\vec{q} = (\pi/a, \ldots, \pi/a)$ part. We only keep the Fourier modes near these wavevectors. $\vec{n}$ and $\vec{l}$ are assumed to be slowly varying and $\vec{n}\cdot\vec{l} = 0$. Then $\Omega_i^2 = 1$ is satisfied to $O(1/s^2)$ if $n^2 = 1$. With this decomposition, we can write

$$\begin{aligned} H &= Js^2\sum_{i,j}\left(-\vec{n}(x_i)\cdot\vec{n}(x_j) + \frac{1}{s^2}\,\vec{l}(x_i)\cdot\vec{l}(x_j)\right) \\ &= Js^2\sum_{i,j}\left((\vec{n}(x_i) - \vec{n}(x_j))^2 + \frac{1}{s^2}\left(\vec{l}(x_i) + \vec{l}(x_j)\right)^2\right) \;+\; \text{const.} \end{aligned} \tag{8.58}$$

going to the continuum limit, we have:

$$H = va^{-d}\int d^d\vec{x}\,\left(\frac{1}{sa}\,\vec{l}^2 + \frac{sa}{4}(\nabla\vec{n})^2\right) \tag{8.59}$$

where $v = 2Jsa$.

The corresponding action is:

$$\begin{aligned} S = a^{-d}\int d^d\vec{x}\,d\tau\,(&-\frac{i}{a}\,\vec{A}\left((-1)^i\vec{n} + \frac{1}{s}\,\vec{l}\right)\cdot\left((-1)^i\frac{d\vec{n}}{d\tau} + \frac{1}{s}\frac{d\vec{l}}{d\tau}\right) \\ &+ \frac{1}{sa}\,v\,\vec{l}^2 + \frac{sa}{4}\,v\,(\nabla\vec{n})^2) \end{aligned} \tag{8.60}$$

Using (8.57) and

$$\begin{aligned} \vec{\nabla}_\Omega \times \left((-1)^i\vec{n}\times\vec{l}\right) &= (-1)^{i+1}\vec{n} + \frac{1}{s}\,\vec{l} \\ &= (-1)^i\vec{n} + \frac{1}{s}\,\vec{l} + (-1)^{i+1}\partial_x\vec{n} \end{aligned}$$

$$\approx \vec{\Omega} \tag{8.61}$$

we can express $\vec{A}$ in terms of $\vec{n}$ and $\vec{l}$ if we drop the gradient term in the penultimate line (this cannot be done in $d = 1$, where it is absolutely crucial, but can in higher dimensions). Neglecting oscillatory terms in the action, we have:

$$S = a^{-d} \int d^d\vec{x}\, d\tau \, \left(\frac{i}{a}\, \vec{n} \times \vec{l} \cdot \frac{d\vec{n}}{d\tau} + \frac{1}{sa}\, v\, \vec{l}^2 + \frac{sa}{4}\, v\, (\nabla \vec{n})^2 \right) \tag{8.62}$$

The functional integral

$$\int \mathcal{D}\vec{l}\, \mathcal{D}\vec{n}\, e^{-S[\vec{l},\vec{n}]} \tag{8.63}$$

is a Gaussian integral in $\vec{l}$, so we can perform the $\vec{l}$ integral. Integrating out $\vec{l}$, we have:

$$S = a^{-d} \int d^d\vec{x}\, d\tau \, \left(\frac{sa}{4v} \left(\frac{d\vec{n}}{d\tau} \right)^2 + \frac{sa}{4}\, v\, (\nabla \vec{n})^2 \right) \tag{8.64}$$

Or, writing $g = a^{d-2}/Js^2$,

$$S = \frac{1}{g} \int d^d\vec{x}\, d\tau \, \left(\frac{1}{2v^2} \left(\frac{d\vec{n}}{d\tau} \right)^2 + \frac{1}{2}\, (\nabla \vec{n})^2 \right) \tag{8.65}$$

This action is called the $O(3)$ *Non-Linear σ Model*, or $O(3)$ NLσM for short. The $O(3)$ refers to the fact that the field $\vec{n}$ is a three-component field which transforms as a vector under the rotation group, $O(3)$. The model is non-linear because there is a non-linear constraint, $n^2 = 1$.

8.3.2 Antiferromagnetic Magnons

Let us, for simplicity work in a system of units in which $v = 1$. We can always rescale our time coordinate at the end of any calculation so as to restore v. Let us also employ the notation $\mu = 0, 1, \ldots, d.$ with 0 referring to the time direction so that $\partial_0 = \partial_t$. Then we can write the action of the $O(3)$ NLσM as:

$$S = \frac{1}{g} \int d^d\vec{x}\, d\tau \, (\partial_\mu \vec{n})^2 \tag{8.66}$$

If, as in the ferromagnetic case, we expand about an ordered state,

$$\vec{n} = (n_x, n_y, \sqrt{1 - n_x^2 - n_y^2}) \tag{8.67}$$

then we can write the action as:

$$S = \frac{1}{g}\int d^d\vec{x}\, d\tau\, \left((\partial_\mu n_i)^2 + \frac{n_i\partial_\mu n_i\ n_j\partial_\mu n_j}{1 - n_i n_i} \right) \tag{8.68}$$

where $i = 1, 2$ and $n_1 = n_x$, $n_2 = n_y$. Let us rescale the fields so that $n_i \to \sqrt{g} n_i$. Then the action becomes:

$$S = \int d^d\vec{x}\, d\tau\, \left((\partial_\mu n_i)^2 + g\, \frac{n_i\partial_\mu n_i\ n_j\partial_\mu n_j}{1 - g n_i n_i} \right) \tag{8.69}$$

In order to do perturbation theory in g, which we can hope to do when it is small, we divide the action into two parts,

$$S_{\rm free} = \int d^d\vec{x}\, d\tau\, (\partial_\mu n_i)^2 \tag{8.70}$$

and

$$S_{\rm int} = \int d^d\vec{x}\, d\tau\ g\, \frac{n_i\partial_\mu n_i\ n_j\partial_\mu n_j}{1 - g n_i n_i} \tag{8.71}$$

Note that $S_{\rm int}$ contains all powers of g when the denominator is expanded in a geometric series.

$S_{\rm free}$ is very similar to the phonon action; as in that case, we can expand n_i in normal modes:

$$n_i(\vec{r}, t) = \int \frac{d^d\vec{k}}{(2\pi)^{d/2}}\, \frac{1}{\sqrt{2k}}\, \left(a_{\vec{k},i} e^{i\vec{k}\cdot\vec{r} + \omega_k^s \tau}\ +\ a^\dagger_{\vec{k},i} e^{-i\vec{k}\cdot\vec{r} - vk\tau} \right) \tag{8.72}$$

$a^\dagger_\pm = a^\dagger_x \pm a^\dagger_y$ create, respectively, up- and down-spin *antiferromagnetic magnons*. Note the difference with the ferromagnetic case, in which there was only one type of magnon. Since there is no net uniform magnetization in the ground state, we can either flip a spin up – thereby creating an up-spin magnon – or flip a spin down – thereby creating a down spin magnon.

We can obtain the antiferromagnetic magnon propagator using this mode expansion or by inverting $S_{\rm free}$:

$$\langle T_\tau\, (n_i(\vec{x}, \tau) n_j(0, 0)) \rangle\ =\ \delta_{ij} \frac{1}{\beta} \sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\, \frac{e^{i(\vec{p}\cdot\vec{x} - \omega_n\tau)}}{\omega_n^2 + p^2} \tag{8.73}$$

or, restoring v,

$$\langle T_\tau\, (n_i(\vec{x}, \tau) n_j(0, 0)) \rangle\ =\ \delta_{ij} \frac{1}{\beta} \sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\, \frac{e^{i(\vec{p}\cdot\vec{x} - \omega_n\tau)}}{\omega_n^2 + v^2 p^2} \tag{8.74}$$

The corresponding spectral function is:

$$B(\omega,\vec{p}) = \frac{1}{2vp}\,\delta(\omega - vp) \;-\; \frac{1}{2vp}\,\delta(\omega + vp) \tag{8.75}$$

With this propagator in hand, we can compute the staggered magnetization in the $g \to 0$ limit. It is given by

$$n_z = \sqrt{1 - g\, n_i n_i} \approx 1 - \frac{1}{2}\, g\, n_i n_i \tag{8.76}$$

Hence,

$$\begin{aligned}
\langle n_z(x,\tau)\rangle &\approx 1 - g\,\frac{1}{2}\,\langle n_i(x,\tau) n_i(x,\tau)\rangle \\
&= 1 - g\,\frac{1}{2}\cdot 2\cdot\frac{1}{\beta}\sum_n \int \frac{d^d\vec{p}}{(2\pi)^d}\,\frac{1}{\omega_n^2 + v^2p^2} \\
&= 1 - g\int \frac{d^d\vec{p}}{(2\pi)^d}\int\frac{d\omega}{2\pi i}\,n_B(\omega)\,2\pi i B(\omega,\vec{p}) \\
&= 1 - g\int \frac{d^d\vec{p}}{(2\pi)^d}\,\frac{1}{2vp}\,\left(n_B\left(vp\right) - n_B\left(-vp\right)\right) \\
&= 1 - g\int \frac{d^d\vec{p}}{(2\pi)^d}\,\frac{1}{2vp}\,\coth\left(\frac{\beta vp}{2}\right)
\end{aligned} \tag{8.77}$$

Unlike in the ferromagnetic case, the second term does not vanish in the $T \to 0$ limit. For $T = 0$, we have:

$$\langle n_z(x,\tau)\rangle \approx 1 - g\int \frac{d^d\vec{p}}{(2\pi)^d}\,\frac{1}{2vp} \tag{8.78}$$

If we approximate the Brillouin zone by a sphere, $|p| < \pi/a$, then we find

$$\langle n_z(x,\tau)\rangle \approx 1 - g\,\frac{1}{d-1}\left(\frac{\pi}{a}\right)^{d-1}\frac{(2\pi)^{\frac{d}{2}}}{(2\pi)^d\Gamma(d/2)} \tag{8.79}$$

Hence, the staggered magnetization of an antiferromagnet is less than 1 even at $T = 0$, unlike the uniform magnetization in the ferromagnetic case. The Néel state, with neighboring spins oppositely oriented, is not the exact ground state. In $d = 1$, the integral is actually logarithmically divergent. This divergence hints at the impossibility of antiferromagnetic order in $d = 1$, which is a consequence of a theorem which we will prove in the next chapter. For T finite, the integral (8.77) is logarithmically divergent at small p in $d = 2$, just as in the ferromagnetic case. Again, it is a sign of the impossibility of antiferromagnetic order at finite temperatures in $d = 2$.

8.3.3 Magnon-Magnon-Interactions

Thus far, we have ignored the higher-order terms in the NLσM. These terms lead to interactions between magnons. To get an idea of the nature of these terms, let's expand S_{int} to $O(g^2)$:

$$S_{g^2} = \int d^d\vec{x}\, d\tau \, \Big((\partial_\mu n_i)^2 + g\, n_i \partial_\mu n_i \; n_j \partial_\mu n_j + g^2 n_k n_k \, n_i \partial_\mu n_i \; n_j \partial_\mu n_j 1 - g n_i n_i + \ldots \Big) \tag{8.80}$$

We need only these terms in order to do computations to order $O(g^2)$. The Feynman rules for this action are:

- Assign a directed momentum and Matsubara frequency to each line. Assign an index $i = 1, 2$ to each line. For external lines, the momentum and frequency are directed into the diagram.

- For each internal line with momentum, frequency $\vec{q}, i\omega_n$ write:

$$-\frac{1}{\beta}\sum_n \int \frac{d^d\vec{p}}{(2\pi)^d} \, \frac{1}{\omega_n^2 + v^2 p^2}$$

- For each 4-leg vertex with momenta, Matsubara frequencies $(\vec{p}_1, \omega_{n_1}), \ldots, (\vec{p}_4, \omega_{n_4})$ directed into the vertex and indices $i_1, \ldots, i_4$ associated to these incoming lines, write:

$$g\, \delta_{i_1 i_2} \delta_{i_3 i_4} \, (\omega_{i_1}\omega_{i_3} + \vec{p}_1 \cdot \vec{p}_3) \, (2\pi)^d \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4) \;\; \beta\, \delta_{n_1+n_2+n_3+n_4,0}$$

- For each 6-leg vertex with momenta momenta, Matsubara frequencies $(\vec{p}_1, \omega_{n_1}), \ldots, (\vec{p}_6, \omega_{n_6})$ directed into the vertex and indices $i_1, \ldots, i_6$ associated to these incoming lines, write:

$$g^2\, \delta_{i_1 i_2} \delta_{i_3 i_4} \delta_{i_5 i_6} \, (\omega_{i_1}\omega_{i_3} + \vec{p}_1 \cdot \vec{p}_3) \, (2\pi)^d \delta(\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \vec{p}_4 + \vec{p}_5 + \vec{p}_6) \;\; \beta\, \delta_{n_1+n_2+n_3+n_4+n_5+n_6,0}$$

- We assign a factor $1/S$ to the diagram if there are S permutations of the vertices and external momenta which leave the diagram invariant.

8.4 Spin Systems at Finite Temperatures

In the previous two sections, we saw that ferromagnetic and antiferromagnetic order are suppressed by thermal fluctuations. Let us examine this

more closely. Let us, for the sake of concreteness, consider the case of antiferromagnetism. Let us re-write the action in the following dimensionless form:

$$S = \frac{a^{d-1}}{gv} \int d^d\vec{y} \int_0^{\beta v/a} d\tau \left(\frac{1}{2} \left(\frac{d\vec{n}}{du} \right)^2 + \frac{1}{2} \left(\nabla \vec{n} \right)^2 \right) \tag{8.81}$$

where $u = v\tau/a$ and $y = x/a$. If we go to momentum space,

$$S = \frac{a^{d-1}}{gv} \int_0^{\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\beta} \sum_n \left(\frac{1}{2} |\omega_n \vec{n}(\omega_n, q)|^2 + \frac{1}{2} |q\vec{n}(\omega_n, q)|^2 \right) \tag{8.82}$$

then the cutoff is just π (for a spherical Brillouin zone; more generally, it's some number of order 1). The Matsubara frequencies are $\omega_n = 2\pi n a/\beta v$. When $a/\beta v \gg 1$ – i.e. at temperatures which are large compared to v/a (the energy of a magnon at the cutoff wavevector) – the configurations with $\omega_n \neq 0$ are strongly suppressed and give very little contribution to the functional integral. Therefore, the functional integral is dominated by configurations which are independent of τ and we can replace $\int_0^\beta d\tau \to \beta$. Hence, we may make the approximation:

$$\begin{aligned} Z &= \int \mathcal{D}\vec{n}\, e^{-\frac{a^{d-1}}{gv} \int_0^\pi \frac{d^d\vec{q}}{(2\pi)^d} \int_0^\beta d\tau \left(\frac{1}{2} (\partial_\tau \vec{n})^2 + \frac{1}{2} |q\vec{n}(q)|^2 \right)} \\ &\approx \int \mathcal{D}\vec{n}\, e^{-\frac{\beta a^{d-1}}{gv} \int_0^\pi d^d\vec{q} \left(\frac{1}{2} |q\vec{n}(\omega_n=0,q)|^2 \right)} \end{aligned} \tag{8.83}$$

We can similarly write the ferromagnetic functional integral with momentum cutoff of order 1 and Matsubara frequencies $\omega_n = 2\pi n a^2/\beta D$:

$$\begin{aligned} Z &= \int \mathcal{D}\vec{\Omega}\, e^{-sa^d \int \frac{d^d\vec{q}}{(2\pi)^d} \int_0^\beta d\tau \left(-i\, \vec{A}(\vec{\Omega}) \cdot \partial_\tau \vec{\Omega} + \frac{1}{2} \left(q\vec{\Omega} \right)^2 \right)} \\ &= \int \mathcal{D}\vec{\Omega}\, e^{-sa^d \beta \int \frac{d^d\vec{q}}{(2\pi)^d} \left(\frac{1}{2} \left(q\vec{\Omega}(\omega_n=0,q) \right)^2 \right)} \end{aligned} \tag{8.84}$$

Hence, the functional integrals for the ferromagnet and antiferromagnet are identical at temperatures large compared to v/a or D/a^2. Both systems are described by the d-dimensional NLσM. The differences between ferromagnets and antiferromagnets, which have to do with their dynamics, are unimportant in the limit of classical statistical mechanics, which is the limit which we have just taken. Thus we can kill two birds with one stone by studying the functional integral

$$Z = \int \mathcal{D}\vec{n}\, e^{-\beta \int d^d\vec{x} \left(\frac{1}{2} (\nabla \vec{n})^2 \right)} \tag{8.85}$$

This functional integral would be a trivial Gaussian integral if it were not for the constraint $n^2 = 1$. To impose this constraint, let's intoduce a Lagrange multiplier:

$$Z = \int \mathcal{D}\vec{n}\,\mathcal{D}\lambda\, e^{-\beta \int d^d\vec{x}\left(\frac{1}{2}(\nabla\vec{n})^2+\lambda(\vec{x})\left(n^2-1\right)\right)} \tag{8.86}$$

Now the functional integral is, indeed, a Gaussian integral in $\vec{n}$ which we can do:

$$\begin{aligned} Z &= \int \mathcal{D}\lambda\, det\left(\nabla^2+\lambda(\vec{x})\right)^{-1/2}\; e^{-\beta\int d^d\vec{x}\,\lambda(\vec{x})} \\ &= \int \mathcal{D}\lambda\, e^{-\frac{1}{2}Tr\ln\left(-\nabla^2+\lambda(\vec{x})\right)-\beta\int d^d\vec{x}\,\lambda(\vec{x})} \end{aligned} \tag{8.87}$$

Unfortunately, we can't do the resulting integral exactly, but we can try to use the saddle-point approximation. The saddle-point of the argument of the exponential is given by:

$$\frac{\delta}{\delta\lambda}\left(Tr\ln\left(-\nabla^2+\lambda(\vec{x})\right)-\beta\int d^d\vec{x}\,\lambda(\vec{x})\right)=0 \tag{8.88}$$

If we look for a saddle-point solution λ_0 for which $\lambda(x)$ is independent of position, then this is simply

$$\frac{d}{d\lambda}\left(\int\frac{d^d\vec{q}}{(2\pi)^d}\ln\left(q^2+\lambda\right)-\beta\lambda\right)=0 \tag{8.89}$$

or,

$$\int\frac{d^d\vec{q}}{(2\pi)^d}\,\frac{1}{q^2+\lambda}=\beta \tag{8.90}$$

If we aproximate the integral by using λ_0 as an infrared cutoff, then we have:

$$\frac{1}{a^{d-2}}-\lambda_0^{\frac{d-2}{2}}\sim\beta \tag{8.91}$$

For $T\to 0$, there is no spatially homogeneous saddle-point solution, λ_0, if $d>2$. For high-temperature, however, there is a solution of the form:

$$\lambda_0^{\frac{d-2}{2}}\sim\frac{1}{a^{d-2}}-\beta \tag{8.92}$$

In $d=2$, there is always a solution:

$$\lambda_0\sim\frac{1}{a^2}\,e^{-(\text{const.})\,\beta} \tag{8.93}$$

When the functional integral is dominated by a non-zero saddle-point value λ_0, we can approximate it by:

$$Z = \int \mathcal{D}\vec{n}\, e^{-\beta \int d^d\vec{x}\left(\frac{1}{2}(\nabla\vec{n})^2+\lambda_0\vec{n}^2\right)} \tag{8.94}$$

which is a Gaussian integral. This Gaussian theory is called a linear σ-model

This describes the high-temperature phase in which thermal fluctuations have disordered the magnet. From (8.94), we can see that

$$\langle\vec{n}(x)\rangle = 0 \tag{8.95}$$

Furthermore, using the real-space Green function which you have calculated in the problem set, we see that correlation functions of the magnetization decay exponentially with distance

$$\langle\vec{n}(\vec{x})\vec{n}(0)\rangle \sim \frac{1}{|x|^{(d-1)/2}}\, e^{-|x|/\xi} \tag{8.96}$$

where we have defined the correlation length $\xi \sim \sqrt{\lambda_0}$. As the temperature is lowered and $\lambda_0 \to 0$, the correlation length grows. Finally, a transition takes place and the magnet orders. In the saddle-point – or mean-field – approximation, this occurs at $\lambda_0 = 0$. The saddle-point approximation would tell us that

$$\langle\vec{n}(\vec{x})\vec{n}(0)\rangle \sim \frac{1}{|x|^{(d-1)/2}} \tag{8.97}$$

at the critical point, $T = T_c$. However, as we will discuss in chapter 11 – and as you have investigated in the problem set – the saddle-point approximation is often incorrect. For temperatures below T_c the magnet is ordered, and we can expand about the ordered state, as we did in the previous two sections.

To summarize, there are 4 regimes for a ferro- or antiferromagnet:

- High temperature, $T > T_c$, where the system is described by a linear σ-model,
$$Z = \int \mathcal{D}\vec{n}\, e^{-\beta \int d^d\vec{x}\left(\frac{1}{2}(\nabla\vec{n})^2+\lambda_0\vec{n}^2\right)}$$
with $\lambda_0 > 0$. Correlation functions fall off exponentially with correlation length $\xi \sim \sqrt{\lambda_0}$.

- The critical point, $T = T_c$, at which correlation functions have power-law falloff.

- The ordered phase, $0 < T < T_c$, where the magnetization (or staggered magnetization) has a non-zero expectation value. This regime is described by the d-dimensional NLσM:

$$Z = \int \mathcal{D}\vec{n}\, e^{-\beta \int d^d\vec{x} \left(\frac{1}{2}(\nabla\vec{n})^2\right)}$$

  which can be expanded perturbatively about the ordered state.

- The ordered state at $T = 0$ which is described by the $d+1$-dimensional NLσM in the antiferromagnetic case,

$$Z = \int \mathcal{D}\vec{n}\, e^{-\frac{1}{g}\int d^d\vec{x}\, d\tau \left(\frac{1}{2v^2}\left(\frac{d\vec{n}}{d\tau}\right)^2 + \frac{1}{2}(\nabla\vec{n})^2\right)}$$

  and by the following functional integral in the ferromagnetic case.

$$Z = Z = \int \mathcal{D}\vec{\Omega}\, e^{-s\int d^d\vec{x}\, d\tau \left(-i\,\vec{A}(\vec{\Omega})\cdot\frac{d\vec{\Omega}}{d\tau} + \frac{1}{2}D\left(\nabla\vec{\Omega}\right)^2\right)} \tag{8.98}$$

8.5 Hydrodynamic Description of Magnetic Systems

In the limit in which magnon-magnon interactions are strong, it is hopeless to try to expand perturbatively about a quadratic action in either the ferromagnetic or antiferromagnetic cases. However, some properties of correlation functions can be deduced from hydrodynamic equations.

A ferromagnet satisfies hydrodynamic equations very similar to those of a conserved particle density which we discussed in chapter 7. The magnetization, $\vec{\Omega}$ is a conserved quantity, so the deviation from an ordered state, $\delta\Omega = \Omega - \Omega_0$ satisfies a conservation law:

$$\frac{\partial}{\partial t}\delta\Omega_i + \vec{\nabla}\cdot\vec{J^i} = 0 \tag{8.99}$$

and a constitutive relation:

$$\vec{J^i} = -\chi_0^{-1}(T)\vec{\nabla}\delta\Omega_i + \vec{\nabla}B_i \tag{8.100}$$

(χ_0 is the static magnetic susceptibility) from which it follows that the magnetization has diffusive correlation functions:

$$\chi_{\delta\Omega_i\delta\Omega_i}(\omega, q) = \frac{q^2}{-i\omega + \chi_0^{-1}(T)q^2} \tag{8.101}$$

In the case of an antiferromagnet, however, the staggered magnetization, $\vec{n}$, is not conserved, so it does not diffuse. If the system is ordered with $\vec{n} = \hat{\mathbf{z}}$, then the correct hydrodynamic equations for $i = 1, 2$ are:

$$\begin{aligned} \frac{\partial l_i}{\partial t} &= \rho_s(T)\vec{\nabla} \cdot \left(\epsilon_{ijk} n_j \vec{\nabla} n_k\right) \\ \frac{\partial}{\partial t}\left(\epsilon_{ijk} n_j \vec{\nabla} n_k\right) &= \chi_\perp^{-1}(T)\vec{\nabla} l_i \end{aligned} \tag{8.102}$$

These equations are the equations of motion for $\vec{n}$, $\vec{l}$ which follow from the NLσM with $\rho_s = 1/g$ and $\chi_\perp = 1/gv^2$. Rotational invariance dictates that the hydrodynamic equations must hold generally albeit with different values of ρ_s and $\chi_\perp$. These equations can be combined to give:

$$\frac{\partial^2}{\partial t^2}\left(\epsilon_{ijk} n_j \vec{\nabla} n_k\right) = \rho_s(T)\chi_\perp^{-1}(T)\,\nabla^2\left(\epsilon_{ijk} n_j \vec{\nabla} n_k\right) \tag{8.103}$$

from which it follows that there is a propagating mode of velocity $\sqrt{\rho_s(T)\chi_\perp^{-1}(T)}$. A more refined analysis, which includes higher-order terms which have been neglected above leads to a small damping of this mode which goes as q^2.

8.6 Spin chains**

8.7 Two-dimensional Heisenberg model**

CHAPTER 9

Symmetries in Many-Body Theory

9.1 Discrete Symmetries

Symmetries constrain the behavior of physical systems and are, therefore, a useful tool in the analysis of a many-body system. In general, a more symmetrical system is more highly constrained and, consequently, more easily solved. The most useful symmetries are continuous symmetries – ie. symmetries which belong to continuous families – which we discuss in the remainder of this chapter. The simplest symmetries are discrete, and we focus on them in this section. We will focus on the archetypal discrete symmetries, parity, P, and time-reversal, T.

A discrete symmetry is a transformation of the fields in a theory, $\varphi(\vec{x},\tau) \rightarrow \varphi'(\vec{x},\tau)$ which leaves the action unchanged. Since the classical equations of motion are just the stationarity conditions on the action, a discrete symmetry takes one solution of the equations of motion and transforms it into another. At the quantum level, we would like such a transformation to be effected by a unitary operator, U:

$$U^\dagger \, \varphi(\vec{x},\tau) \, U = \varphi'(\vec{x},\tau) \tag{9.1}$$

Parity is an example of such a symmetry. We will call any transformation of the form

$$\varphi^a(\vec{x},\tau) \rightarrow M_{ab} \, \varphi^b(-\vec{x},\tau) \tag{9.2}$$

a parity transformation, but M_{ab} is typically $\pm\delta_{ab}$. Let us consider, as an example, our action for interacting phonons.

$$S = \int d\tau d^3\vec{r} \left[\frac{1}{2}\rho(\partial_t u_i)^2 + \mu u_{ij} u_{ij} + \frac{1}{2}\lambda u_{kk}^2 + \frac{g}{4!}(\partial_k u_k)^4 \right] \tag{9.3}$$

The parity transformation,

$$u_i(\vec{x},\tau) \to -u_i(-\vec{x},\tau) \tag{9.4}$$

leaves (9.3) invariant. If we define the unitary operator U_P by

$$U_P^\dagger u_i(\vec{x},\tau) U_P = -u_i(-\vec{x},\tau) \tag{9.5}$$

then U has the following effect on the creation and annihilation operators:

$$\begin{aligned} U_P^\dagger a_{\vec{k},s} U_P &= -a_{-\vec{k},s} \\ U_P^\dagger a_{\vec{k},s}^\dagger U_P &= -a_{-\vec{k},s}^\dagger \end{aligned} \tag{9.6}$$

Hence the vacuum state of a free phonon system, $g = 0$, which is defined by:

$$a_{\vec{k},s}|0\rangle = 0 \tag{9.7}$$

is invariant under parity:

$$U|0\rangle = |0\rangle \tag{9.8}$$

If we assume that the ground state evolves continuously as g is increased from 0 so that the $g \neq 0$ ground state is also invariant under parity, then parity constrains the correlation functions of the interacting theory:

$$\langle 0|T_\tau\left(u_{i_1}(x_1,\tau_1)\ldots u_{i_n}(x_n,\tau_n)\right)|0\rangle = (-1)^n \langle 0|T_\tau\left(u_{i_1}(-x_1,\tau_1)\ldots u_{i_n}(-x_n,\tau_n)\right)|0\rangle \tag{9.9}$$

Note that

$$u_i(\vec{x},\tau) \to -u_i(\vec{x},\tau) \tag{9.10}$$

is also a symmetry of (9.3), so we can take any combination of this and parity, such as

$$u_i(\vec{x},\tau) \to u_i(-\vec{x},\tau) \tag{9.11}$$

It doesn't really matter what we call these various symmetries so long as we realize that there are two independent symmetries. Realistic phonon Lagrangians have cubic terms which are not invariant under (9.10), so usually

the parity transformation (9.4) is the only symmetry. The symmetry (9.10), when it is present, leads to the relation

$$\langle 0\,|T_\tau\left(u_{i_1}(x_1,\tau_1)\ldots u_{i_n}(x_n,\tau_n)\right)|\,0\rangle = (-1)^n\ \langle 0\,|T_\tau\left(u_{i_1}(x_1,\tau_1)\ldots u_{i_n}(x_n,\tau_n)\right)|\,0\rangle \tag{9.12}$$

which implies that correlation functions of odd numbers of phonon fields must vanish.

A spin, on the other hand, transforms as:

$$U_P^\dagger\,\vec{\Omega}(\vec{x},t)\,U_P = \vec{\Omega}(-\vec{x},t) \tag{9.13}$$

The time-derivative term in the Lagrangian is not invariant under $\Omega \to -\Omega$, so there is no arbitrariness in our choice of parity transformation.

Time-reversal is a symmetry which does not quite fit into this paradigm. Reversing the direction of time, $t \to -t$ takes one solution of the equations of motion into another, but it does not necessarily leave the action $S = \int_{\tau_i}^{\tau_f} d\tau L(\tau)$ invariant. Nevertheless, we might expect that there is a unitary operator, U_T, which transforms the phonon field $u_i(t) \to u_i(-t)$

$$U_T^{-1}\,u_i(t)\,U_T = u_i(-t) \tag{9.14}$$

In fact, this operator cannot be unitary. To see this, differentiate both sides of (9.14):

$$U_T^{-1}\,\partial_t u_i(t)\,U_T = -\partial_t u_i(-t) \tag{9.15}$$

Act on

$$\left[u_i(x,t),\rho\partial_t u_j(x',t)\right] = i\,\delta_{ij}\delta(x-x') \tag{9.16}$$

with U_T^{-1} and U_T:

$$\begin{aligned} U_T^{-1}\,\left[u_i(x,t),\rho\partial_t u_j(x',t)\right]\,U_T &= U_T^{-1}\,i\,\delta_{ij}\delta(x-x')\,U_T \\ \left[u_i(x,-t),-\rho\partial_t u_j(x',-t)\right] &= i\,\delta_{ij}\delta(x-x')\,U_T^{-1}U_T \\ -\left[u_i(x,t),\rho\partial_t u_j(x',t)\right] &= i\,\delta_{ij}\delta(x-x') \\ -i\,\delta_{ij}\delta(x-x') &= i\,\delta_{ij}\delta(x-x') \end{aligned} \tag{9.17}$$

which is a contradiction.

The time-reversal operator must actually be an *antiunitary* operator. An antiunitary operator is a type of antilinear operator. While a linear operator $\mathcal{O}$ satisfies:

$$\mathcal{O}\left(\alpha|\psi\rangle\ +\ \beta|\chi\rangle\right) = \alpha\mathcal{O}|\psi\rangle\ +\ \beta\mathcal{O}|\chi\rangle \tag{9.18}$$

an antilinear operator satisfies

$$\mathcal{O}\left(\alpha|\psi\rangle\ +\ \beta|\chi\rangle\right) = \alpha^*\mathcal{O}|\psi\rangle\ +\ \beta^*\mathcal{O}|\chi\rangle \tag{9.19}$$

An antiunitary operator is an antilinear operator $\tilde{\mathcal{O}}$ which satisfies

$$\left(\tilde{\mathcal{O}}\chi, \tilde{\mathcal{O}}\psi\right) = (\chi, \psi) \tag{9.20}$$

where we have used the notation (χ, ψ) to denote the inner product between the states $|\psi\rangle$ and $|\chi\rangle$.

The time-reversal operator, U_T, is an anti-unitary operator, which explains how the paradox (9.17) is avoided. While the phonon field has the time-reversal property (9.14), a spin must transform under time-reversal so as to leave invariant the time-derivative term in the action:

$$s \int_0^\beta d\tau \int_0^1 dr \left(-i \frac{d\vec{\Omega}}{dr} \cdot \vec{\Omega} \times \frac{d\vec{\Omega}}{d\tau} \right) \tag{9.21}$$

Evidently, the correct transformation property is:

$$U_T^{-1} \, \vec{\Omega}(t) \, U_T = -\vec{\Omega}(-t) \tag{9.22}$$

Hence, the ferromagnetic, $\langle \vec{\Omega}(x,t) \rangle = \vec{\Omega}_0$, and antiferromagnetic, $\langle \vec{\Omega}(x,t) \rangle = (-1)^i \vec{n}_0$, ground states are not time-reversal invariant, i.e. they spontaneously break time-reversal invariance, unlike the phonon ground state which does not. The antiferromagnetic state also breaks the discrete symmetry of translation by one lattice spacing. However, the product of T and a translation by one lattice spacing is unbroken.

9.2 Noether's Theorem: Continuous Symmetries and Conservation Laws

Before looking at continuous symmetries in quantum systems, let us review one of the basic results of classical field theory: Noether's theorem. This theorem relates symmetries of the action to the existence of conserved currents.

Suppose we have a classical field theory defined by an action and Lagrangian density:

$$S = \int dt d^3\vec{r} \, \mathcal{L}(\phi, \partial\phi, \vec{r}) \tag{9.23}$$

where ϕ is the classical field. Consider a transformation

$$\phi(\vec{r}, t) \to \phi(\vec{r}, t, \lambda) \,, \;\; \phi(\vec{r}, t, 0) = \phi(\vec{r}, t) \tag{9.24}$$

and define the infinitesimal transformation

$$DK = \left(\frac{\partial K}{\partial \lambda}\right)_{\lambda=0} \tag{9.25}$$

Then, *this transformation is a symmetry of the action if and only if*

$$D\mathcal{L} = \partial_\mu F_\mu \tag{9.26}$$

for any ϕ (i.e. not only for ϕ satisfying the equations of motion). (Greek indices take the values $0, 1, 2, 3$ where $0 = i\tau$; relativistic invariance is *not* implied.)

Now, a general expression for $D\mathcal{L}$ can be obtained from the chain rule:

$$\begin{aligned} D\mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi} D\phi + \pi_\mu D\left(\partial_\mu \phi\right) \\ &= \partial_\mu \pi_\mu D\phi + \pi_\mu D\left(\partial_\mu \phi\right) \\ &= \partial_\mu \left(\pi_\mu D\phi\right) \end{aligned} \tag{9.27}$$

We used the equations of motion to go from the first line to the second and the equality of mixed partials, $D\partial\phi = \partial D\phi$, to go from the second to the third.

Setting these two expressions equal to each other, we have

Noether's theorem: for every transformation which is a symmetry of the action – i.e. $D\mathcal{L} = \partial_\mu F_\mu$ – there is a current $j_\mu = (\rho, \vec{j})$,

$$j_\mu = \pi_\mu D\phi - F_\mu \tag{9.28}$$

which is conserved,

$$\partial_\mu j_\mu = \partial_t \rho + \nabla \cdot \vec{j} = 0 \tag{9.29}$$

The extension to theories with multiple fields is straightforward and can be accomodated by decorating the preceding formulas with extra indices.

As an example, let's consider space and time translations:

$$\phi(x_\mu) \to \phi(x_\mu + \lambda\, e_\mu) \tag{9.30}$$

where $x_\mu = (t, \vec{r})$ and e_μ is an arbitrary 4-vector. Then,

$$\begin{aligned} D\phi &= e_\alpha \partial_\alpha \phi \\ D\mathcal{L} &= \partial_\alpha \left(e_\alpha \mathcal{L}\right) \end{aligned} \tag{9.31}$$

Hence, the conserved current is

$$j_\beta = e_\alpha T_{\alpha\beta} \tag{9.32}$$

where

$$T_{\alpha\beta} = \pi_\alpha \partial_\beta \phi - \delta_{\alpha\beta}\mathcal{L} \tag{9.33}$$

This is the stress-energy tensor. $T_{0\mu}$ is the 4-current corresponding to time-translation invariance: T_{00} is the energy density and T_{0i} is the energy 3-current. $T_{i\mu}$ are the 4-currents corresponding to spatial translational invariance: T_{i0} are the momentum densities and T_{ij} are the momentum currents.

In our theory of an elastic medium, $T_{\alpha\beta}$ is given by:

$$\begin{aligned} T_{00} &= \mathcal{H} \\ T_{0i} &= \rho\, \partial_t u_j \partial_i u_j \\ T_{ij} &= -2\mu\, \partial_i u_k \partial_j u_k - \partial_k u_k \partial_j u_i - \delta_{ij}\mathcal{L} \end{aligned} \tag{9.34}$$

T_{ij} is the stress tensor of the elastic medium.

Our action for a spin – as well as our actions for ferro- and anti-ferromagnets – is invariant under spin rotations, $\Omega_a \to \mathcal{R}_{ab}\Omega_b$. In the case of an ferromagnet, this leads to 3 conserved quantities corresponding to spin rotations about the three different axes:

$$\left(J_0^i, \vec{J}^i\right) = \left(\Omega^i, D\epsilon_{ijk}\Omega^j \vec{\nabla}\Omega^k\right) \tag{9.35}$$

In the antiferromagnetic case, they are:

$$J_\mu^i = \epsilon_{ijk} n_j \partial_\mu n_k \tag{9.36}$$

In general, symmetries are related to unobservable quantities. In the above, the conservation of momentum follows from the unobservability of absolute position; the conservation of energy, from the unobservability of absolute temporal position. Angular momentum is a consequence of the unobservability of absolute direction.

9.3 Ward Identities

In the previous section, we discussed the consequences of continuous symmetries and conservations laws for classical systems. We now turn to the quantum theory, where the existence of continuous symmetries and their associated conservation laws leads to important constraints on correlation functions. These constraints are called Ward identities. The Ward identity relates the divergence of a time-ordered correlation function of a conserved current, j_μ, with some other fields, φ_i to the variations of those field under the symmetry generated by j_0. The variation of $\varphi(x,t)$ under such a

symmetry operation is:

$$D\varphi(x,t) = \int d^d x' \, \left[J^0(x',t), \varphi(x,t) \right] \tag{9.37}$$

To derive the Ward identities, we consider a correlation function of j_μ with the φ_i's:

$$\begin{aligned}
&\langle T_\tau \left(j_\mu(x,\tau) \, \varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle = \\
&\qquad \theta(\tau-\tau_1)\theta(\tau_1-\tau_2)\ldots\theta(\tau_{n-1}-\tau_n) \, \langle j_\mu(x,\tau) \, \varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \rangle + \\
&\qquad \theta(\tau_1-\tau)\theta(\tau_1-\tau_2)\ldots\theta(\tau_{n-1}-\tau_n) \, \langle \varphi_1(x_1,\tau_1) \, j_\mu(x,\tau) \, \ldots \, \varphi_n(x_n,\tau_n) \rangle \\
&\qquad + \ldots
\end{aligned} \tag{9.38}$$

If we differerentiate this with respect to x_μ, the derivative operator can act on a θ-function which has τ in its argument or it can act on $j_\mu(x,\tau)$. If the symmetry is conserved in the classical field theory, then we can ordinarily conclude that $\partial_\mu j_\mu(x,\tau) = 0$. However, it is possible for this equation to be violated in the quantum theory when there is a cutoff (the conservation law can be violated when the conserved quantity flows to wavevectors beyond the cutoff) and this violation can remain finite even as the cutoff is taken to infinity. Such a symmetry is called *anomalous*. If the symmetry is not anomalous, then the right-hand-side contains only terms resulting from the derivative acting on the θ-function to give a δ-function:

$$\begin{aligned}
&\partial_\mu \langle T_\tau \left(j_\mu(x,\tau) \, \varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle \\
&\qquad = \delta(\tau-\tau_1)\theta(\tau_1-\tau_2)\ldots\theta(\tau_{n-1}-\tau_n) \, \langle j_\mu(x,\tau) \, \varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \rangle \\
&\qquad\quad - \delta(\tau_1-\tau)\theta(\tau_1-\tau_2)\ldots\theta(\tau_{n-1}-\tau_n) \, \langle \varphi_1(x_1,\tau_1) \, j_\mu(x,\tau) \, \ldots \, \varphi_n(x_n,\tau_n) \rangle + \ldots \\
&\qquad = \delta(\tau-\tau_1)\theta(\tau_1-\tau_2)\ldots\theta(\tau_{n-1}-\tau_n) \, \langle [j_\mu(x,\tau), \varphi_1(x_1,\tau_1)] \, \ldots \, \varphi_n(x_n,\tau_n) \rangle + \ldots \\
&\qquad = \delta(x-x_1)\delta(\tau-\tau_1) \, \langle T_\tau \left(D\varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle \\
&\qquad\quad + \delta(x-x_1)\delta(\tau-\tau_1) \, \langle T_\tau \left(\varphi_1(x_1,\tau_1) \, D\varphi_2(x_2,\tau_2) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle + \ldots
\end{aligned} \tag{9.39}$$

The final equality is the Ward identity:

$$\begin{aligned}
&\partial_\mu \langle T_\tau \left(j_\mu(x,\tau) \, \varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle \\
&\qquad = \delta(x-x_1)\delta(\tau-\tau_1) \, \langle T_\tau \left(D\varphi_1(x_1,\tau_1) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle \\
&\qquad\quad + \delta(x-x_2)\delta(\tau-\tau_2) \, \langle T_\tau \left(\varphi_1(x_1,\tau_1) \, D\varphi_2(x_2,\tau_2) \, \ldots \, \varphi_n(x_n,\tau_n) \right) \rangle \\
&\qquad\quad + \ldots
\end{aligned} \tag{9.40}$$

As an example of the Ward identity, consider an antiferromagnet, for which the spin currents are:

$$J^i_\mu = \epsilon_{ijk} n_j \partial_\mu n_k \tag{9.41}$$

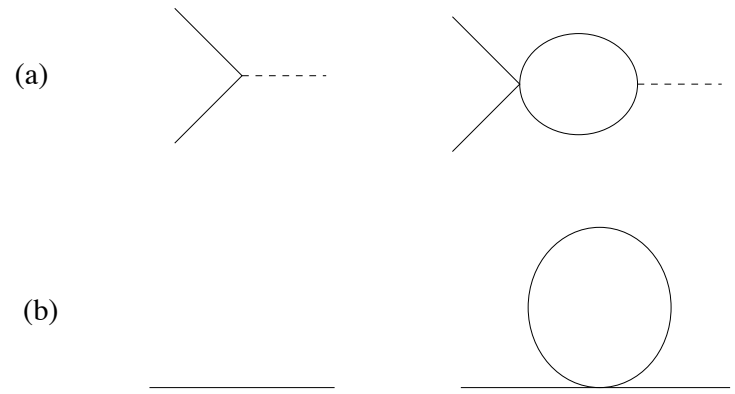


Figure 9.1: Diagrams contributing to the (a) left-hand-side and (b) right-hand side of the Ward identity (9.42).

Then the Ward identity tells us that:

$$\begin{aligned}\partial_\mu \langle T_\tau \left(\epsilon_{ijs} n_j(x,\tau)\partial_\mu n_s(x,\tau)\, n_k(x_1,\tau_1)\, n_l(x_2,\tau_2)\right)\rangle \\ = \delta(x-x_1)\delta(\tau-\tau_1)\epsilon_{ikm}\langle T_\tau \left(n_m(x_1,\tau_1)\, n_l(x_2,\tau_2)\right)\rangle \\ + \delta(x-x_2)\delta(\tau-\tau_2)\epsilon_{ilr}\langle T_\tau \left(n_l(x_1,\tau_1)\, n_r(x_2,\tau_2)\right)\rangle\end{aligned} \tag{9.42}$$

This is a non-trivial constraint when imposed order-by-order in perturbation theory, since the correlation function on the left-hand-side is given by diagrams such as those in figure 9.1a while the right-hand-side is given by diagrams such as those of 9.1b.

As another example, consider a ferromagnet which is ordered along the $\hat{\mathbf{z}}$ axis, $\langle \Omega_z \rangle = 1$:. Ω_x generates rotations about the x-axis, so the following correlation function is of the form for which the Ward identity is applicable:

$$\langle \Omega_x(i\omega_n, 0)\Omega_y(i\omega_n, 0)\rangle \tag{9.43}$$

with $J_0 = \Omega_x$ and $\partial_\mu \to p_\mu = (i\omega_n, 0)$. Hence, the Ward identity tells us that:

$$i\omega_n \langle \Omega_x(i\omega_n, 0)\Omega_y(i\omega_n, 0)\rangle = \langle \Omega_z \rangle = 1 \tag{9.44}$$

or,

$$\langle \Omega_x(i\omega_n, 0)\Omega_y(i\omega_n, 0)\rangle = \frac{1}{i\omega_n} \tag{9.45}$$

We found the same result earlier for a linearized theory in which magnon-magnon interactions. The Ward identity shows that this result is *exact*,

i.e. the tree-level result is unchanged by the inclusion of magnon-magnon interactions. The divergence of this correlation function at low frequency is an example of Goldstone's theorem in action, as we will see in the next section.

9.4 Spontaneous Symmetry-Breaking and Goldstone's Theorem

Often, the ground state is invariant under the symmetries of the Lagrangian. Our phonon Lagrangian, for instance, is invariant under parity and time-reversal, and the ground state is as well. However, this is not the only possibility, as we have already seen.It is possible that there is not an invariant ground state, but rather a multiplet of degenerate symmetry-related ground states, in which case we say that the symmetry is *spontaneously broken*. In terms of correlation functions, the statement of spontaneous symmetry-breaking is

$$\langle \phi(x,\tau) \rangle \neq 0 \tag{9.46}$$

where $\phi(x,\tau)$ is a field which is not invariant under the symmetry, $\phi(x,\tau) \neq U^\dagger \, \phi(x,\tau) \, U$. Such a field is called an *order parameter*.

For instance, our field theory for the Ising model,

$$\mathcal{Z} = N \int \mathcal{D}\varphi \, e^{-\int d^d x \left(\frac{1}{2} K (\nabla \varphi)^2 + \frac{1}{2} r \varphi^2 + \frac{1}{4!} u \varphi^4 \right)} \tag{9.47}$$

is invariant under the Z_2 symmetry $\varphi \to -\varphi$ which is broken for $T < T_c$ (i.e. $r < 0$), $\langle \varphi \rangle = \pm \sqrt{6r/u}$ and unbroken for $T > T_c$, $\langle \varphi \rangle = 0$. As the temperature is lowered through T_c, the system spontaneously chooses one of the two symmetry-related configurations $\langle \varphi \rangle = \pm \sqrt{6r/u}$, either as a result of a random fluctuation or some weak external perturbation. The ferromagnetic and antiferromagnetic ground states are two more examples of spontaneous symmetry breaking: the Heisenberg model and the field theories derived from it,

$$S = s \int d^d \vec{x} \, d\tau \, \left(-i \, \vec{A} \left(\vec{\Omega} \right) \cdot \frac{d\vec{\Omega}}{d\tau} + \frac{1}{2} D \left(\nabla \vec{\Omega} \right)^2 \right) \tag{9.48}$$

and

$$S = \frac{1}{g} \int d^d \vec{x} \, d\tau \, \left(\frac{1}{2v^2} \left(\frac{d\vec{n}}{d\tau} \right)^2 + \frac{1}{2} \left(\nabla \vec{n} \right)^2 \right) \tag{9.49}$$

are invariant under $SU(2)$ spin-rotational symmetry, $\Omega_i \to \mathcal{R}_{ij}\Omega_j$, $n_i \to \mathcal{R}_{ij}n_j$, but the ground states are not invariant since the magnetization or staggered magnetization chooses a particular direction. The signal of spontaneous symmetry breakdown is the non-invariant expectation value $\langle\vec{\Omega}\rangle \neq 0$ or $\langle\vec{n}\rangle \neq 0$. At high-temperature, $T > T_c$, the symmetry is restored and $\langle\vec{\Omega}\rangle = 0$ or $\langle\vec{n}\rangle = 0$. These expectation values also break the discrete T symetry. A ferromagnetic in a magnetic field does not have an $SU(2)$ or T-invariant Lagrangian, so its ferromagnetic ground state is an example of an explicitly broken symmetry rather than a spontaneously broken one. The $\mu\vec{B}\cdot\vec{\Omega}$ term is called a symmetry-breaking term. The phonon Lagrangian is actually another example: the translational symmetry of the continuum is broken to the discrete translational symmetry of the lattice. At high temperature (when our continuum elastic theory is no longer valid), the lattice melts and the resulting fluid state has the full translational symmetry.

In the first example, the Ising model, the broken symmetry is discrete, and there are no gapless excitations in the symmetry-broken phase. In the other two examples, magnets and ionic lattices, the broken symmetry, since it is continuous, leads to gapless excitations – magnons and phonons. This is a general feature of field theories with broken symmetries: broken continuous symmetries lead to gapless excitations – called Goldstone modes or Goldstone bosons – but broken discrete symmetries do not.

Physically, the reason for the existence of Goldstone bosons is that by applying the generator of the broken symmetry, we obtain another state with the same energy as the ground state. For instance, if we take a magnet aligned along the $\hat{\mathbf{z}}$ axis and rotate all of the spins away from the $\hat{\mathbf{z}}$ axis then we obtain another state of the same energy. If the spins instead vary slowly in space with wavevector $\vec{q}$, then the energy of the resulting state vanishes as $\vec{q} \to 0$. These states are the Goldstone modes.

The number of Goldstone modes is at most $\dim G - \dim H$ if G is the symmetry group of the theory and H is subgroup of G which is left unbroken. If $H = G$, i.e. the symmetry is completely unbroken, then there no Goldstone bosons. In the case of an antiferromagnet, $G = SU(2)$ and $H = U(1)$ – the group of rotations about staggered magnetization axis – so there are $\dim G - \dim H = 2$ gapless modes. A ferromagnet in zero field has only one Goldstone mode while $\dim G - \dim H = 2$. A ferromagnet in a finite field has no Goldstone modes; $G = H = U(1)$, the group of rotations about the direction of the field, so $\dim G - \dim H = 0$. A crytal has only three Goldstone modes, the u_i's, while G is the group of translations and rotations, $\dim G = 6$, and H is a discrete subgroup, $\dim H = 0$.

We will now give a precise statement and proof of Goldstone's theorem.

Suppose we have a conserved quantity, J^0, and its associated current, J^i, so that $\partial_\mu J^\mu = 0$. Let $\varphi(x,t)$ be some field in the theory. $\varphi(x,t)$ transforms as

$$D\varphi(x,t) = \int d^d x' \, \left[J^0(x',t), \varphi(x,t)\right] \tag{9.50}$$

under an infinitesimal symmetry operation corresponding to the conserved quantity J^0. Then, the following theorem holds.

Goldstone's Theorem: If there is an energy gap, Δ, then

$$\langle 0 \left| D\varphi(k=0,t) \right| 0 \rangle = 0 \tag{9.51}$$

Conversely, if $\langle 0|D\varphi(k=0,t)|0\rangle \neq 0$, then there must be gapless excitations. These gapless excitations are the Goldstone modes.

The proof proceeds by constructing a spectral representation for $\langle 0|J^0(x',t')\varphi(x,t)|0\rangle$:

$$\left\langle 0 \left| J^0(x',t')\varphi(x,t) \right| 0 \right\rangle = \int d^d k \, d\omega \, e^{-i(k\cdot(x-x')-\omega(t-t'))} \, \rho(\omega,k) \tag{9.52}$$

where

$$\rho(\omega,k) = \sum_n \left\langle 0 \left| J^0(0,0) \right| n \right\rangle \langle n \left| \varphi(0,0) \right| 0 \rangle \, \delta(\omega - E_n) \, \delta(\vec{k} - \vec{P}_n) \tag{9.53}$$

By unitarity, $\rho(\omega,k) \geq 0$. The existence of an energy gap, Δ, implies that $\rho(\omega,k) = 0$ for $\omega < \Delta$. Applying the conservation law to the correlation function of (9.52), we have:

$$\left\langle 0 \left| \partial_\mu J^\mu(x',t')\varphi(x,t) \right| 0 \right\rangle = 0 \tag{9.54}$$

Fourier transforming and taking the $\vec{k} \to 0$ limit, we have:

$$\omega \, \left\langle 0 \left| J^0(k=0,-\omega) \, \varphi(k=0,\omega) \right| 0 \right\rangle = 0 \tag{9.55}$$

The left-hand-side can be rewritten using our spectral representation (9.52):

$$\omega \, \rho(k=0,\omega) = 0 \tag{9.56}$$

which implies that

$$\rho(k=0,\omega) = 0 \tag{9.57}$$

or

$$\omega = 0 \tag{9.58}$$

Hence, $\rho(k=0,\omega)=0$ for all $\omega>0$. However, the existence of an energy gap, Δ implies that $\rho(k=0,\omega)=0$ for $\omega<\Delta$ and, in particular, for $\omega=0$. Hence, $\rho(k=0,\omega)=0$ for all ω. Therefore,

$$\left\langle 0\left|J^0(x',t')\varphi(k=0,t)\right|0\right\rangle=0 \tag{9.59}$$

Similarly,

$$\left\langle 0\left|\varphi(k=0,t)J^0(x',t')\right|0\right\rangle=0 \tag{9.60}$$

and, consequently,

$$\langle 0\left|D\varphi(k=0,t)\right|0\rangle=0 \tag{9.61}$$

When the symmetry *is* broken, $\rho(\omega,k=0)$ does not vanish when $\omega=0$; instead, there is a contribution to $\rho(\omega,k=0)$ coming from the Goldstone modes of the form $\rho(\omega,k=0)=\sigma\delta(\omega)$. Then $\langle 0\left|D\varphi(k=0,t)\right|0\rangle=\sigma$. Note that this proof depended on unitarity and translational invariance.

A field φ which satisfies

$$\langle 0\left|D\varphi(k=0,t)\right|0\rangle\neq 0 \tag{9.62}$$

is an *order parameter*: it signals the development of an ordered state. The order parameter of a ferromagnet is $\vec{\Omega}$ while the order parameter of an antiferromagnet is $\vec{n}$. It is not necessary for the order parameter of a theory to be the fundamental field of the theory. The order parameters of broken translational invariance in a crystal are:

$$\rho_G=e^{i\vec{G}\cdot\vec{u}(x,\tau)} \tag{9.63}$$

where G is a reciprocal lattice vector of the crystal and $\vec{u}$ is the phonon field.

If the order parameter, φ, is itself a conserved quantity, $J_0=\varphi$, which generates a spontaneously broken symmetry, then $D\varphi$ vanishes identically and the associated Goldstone boson doesn't exist. This is the reason why a ferromagnet has only 1 Goldstone mode. If the ferromagnet is ordered along the $\hat{\mathbf{z}}$ axis, then the symmetries generated by Ω_x and Ω_y are broken. If we look at the spectral functions for $\langle\Omega_x\,\Omega_y\rangle$ and $\langle\Omega_y\,\Omega_y\rangle$, only the former has a $\delta(\omega)$ contribution; the latter vanishes. In the case of an antiferromagnet, the spectral functions for both $\langle L_x\,n_y\rangle$ and $\langle L_y\,n_y\rangle$, have $\delta(\omega)$ contributions.

9.4.1 Order parameters**

9.4.2 Conserved versus nonconserved order parameters**

9.5 Absence of broken symmetry in low dimensions**

9.5.1 Discrete symmetry**

9.5.2 Continuous symmetry: the general strategy**

9.5.3 The Mermin-Wagner-Coleman Theorem

In chapter 9, we encountered hints that neither ferromagnets nor antiferromagnets could order at finite T in $d \leq 2$ and that antiferromagnets could not even order at zero temperature in $d = 1$. Let us now discuss the difficulties involved in breaking a symmetry in a low dimensional system. Consider the simplest example, namely the Ising model. Suppose the system is ordered with all of the spins pointing up. What is the energy cost to create a size L^d region of down spins in d-dimensions? It is simply

$$E_{\text{fluct}} \sim L^{d-1} \tag{9.64}$$

i.e. the energy cost of a domain of reversed spins is proportional to the surface area of the *domain wall* since that is the only place where unlike spins are neighbors. For $d > 1$, this grows with L, so large regions of reversed spins are energetically disfavored. At low temperature, this energy cost implies that such fluctuations occur with very low probability

$$P_{\text{fluc}} \sim e^{-(\text{const.})\beta L^{d-1}} \tag{9.65}$$

and hence the orderd phase is stable. In $d = 1$, however, the energy cost of a fluctuation is independent of the size of the fluctuation. It is simply $4J$. Hence, a fluctuation in which a large fraction of the system consists of reversed spins can occur with probability $\sim \exp(-4\beta J)$. As a result of these fluctuations, it is impossible for the system to order at any finite temperature. This is clearly true for any discrete symmetry.

Let us now consider a continuous symmetry at finite temperature. For the sake of concreteness, let us consider a d-dimensional magnet in an ordered phase in which the magnetization (or staggered magnetization) is aligned along the $\hat{\mathbf{z}}$ axis. (Recall that ferro- and antiferromagnets have the same description at finite temperature.) The energy cost for a size L^d region of reversed magnetization is less than in the case of a discrete symmetry

since the magnetization at the domain wall need not jump from one degenerate ground state to another. Rather, the spins can interpolate continuously between one ground state and another. The energy cost will be the gradient energy,

$$\int d^d\vec{x} \left(\frac{1}{2}(\nabla\vec{n})^2\right) \tag{9.66}$$

For a fluctuation of linear size L, $(\nabla\vec{n})^2 \sim 1/L^2$, so

$$\int_{L^d} d^d\vec{x} \left(\frac{1}{2}(\nabla\vec{n})^2\right) \sim L^{d-2} \tag{9.67}$$

Hence,

$$P_{\text{fluc}} \sim e^{-(\text{const.})\beta L^{d-2}} \tag{9.68}$$

and we conclude that a continuous symmetry can be broken for $T > 0$ in $d > 2$, but not for $d \leq 2$. This confirms our suspicion that magnets can't order for $T > 0$ in $d = 2$. The dimension below which symmetry-breaking is impossible is called the *lower critical dimension*. For $T > 0$, the lower critical dimension is 2 for continuous symmetries and 1 for discrete dymmetries.

These qualitative considerations can be made rigorous. Let us consider our finite-temperature order parameter, $\varphi(x, \omega = 0)$, in $d = 2$. We will show that

$$\langle D\varphi\rangle = 0 \tag{9.69}$$

i.e. the symmetry is necessarily unbroken for $d = 2$. Define:

$$\begin{aligned} F(k) &= \int d^2x e^{ik\cdot x} \ \langle\varphi(x)\varphi(0)\rangle \\ F_i(k) &= \int d^2x e^{ik\cdot x} \ \langle j_i(x)\varphi(0)\rangle \\ F_{il}(k) &= \int d^2x e^{ik\cdot x} \ \langle j_i(x) j_l(0)\rangle \end{aligned} \tag{9.70}$$

where $i = 0, 1$. The conservation law $k_i F_i = 0$ implies that

$$F_i(k) = \sigma\, k_i \delta(k^2) + \epsilon_{ij} k_j \rho(k) \tag{9.71}$$

This decomposition is clearly special to two dimensions. Substituting this decomposition into the definition of $D\varphi$, we have:

$$\langle D\varphi\rangle = \sigma \tag{9.72}$$

By unitarity,

$$\int d^2x\, h(x)\ (aj_0(x) + b\varphi(x))\ |0\rangle \tag{9.73}$$

has positive norm. From the special cases $a = 0$ and $b = 0$, we see that

$$\begin{aligned} \int d^2k\, |h(k)|^2\, F(k)\ &\geq 0 \\ \int d^2k\, |h(k)|^2\, F_{00}(k)\ &\geq 0 \end{aligned} \tag{9.74}$$

positivity of the norm also implies that

$$\left(\int d^2k\, |h(k)|^2\, F(k)\right)\ \left(\int d^2k\, |h(k)|^2\, F_{00}(k)\right)\ \geq\ \left(\int d^2k\, |h(k)|^2\, F_0(k)\right)^2 \tag{9.75}$$

If we take an $h(k)$ which is even in x_1 – and, therefore, even in k_1 – then the right-hand-side will be:

$$\sigma \int d^2k\, |h(k)|^2\, k_0\, \delta(k^2) = \sigma \int dk_0 |h(k_0, |k_0|)|^2 \tag{9.76}$$

We can make the left-hand-side vanish by making $|h(k)|^2$ sharply peaked at very high k where $F(k)$ and $F_{00}(k)$ must vanish. Consequently, $\sigma = 0$ and the symmetry is unbroken. The proof works by essentially taking the spatial points very far apart in the correlation functions on the left-hand-side. In the presence of long-range forces, the left-hand-side need not vanish, and spontaneous symmetry-breaking is possible.

Thus far, our discussion has focussed on thermal fluctuations. Can quantum fluctuations prevent order at $T = 0$? In the case of an antiferromagnet, the answer is clearly yes. The quantum theory of an d-dimensional antiferromagnet at $T = 0$ is the same as the classical statistical theory of a magnet in $d + 1$-dimensions. Hence, we conclude that a quantum antiferromagnet can order at $T = 0$ in $d + 1 > 2$, i.e. in $d > 1$, but not in $d = 1$.

A ferromagnet, on the other hand, can order in any number of dimensions. The exact ground state of a Heisenberg ferromagnet is a state in which all of the spins are aligned. The reason that the above arguments about fluctuations do not apply to the ferromagnet is that it has a *fluctuationless* ground state. This can be said somewhat differently as follows. The order parameter for a ferromagnet, $\Omega(q = 0, \omega_n = 0)$ is a conserved quantity: the components of $\Omega(q = 0, \omega_n = 0)$ are the components of the total spin along the different axes. Thus, while it is true that there is very little energy cost for a state with reversed spins, such a state will never be

reached at $T = 0$ since the dynamics conserves the the total spin. In the case of an antiferromagnet, on the other hand, $\vec{n}$ is not conserved; hence, the dynamics of the system can lead to fluctuations which destroy the order. At finite temperature, we must average over *all* of the states in the canonical ensemble, so the fluctuations can destroy the ordered state of the ferromagnet. To summarize, if the order parameter is a conserved quantity, then there can always be order at $T = 0$ in any d. If it is not, then quantum fluctuations can destroy the order at $T = 0$. In the case of antiferomagnets – or phonons – this occurs in $d = 1$. More generally, it occurs when $d+z = 2$ for a continuous symmetry or $d + z = 1$ for a discrete symmetry.

9.5.4 Absence of magnetic order**

9.5.5 Absence of crystalline order**

9.5.6 Generalizations**

9.5.7 Lack of order in the ground state**

9.6 Proof of existence of order**

9.6.1 Infrared bounds**

Part IV

Critical Fluctuations and Phase Transitions

CHAPTER 10

The Renormalization Group and Effective Field Theories

10.1 Low-Energy Effective Field Theories

In our earlier discussions, we focussed on the low (compared to some cutoff Λ) T, low ω, q properties of the systems at which we looked. Why? The principal reason is that these properties are *universal* – i.e. independent of many of the details of the systems. Sometimes universal properties are the most striking and interesting aspect of a physical system, but not always (certainly not for many practical, e.g. engineering, applications). We like universal properties because we can understand them using *effective field theories*.

Suppose we have a system defined by the following functional integral:

$$Z = \int \mathcal{D}\phi \; e^{-S[\phi]} \tag{10.1}$$

with correlation functions:

$$\langle \phi(p_1) \dots \phi(p_n) \rangle = \int \mathcal{D}\phi \; \phi(p_1) \dots \phi(p_n) \; e^{-s[\phi]} \tag{10.2}$$

The long-wavelength, universal properties of the system are determined by these correlation functions in the $p_i \to 0$ limit, i.e. in the limit that the p_i's are smaller than any other scales in the problem.

Z contains a great deal of 'extraneous' information about correlation functions for large p_i. We would like an *effective action*, $S_{\rm eff}$, so that

$$Z_{\rm eff} = \int \mathcal{D}\phi \; e^{-S_{\rm eff}[\phi]} \tag{10.3}$$

only contains the information necessary to compute the long-wavelength correlation functions. The reason that this a worthwhile program to pursue is that $Z_{\rm eff}$ is often simple or, at least, simpler than Z. On the other hand, this is not a comletely straightforward program because no one tells us how to derive $Z_{\rm eff}$. We have to use any and all tricks available to us (sometimes we can find a small parameter which enables us to get $Z_{\rm eff}$ approximately and often we simply have to guess.

At a formal level, we can make the division:

$$\begin{aligned} \phi_L(p) &= \phi(p)\,\theta(\Lambda' - |p|) \\ \phi_H(p) &= \phi(p)\,\theta(|p| - \Lambda') \end{aligned} \tag{10.4}$$

so that

$$\phi(p) = \phi_L(p) + \phi_H(p) \tag{10.5}$$

where Λ' is some scale such that we're interested in $|p| < \Lambda'$. Then

$$Z = \int \mathcal{D}\phi_L \mathcal{D}\phi_H \; e^{-S[\phi_L,\phi_H]} \tag{10.6}$$

The effective field theory for long-wavelength correlation functions, i.e. correlation functions of ϕ_L, is

$$Z_{\rm eff} = \int \mathcal{D}\phi_L \; e^{-S_{\rm eff}[\phi_L]} \tag{10.7}$$

where

$$e^{-S_{\rm eff}[\phi_L]} = \int \mathcal{D}\phi_H \; e^{-S[\phi_L,\phi_H]} \tag{10.8}$$

$S_{\rm eff}[\phi_L]$ has cutoff Λ'

Occasionally, we will be in the fortunate situation in which

$$S[\phi_L, \phi_H] = S_L[\phi_L] + S_H[\phi_H] + \lambda S_{\rm int}[\phi_L, \phi_H] \tag{10.9}$$

with λ small so that we can compute $S_{\rm eff}$ perturbatively in λ:

$$S_{\rm eff} = S_L[\phi_L] + \lambda S_1[\phi_L] + \lambda^2 S_2[\phi_L] + \ldots \tag{10.10}$$

In general, we have no such luck, and we have to work much harder to derive $S_{\rm eff}$. However, even without deriving $S_{\rm eff}$, we can make some statements about it on general grounds.

10.2 Renormalization Group Flows

Let's suppose that we have somehow derived S_{eff} with cutoff Λ. Let's call it $S_\Lambda[\phi]$. $S[\phi]$ itself may have had all kinds of structure, but this doesn't interest us now; we're only interested in $S_\Lambda[\phi]$.

We expand $S_\Lambda[\phi]$ as

$$S_\Lambda[\phi] = \sum_i g_i \mathcal{O}_i \tag{10.11}$$

where the g_i's are 'coupling constants' and the $\mathcal{O}_i$ are local operators. For instance, our phonon Lagrangian can be written as:

$$S = \rho\mathcal{O}_1 + \mu\mathcal{O}_2 + \lambda\mathcal{O}_3 + g\mathcal{O}_4 \tag{10.12}$$

with

$$\begin{aligned}
\mathcal{O}_1 &= \frac{1}{2}\int d\tau d^3\vec{x}(\partial_t u_i)^2 \\
\mathcal{O}_2 &= \int d\tau d^3\vec{x}\, u_{ij}u_{ij} \\
\mathcal{O}_3 &= \frac{1}{2}\int d\tau d^3\vec{x}\, u_{kk}^2 \\
\mathcal{O}_4 &= \frac{1}{4!}\int d\tau\, d^3\vec{x}\,(\partial_k u_k)^4
\end{aligned} \tag{10.13}$$

while the NLσM for an antiferromagnet can be written as:

$$S = \mathcal{O}_0 + g\mathcal{O}_1 + g^2\mathcal{O}_2 + \ldots \tag{10.14}$$

with

$$\begin{aligned}
\mathcal{O}_0 &= \int d^d\vec{x}\, d\tau\, (\partial_\mu n_i)^2 \\
\mathcal{O}_1 &= \int d^d\vec{x}\, d\tau\ n_i\partial_\mu n_i\ n_j\partial_\mu n_j \\
\mathcal{O}_2 &= \int d^d\vec{x}\, d\tau\ n_i\partial_\mu n_i\ n_j\partial_\mu n_j\, n_i n_i \\
&\vdots
\end{aligned} \tag{10.15}$$

We now pick one term in the action – call it $\mathcal{O}_{\text{free}}$ even though it need not be quadratic – and use this term to assign dimensions to the various fields in the theory by requiring $\mathcal{O}_{\text{free}}$ to be dimensionless. For instance, if we choose $\mathcal{O}_{\text{free}} = \mathcal{O}_1$ in our phonon theory, then $[u_i] = 1$. If we choose $\mathcal{O}_{\text{free}} = \mathcal{O}_4$ then $[u_i] = 0$. Typically, we choose the term which we believe to be most

'important'. If we choose the 'wrong' one (i.e. an inconvenient one) then we will find out in the next step. Let's call δ_ϕ the dimension of the field ϕ and δ_i the dimension of the operator $\mathcal{O}_i$. $\delta_{\rm free} = 0$ by construction.

We now rescale all momenta and fields by the cutoff Λ,

$$\begin{aligned} q &\to q\Lambda \\ \phi &\to \phi\Lambda^{\delta_\phi} \end{aligned} \tag{10.16}$$

so that the momenta and fields are now dimensionless. Then

$$S_\Lambda[\phi] = \sum_i g_i \Lambda^{\delta_i}\, \mathcal{O}_i = \sum_i \lambda_i\, \mathcal{O}_i \tag{10.17}$$

Ordinarily, the dimensionless couplings λ_i will be $O(1)$. On dimensional grounds, at energy scale ω, $\mathcal{O}_i \sim (\omega/\Lambda)^{\delta_i}$, and the action

$$S_\Lambda[\phi] = \sum_i \lambda_i \left(\frac{\omega}{\Lambda}\right)^{\delta_i}, \tag{10.18}$$

If $\delta_i > 0$, this term becomes less important at low energies. Such a term is called *irrelevant*. If $\delta_i = 0$, the term is called *marginal*; it remains constant as $\omega \to 0$. If $\delta_i < 0$, the term is relevant; it grows in importance at low energies. If $S_{\rm eff}$ is simple, it is only because there might be a finite number of *relevant* operators. At lower and lower energies, $\omega \ll \Lambda$, it becomes a better and better approximation to simply drop the irrelevant operators.

Let's formalize this by putting together the notion of a low-energy effective field theory with the above considerations about scaling:

- We have an effective action, $S_\Lambda[\phi]$ and a choice of $\mathcal{O}_{\rm free}$.
- We divide

  $$\phi(p) = \phi_L(p)\,\theta(b\Lambda' - |p|) \;+\; \phi_H(p)\,\theta(|p| - b\Lambda')$$

  where $b < 1$.
- The next step is to obtain (by hook or by crook) $S_{b\Lambda}$

  $$e^{-S_{b\Lambda}[\phi_L]} = \int \mathcal{D}\phi_H \; e^{-S_\Lambda[\phi_L,\phi_H]}$$

- We now rescale

  $$q \to qb$$

$$\omega \to \omega b^z$$
$$\phi \to \phi b^\zeta$$

where ζ and z are chosen to preserve $\mathcal{O}_{\text{free}}$. In general, ζ and z will depend on the couplings g_i, $\zeta = \zeta(g_i)$, $z = z(g_i)$. In equilibrium classical statistical mechanics, there are no frequencies, so we do not need to worry about g; in the theories which have examined thus far ω and q are on the same footing, so z is fixed to $z = 1$. In general, however, one must allow for arbitrary z. The rescaling yields $S^b_\Lambda[\phi]$ which also has cutoff Λ.

- The physics of the system can be obtained from $S^b_\Lambda[\phi]$ by a rescaling. For instance, the correlation length is given by

$$(\xi)_{S_\Lambda} = \frac{1}{b}\,(\xi)_{S^b_\Lambda}$$

- If

$$S_\Lambda[\phi] = \sum_i g_i^0 \mathcal{O}_i$$

then

$$S^b_\Lambda[\phi] = \sum_i g_i(b)\mathcal{O}_i$$

where $g_i(1) = g_i^0$. Let $b = e^{-\ell}$. Then we can define *flow equations*:

$$\frac{dg}{d\ell} = -\delta_i\, g_i + \dots$$

which describe the evolution of S_Λ under an infinitesimal transformation. These equations are called *Renormalization Group* (RG) equations or *flow equations*.

If we can neglect the ..., then for $\delta_i < 0$, g_i grows as ℓ increases, i.e. $\mathcal{O}_i$ is more important at low energies ($\ell \to \infty$. For $\delta_i > 0$, g_i decreases, i.e. $\mathcal{O}_i$ is less important at low energies. Of course, the ... need not be small. In fact, it can dominate the first term. In the case of a marginal operator, $\delta_i = 0$, the ... is the whole story. For example, if

$$\frac{dg}{d\ell} = g^2 \tag{10.19}$$

Then

$$g(\ell) = \frac{g_0}{1 - g_0(\ell - \ell_0)} \tag{10.20}$$

so $g(\ell)$ grows as ℓ grows.

10.3 Fixed Points

If, for some values of the couplings, $g_i = g_i^*$,

$$\left(\frac{dg_i}{d\ell}\right)_{g_k=g_k^*} = 0 \tag{10.21}$$

then we call $g_k = g_k^*$ a *fixed point*. At a fixed point, $S_\Lambda = S_\Lambda^{b'}$, so the physics is the same at all scales. Hence,

$$\xi = \frac{1}{b}\,\xi \tag{10.22}$$

i.e. $\xi = \infty$ – the low-energy physics is gapless – or $\xi = 0$ – there is no low-energy physics.

The notion of *universality* is encapsulated by the observation that different physical systems with very different 'microscopic' actions S_Λ, S'_Λ, S''_Λ can all flow into the same fixed point action, S^*_Λ. If this happens, these systems have the same asymptotic long-wavelength physics, i.e. the same universal behavior.

At a fixed point, we can linearize the RG equations:

$$\frac{d}{d\ell}\left(g_i - g_i^*\right) = A_{ij}\left(g_j - g_j^*\right) \tag{10.23}$$

This can be diagonalized to give:

$$\frac{du_i}{d\ell} = y_i\, u_i \tag{10.24}$$

where $u_i = O_{ij}(g_j - g_j^*)$. The corresponding operators, $\tilde{\mathcal{O}}_i = O_{ij}\mathcal{O}_i j$,

$$S_\Lambda[\phi] = \sum_i u_i \tilde{\mathcal{O}}_i \tag{10.25}$$

are called *eigenoperators*. If $y_i > 0$, we say that u_i and $\tilde{\mathcal{O}}_i$ are relevant at this fixed point. If $y_i = 0$, we say that u_i is marginal. If $y_i < 0$, u_i is irrelevant at this fixed point.

Earlier, we characterized $\mathcal{O}_i$ as relevant, marginal, or irrelevant according to whether $\delta_i < 0$, $\delta_i = 0$, or $\delta_i > 0$. What this really means is that $\mathcal{O}_i$ has this property at the fixed point $S^* = \mathcal{O}_{\text{free}}$. It is possible for a coupling constant, g, to be relevant at one fixed point, S_1^*, but irrelevant at another fixed point, S_2^*, as shown in figure 10.1.

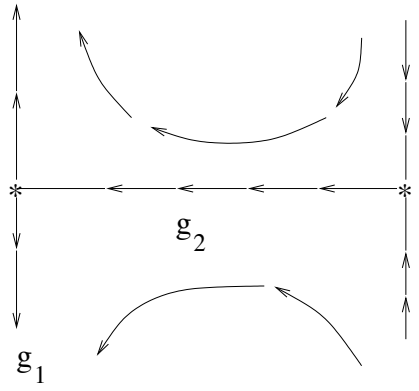


Figure 10.1: The coupling g_1 is relevant at the fixed point on the left but irrelevant at the fixed point on the right.

10.4 Phases of Matter and Critical Phenomena

If $y_i < 0$ for all i at a given fixed point, then we call this fixed point an attractive or stable fixed point. Theories will generically flow into such a fixed point. Stable fixed points represent phases of matter. In this course, we have already looked at a number of stable fixed points. Our phonon theory,

$$S_0 = \int dt d^3\vec{x}\mathcal{L} = \frac{1}{2}\int dt d^3\vec{r}\left[\rho(\partial_t u_i)^2 - 2\mu u_{ij}u_{ij} - \lambda u_{kk}^2\right] \tag{10.26}$$

is a stable fixed point (you can check that g is an irrelevant coupling) at $T = 0$ for $d > 1$. This stable fixed point corresponds to a stable phase of matter: a crystal. For $T > 0$, it is stable for $d > 2$. Our theories of non-interacting magnons

$$S = s\int d^d\vec{x}\, d\tau\, \left(\frac{1}{2}\, m_+\frac{\partial m_-}{\partial\tau} + \frac{1}{2}D\,\vec{\nabla}m_+\cdot\vec{\nabla}m_-\right) \tag{10.27}$$

$$S = \int d^d\vec{x}\, d\tau\, (\partial_\mu n_i)^2 \tag{10.28}$$

are also stable fixed points corresponding, repectively, to ferromagnetic and antiferromagnetic phases. The ferromagnet is always stable at $T = 0$, while the antiferromagnet is stable at $T = 0$ for $d > 1$. For $T > 0$, both are stable for $d > 2$. Similarly, XY magnets and superfluid ^{4}He

$$S = \int d^d\vec{x}\, d\tau\, (\partial_\mu\theta)^2 \tag{10.29}$$

are phases of matter at $T = 0$ for $d > 1$. For $T > 0$, they are stable for $d \geq 2$ ($d = 2$ is a special case).

The stable phases described above are all characterized by gapless modes – i.e. $\xi = \infty$ which are a consequence of spontaneous symmetry breaking. There are also stable phases without gapless modes – i.e. with $\xi = 0$. The ^{4}He action with $\mu < \mu_c$ (in the saddle-point approximation, $\mu_c = 0$) describes an empty system with a gap $\mu_c - \mu$ to all excitations.

$$S = \int d\tau d^d x \left(\psi^* \partial_\tau \psi + |\nabla \psi|^2 + V\Big(|\psi|^2 - \frac{\mu}{2V}\Big)^2 \right) \tag{10.30}$$

Similarly, φ^4 theory

$$\int d^d x \left(\frac{1}{2} K (\nabla \varphi)^2 + \frac{1}{2} r \varphi^2 + \frac{1}{4!} u \varphi^4 \right) \tag{10.31}$$

has two stable phases – ordered and disordered phases – corresponding to the fixed points $r \to \pm\infty$. At both of these fixed points, $\xi = 0$. Similarly, the high-temperature disordered states of magnets are stable phases with gaps.

It makes sense to do perturbation theory in an irrelevant coupling because this perturbation theory gets better at low q, ω. Essentially, the expansion parameter for perturbation theory is the dimensionless combination $g\omega^{-y_g}$ for some correlation function at characteristic frequency ω. Hence, our perturbative calculations of correlation functions in the phonon and magnon theories were sensible calculations. Similarly, perturbation theory in the coupling u in φ^4 is sensible for $d > 4$ (the Ginzburg criterion) as you showed in the problem set. However, it does not make sense to perturb in a relevant coupling such as u in φ^4 for $d < 4$. In such a case, the effective expansion parameter grows at low q, ω. The low q, ω physics is, in fact controlled by some other fixed point.

If some of the $y_i > 0$ then the fixed point is repulsive or unstable. The relevant couplings must be tuned to zero in order for the theory to flow into an unstable fixed point. Unstable fixed points represent (multi-)critical points separating these phases. The unstable directions correspond to the parameters which must be tuned to reach the critical point. Superfluid ^{4}He

$$S = \int d\tau d^d x \left(\psi^* \partial_\tau \psi + |\nabla \psi|^2 + V\Big(|\psi|^2 - \frac{\mu}{2V}\Big)^2 \right) \tag{10.32}$$

has a critical point at $\mu = \mu_c$. The corresponding fixed point is at $\mu = \mu^*$, $V = V^*$. This critical point separates two stable phases: the superfluid

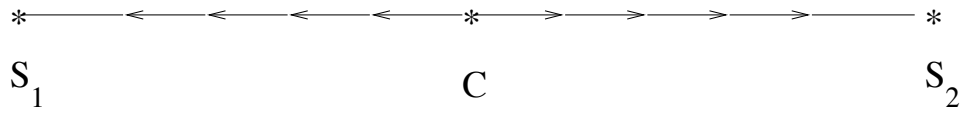


Figure 10.2: The flow diagram of a critical point C and two stable fixed points S_1, S_2 which it separates.

and the empty system. There is one relevant direction at this fixed point. By tuning this relevant direction, we can pass from one phase through the critical point to the other phase. Similarly, the Ising model has a fixed point with one relevant direction which we discuss in a later section. By tuning the relevant coupling, we can pass from the ordered state through the critical point and into the disordered state. Figure 10.2 depicts the flow diagram for a critical point and two stable fixed points which it separates.

10.5 Infinite number of degrees of freedom and the nonanalyticity of the free energy**

10.5.1 Yang-Lee theory**

10.6 Scaling Equations

Let us consider the the implications of this framework for physical quantities. Suppose $C(p_i, g_i)$ is some physical quantity such as a correlation function of n fields φ. It will, in general, depend on some momenta p_i and on the coupling constants of the system. Suppose that the couplings are all close to their fixed point values, $g_i \approx g_i^*$, so we will write C as $C(p_i, g_i - g_i^*)$ and suppose that the linearized flow equations read:

$$\frac{d}{d\ell}\left(g_i - g_i^*\right) = \lambda_i \, \left(g_i - g_i^*\right) \tag{10.33}$$

Then we can perform an RG transformation, according to which:

$$C\left(p_i, g_i - g_i^*\right) = b^{-n\zeta^*} \, C\left(\frac{p_i}{b}, \left(g_i - g_i^*\right) b^{-\lambda_i}\right) \tag{10.34}$$

Suppose that we are in the vicinity of a stable fixed point, so that all of the $\lambda_i < 0$. Then, in the $b \to 0$ limit

$$C\left(p_i, g_i - g_i^*\right) = b^{-n\zeta^*} \, C\left(\frac{p_i}{b}, 0\right) \tag{10.35}$$

If, for instance, we are interested in the two-point correlation function at low p, we can take $b = p$ and:

$$C\left(p, g_i - g_i^*\right) \to p^{-2\zeta^*} C\left(1, 0\right) \tag{10.36}$$

A similar result follows if we are in the vicinity of an unstable fixed point, but we have set all of the relevant couplings equal to zero. This scaling relation may seem to contradict simple dimensional analysis, which would predict $C\left(p, g_i - g_i^*\right) \sim p^{-2\delta_\phi}$. In fact, there is no contradiction. The missing powers of p are made up by the dependence on the cutoff:

$$C\left(p, g_i - g_i^*\right) \sim p^{-2\zeta^*} \Lambda^{-2\delta_\phi + 2\zeta^*} \tag{10.37}$$

These observations can be reformulated as follows. Consider a correlation function $C_n(p_i, g_i)$ of n φ fields. Up to a rescaling, this correlation function is equal to its value after an RG transformation:

$$C_n(p_i, g_i) = \left(e^\ell\right)^{n\zeta(g_i)} C_n(p_i e^\ell, g_i(\ell)) \tag{10.38}$$

The left-hand-side is independent of ℓ, so differentiating both sides with respect to ℓ yields the *RG equation*:

$$\left(\frac{\partial}{\partial \ell} + n\left(\delta_\varphi - \frac{1}{2}\eta\left(g_j\right)\right) + \beta_i\left(g_j\right)\frac{\partial}{\partial g_i}\right) C_n(p_i e^\ell, g_i(\ell)) = 0 \tag{10.39}$$

where

$$2\delta_\varphi - \eta\left(g_j\right) = \frac{d}{d\ell}\left(e^\ell\right)^{\zeta(g_i)} \tag{10.40}$$

δ_φ is the naive scaling dimension of φ. η is called the *anomalous dimension* of φ. The β functions are the right-hand-sides of the flow-equations for the couplings:

$$\beta_i\left(g_j\right) = \frac{dg_i}{d\ell} \tag{10.41}$$

At a fixed point, the β-functions vanish, $\beta_i = 0$ and η is a constant,

$$2\delta_\varphi - \eta = 2\zeta\left(g_i^*\right) \tag{10.42}$$

so the RG equation reads:

$$\left(\frac{\partial}{\partial \ell} - n\left(\delta_\varphi - \frac{1}{2}\eta\right)\right) C_n(p_i e^{-\ell}) = 0 \tag{10.43}$$

In other words, the correlation function is a power-law in p_i with exponent $n\delta_\varphi - n\eta/2$.

Suppose, instead, that we are near a fixed point with one relevant direction. Call this coupling u and the other irrelevant couplings g_i. Then,

$$C\left(p_i, u-u^*, g_i-g_i^*\right) = b^{-n\zeta^*}\, C\left(p_i/b, (u-u^*)\, b^{-\lambda_u}, (g_i-g_i^*)\, b^{-\lambda_i}\right) \tag{10.44}$$

If $u-u^*$ is small, then we can take $b=(u-u^*)^{1/\lambda_u}$ and be in the $b\to 0$ limit:

$$C\left(p_i, u-u^*, g_i-g_i^*\right) \to \frac{1}{(u-u^*)^{n\zeta^*/\lambda_u}} C\left(p_i(u-u^*)^{-1/\lambda_u}, 1, 0\right) \tag{10.45}$$

or

$$C\left(p_i, u-u^*, g_i-g_i^*\right) \to \frac{1}{(u-u^*)^{n\zeta^*/\lambda_u}} F\left(p_i(u-u^*)^{-1/\lambda_u}\right) \tag{10.46}$$

where $F(x)$ is called a *scaling function*. If the stable phase to which the system flows is trivial – i.e. has a gap – then $C\left(p_i(u-u^*)^{-1/\lambda_u}, 1, 0\right)$ can be calculated and does not have any interesting structure. The non-trivial physics is entirely in the prefactor.

If we are interested in a correlation function at $p_i=0$ such as the magnetization of a ferromagnet, then we can write:

$$C_0\left(u-u^*, g_i-g_i^*\right) \to \frac{1}{(u-u^*)^{n\zeta^*/\lambda_u}} F\left((g_i-g_i^*)\,(u-u^*)^{-\lambda_i/\lambda_u}\right) \tag{10.47}$$

Now imagine that there is a second relevant coupling, g, or, even that g is the leading irrelevant coupling (i.e. the *least* irrelevant of the irrelevant couplings) so that we are interested in the g dependence of the correlation function. Then, setting the other couplings to their fixed point values in the small $u-u^*$ limit:

$$C_0\left(u-u^*, g_i-g_i^*\right) \to \frac{1}{(u-u^*)^{n\zeta^*/\lambda_u}}\, F\left(\frac{g_i-g_i^*}{(u-u^*)^{-\lambda_i/\lambda_u}}\right) \tag{10.48}$$

10.7 Analyticity of β-functions**

10.8 Finite-Size Scaling

Temperature plays a very different role in classical and quantum statistical mechanics. In the classical theory, temperature is one of the couplings in the theory. The temperature dependence of physical quantities can be

determined from the scaling behavior of the temperature. Classical statistical mechanics can be used to calculate a correlation function at wavevector the temperature is larger than the important excitation energies since the $n \neq 0$ Matsubara frequencies can then be ignored. In the quantum theory, temperature is the size of the system in the imaginary time direction, and the temperature dependence of physical quantities can be determined from finite-size scaling which we discuss below. Finite-size scaling can be used in the limit in which β is large. An alternative, related way of dealing with finite-temperature is dicussed in the context of the NLσM in the last section of this chapter.

Finite-size scaling is also useful for dealing with systems which are finite in one or more spatial directions. Since numerical calculations must be done in such systems, finite-size scaling allows us to compare numerics to analytical calculations which are more easily done in the infinite-size limit.

Since renormalization group equations describe the evolution of effective Lagrangians as one integrates out short-distance physics, it is clear that these equations are insensitive to finite-size effects so long as the finite-size is much larger than the inverse of the cutoff. While the equations themselves are unchanged, the solutions are modified because they depend on an additional dimensionful parameter, namely the size of the system (in our case, β). For simplicity, let us consider a theory with a single relevant coupling (say, ϕ^4 theory), which satisfies a renormalization group equation with a low-energy fixed point:

$$\left(\frac{\partial}{\partial \ell} + \beta(g)\frac{\partial}{\partial g} + n\left(\delta_\varphi - \frac{1}{2}\,\eta\,(g)\right)\right) G^{(n)}(p_i e^{\ell}, g(\ell), L e^{-\ell}) = 0 \qquad (10.49)$$

$G^{(n)}$ is an n-point Green function, L is the finite size of the system. We may take $e^{\ell} = L$, and we find

$$G^{(n)}(p_i, g, L) = L^{n\left(\delta_\varphi - \frac{1}{2}\,\eta(g_j)\right)} G^{(n)}(p_i L, g(\ln L), 1) \qquad (10.50)$$

Then in the large-size limit, $L \to \infty$, we have $g(\ln L) \to g^*$. As a result, we have the scaling form:

$$G^{(n)}(p_i, g, L) = L^{n\left(\delta_\varphi - \frac{1}{2}\,\eta(g_j)\right)} G^{(n)}(p_i L, g^*, 1) \qquad (10.51)$$

We will be primarily concerned with the case in which the finite size, L, will be the inverse temperature, β, so (10.51) will give the temperature dependence of Green functions in the low-temperature limit.

$$G^{(n)}(p_i, g, \beta) \sim \beta^{n\left(\delta_\varphi - \frac{1}{2}\,\eta\right)} G^{(n)}(p_i \beta, g^*, 1) \qquad (10.52)$$

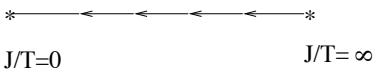


Figure 10.3: The flow diagram of the $1D$ Ising model.

10.9 Non-Perturbative RG for the 1D Ising Model

In the next two sections, we will look at two examples of RG transformations, one non-perturbative and one perturbative. First, we look at the $1D$ Ising model,

$$H = J\sum_i \sigma_i \sigma_{i+1} \tag{10.53}$$

Our RG transformation will be done in 'real space' by integrating out the spins on the even sites. This procedure is called *decimation*. Whereas the original model has wavevectors $-\pi/a < k < \pi/a$, the resulting theory has $-\pi/2a < k < \pi/2a$. We can then rescale momenta by 2 to obtain an RG tranformation.

$$\begin{aligned}
\mathcal{Z} &= \sum_{\sigma_i=\pm 1} \prod_i e^{\frac{J}{T}\sigma_i\sigma_{i+1}} \\
&= \sum_{\sigma_{2+1}=\pm 1} \prod_i \left(2\cosh \frac{J}{T} \left(\sigma_{2i+1} + \sigma_{2(i+1)+1} \right) \right) \\
&= \sum_{\sigma_{2+1}=\pm 1} \prod_i K e^{\left(\frac{J}{T}\right)' \sigma_{2i+1}\sigma_{2(i+1)+1}}
\end{aligned} \tag{10.54}$$

where $K = 2e^{\left(\frac{J}{T}\right)'}$ and

$$\left(\frac{J}{T}\right)' = \frac{1}{2} \ln \cosh 2\frac{J}{T} \tag{10.55}$$

This RG transformation has only 2 fixed points, $\frac{J}{T} = 0$ and $\frac{J}{T} = \infty$. $\frac{J}{T}$ is relevant at the $\frac{J}{T} = \infty$ fixed point but irrelevant at the $\frac{J}{T} = 0$ fixed point. The flow diagram is shown in 10.3. This flow diagram shows that for any $T > 0$, the system is controlled by the disordered $\frac{J}{T} = 0$ fixed point. Only at $T = 0$ can the system be ordered.

10.10 Dimensional crossover in coupled Ising chains**

10.11 Real-space RG**

10.12 Perturbative RG for φ^4 Theory in $4-\epsilon$ Dimensions

Our second example is φ^4 theory.

$$S = \int \frac{d^d q}{(2\pi)^d} \frac{1}{2} q^2 |\varphi(q)|^2 + \int \frac{d^d q}{(2\pi)^d} \frac{1}{2} r \, |\varphi(q)|^2 + \frac{u}{4!} \int \frac{d^d q_1}{(2\pi)^d} \frac{d^d q_2}{(2\pi)^d} \frac{d^d q_3}{(2\pi)^d} \varphi(q_1) \, \varphi(q_2) \, \varphi(q_3) \, \varphi(-q_1 - q_2 - q_3) \tag{10.56}$$

We take the first term as $\mathcal{O}_{\text{free}}$. Under a rescaling $q \to qb$, we must take

$$\varphi \to \varphi \, b^{-\frac{d+2}{2}} \tag{10.57}$$

Using this rescaling, we immediately see that the leading terms in the RG equations are:

$$\begin{aligned} \frac{dr}{d\ell} &= 2r + \dots \\ \frac{du}{d\ell} &= (4-d)\, u + \dots \end{aligned} \tag{10.58}$$

We immediately find one fixed point, the Gaussian fixed point: $r = u = 0$. At this fixed point, r is always relevant while u is irrelevant for $d > 4$ and relevant for $d < 4$. You will recognize that this is the same as the Ginzburg criterion which determines when the saddle-point approximation is valid for this theory: the saddle-point approximation is valid when the quartic interaction is irrelevant. When the quartic interaction is irrelevant, the correct theory of the critical point is simply.

$$S = \int \frac{d^d q}{(2\pi)^d} \, q^2 |\varphi(q)|^2 \tag{10.59}$$

which has critical correlation functions

$$\langle \varphi(\vec{x}) \, \varphi(0) \rangle \sim \frac{1}{|x|^{d-2}} \tag{10.60}$$

The one relevant direction at the Gaussian fixed point in $d > 4$ is the temperature, r. At the Gaussian fixed point, r has scaling dimension 2.

Hence, $\xi \sim 1/\sqrt{r}$. As we discussed in the context of the Ising model, $r \sim T - T_c$. Hence,

$$\xi \sim \frac{1}{|T - T_c|^{1/2}} \tag{10.61}$$

Of course, we should also allow φ^6, φ^8, etc. terms. If we don't include them initially in our action, they will be generated by the RG transformation. However, the φ^6 operator is only relevant below 3 dimensions, the φ^8 operator is only relevant below 8/3 dimensions, etc. Hence, for $d > 3$, we can ignore the higher order terms in the asymptotic $q \to 0$ limit because they are irrelevant. (Actually, we have only shown that they are irrelevant at the Gaussian fixed point; in fact, they are also irrelevant at the fixed point which we find below.)

For $d < 4$, the Gaussian fixed point has two unstable directions, r and u. We can compute the RG equations to the one-loop level to find other fixed points. We make the division of φ into φ_L and φ_H and integrate out φ_H at the one-loop level. At a schematic level, this works as follows:

$$e^{-S^b_\Lambda[\varphi_L]} = e^{-S_L[\varphi_L]} \int \mathcal{D}\varphi_H \, e^{-S^0_H[\varphi_H]} \, e^{-S_{\text{int}}[\varphi_L, \varphi_H]} \tag{10.62}$$

where

$$\begin{aligned} S_L[\varphi_L] = &\int_0^{b\Lambda} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2} \, q^2 |\varphi_L(q)|^2 + \int_0^{b\Lambda} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2} \, r \, |\varphi_L(q)|^2 \\ &+ \frac{u}{4!} \int_0^{b\Lambda} \frac{d^d q_1}{(2\pi)^d} \, \frac{d^d q_2}{(2\pi)^d} \, \frac{d^d q_3}{(2\pi)^d} \, \varphi_L(q_1) \, \varphi_L(q_2) \, \varphi_L(q_3) \, \varphi_L(q_4) \end{aligned} \tag{10.63}$$

$$S^0_H[\varphi_H] = \int_{b\Lambda}^{\Lambda} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2} \, q^2 |\varphi_H(q)|^2 + \int_{b\Lambda}^{\Lambda} \frac{d^d q}{(2\pi)^d} \, \frac{1}{2} \, r \, |\varphi_H(q)|^2 \tag{10.64}$$

$$\begin{aligned} S_{\text{int}}[\varphi_L, \varphi_H] = &\frac{u}{4!} \int \frac{d^d q_1}{(2\pi)^d} \, \frac{d^d q_2}{(2\pi)^d} \, \frac{d^d q_3}{(2\pi)^d} \, \varphi_H(q_1) \, \varphi_H(q_2) \, \varphi_H(q_3) \, \varphi_H(q_4) \\ &+ \frac{u}{4} \int \frac{d^d q_1}{(2\pi)^d} \, \frac{d^d q_2}{(2\pi)^d} \, \frac{d^d q_3}{(2\pi)^d} \, \varphi_H(q_1) \, \varphi_H(q_2) \, \varphi_L(q_3) \, \varphi_L(q_4) \end{aligned} \tag{10.65}$$

S_{int} also contains $\varphi_H \varphi_H \varphi_H \varphi_L$ and $\varphi_L \varphi_L \varphi_L \varphi_H$ terms, but the phase space for these terms is very small since it is difficult for three large momenta to add up to a small momentum or the reverse. Hence, we can safely ignore these terms. Expanding perturbatively, there is a contribution of the form:

$$e^{-S^b_\Lambda[\varphi_L]} = e^{-S_L[\varphi_L]} \int \mathcal{D}\varphi_H \, e^{-S^0_H[\varphi_H]} \, (1 -$$

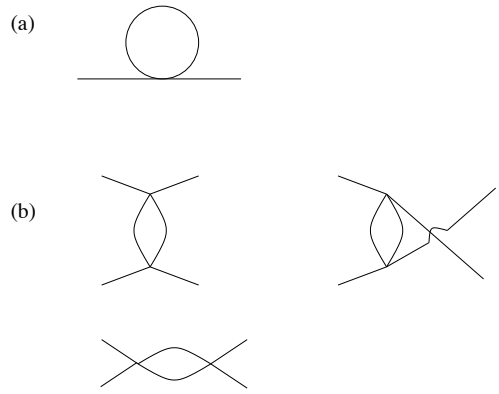


Figure 10.4: The one-loop diagrams which determine the RG equations for (a) r and (b) u.

$$\begin{aligned}
&\frac{u}{4}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\,\varphi_H(q_1)\,\varphi_H(q_2)\,\varphi_L(q_3)\,\varphi_L(q_4)+\ldots)\\
&= e^{-S_L[\varphi_L]}\left(1-\frac{u}{4}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\,\langle\varphi_H(q_1)\,\varphi_H(q_2)\rangle\,\varphi_L(q_3)\,\varphi_L(q_4)+\ldots\right)\\
&= e^{-\left(S_L[\varphi_L]+\frac{u}{4}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\,\langle\varphi_H(q_1)\,\varphi_H(q_2)\rangle\,\varphi_L(q_3)\,\varphi_L(q_4)\right)}+O(u^2)
\end{aligned}\tag{10.66}$$

We can do this diagrammatically by computing one-loop diagrams with internal momenta restricted to the range $b\Lambda < |q| < \Lambda$. The external legs must be φ_L fields, i.e. must have momenta $q < b\Lambda$. (Note that the contribution to $S_{b\Lambda}$ is the negative of the value of the diagram since we are absorbing it into $e^{-S^b_\Lambda[\varphi_L]}$.) The contribution described above results from the first-order diagram with two external legs. Such diagrams give a contribution to $S_{b\Lambda}$ of the form:

$$\int \frac{d^dq}{(2\pi)^d}\,c(q)\,|\varphi_L(q)|^2 \tag{10.67}$$

where $c(q) = c_0 + c_2q^2 + \ldots$. Diagrams with four external legs give a contribution to $S_{b\Lambda}$ of the form:

$$\frac{1}{4!}\int \frac{d^dq_1}{(2\pi)^d}\frac{d^dq_2}{(2\pi)^d}\frac{d^dq_3}{(2\pi)^d}\,v(q_1,q_2,q_3)\,\varphi(q_1)\,\varphi(q_2)\,\varphi(q_3)\,\varphi(-q_1-q_2-q_3) \tag{10.68}$$

The one-loop contribution to the RG equations is given by the diagrams of figure 10.4. The diagram of 10.4(a) gives the contribution of (10.66),

namely

$$\delta S_{b\Lambda} = -\frac{1}{2}\,u \int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^2+r} \int \frac{d^dp}{(2\pi)^d}\,\frac{1}{2}|\varphi(p)|^2 \tag{10.69}$$

Dropping higher-order terms in r (because we are interested in the vicinity of the Gaussian fixed point, $r = u = 0$), we can rewrite the integral as

$$\begin{aligned}
\int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^2+r} &= \int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^2} - r\int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^2(q^2+r)} \\
&= \int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^2} - r\int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{q^4} + O(r^2) \\
&= \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{d-2}\left(1-b^{d-2}\right) \\
&\quad -\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,r\frac{1}{d-4}\left(1-b^{d-4}\right)
\end{aligned} \tag{10.70}$$

Meanwhile, the three diagrams of figure 10.4(b) each give a contribution

$$-\frac{1}{2}\,u^2 \int_b^1 \frac{d^dq}{(2\pi)^d}\,\frac{1}{(q^2+r)^2} \int \frac{d^dq_1}{(2\pi)^d}\,\frac{d^dq_2}{(2\pi)^d}\,\frac{d^dq_3}{(2\pi)^d}\,\frac{1}{4!}\,\varphi(q_1)\;\varphi(q_2)\;\varphi(q_3)\;\varphi(-q_1-q_2-q_3) \tag{10.71}$$

or, adding them together and evaluating the integral in the $r = 0$ limit,

$$-\frac{3}{2}\,u^2\,\frac{1}{(2\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{d-4}\left(1-b^{d-4}\right) \int \frac{d^dq_1}{(2\pi)^d}\,\frac{d^dq_2}{(2\pi)^d}\,\frac{d^dq_3}{(2\pi)^d}\,\frac{1}{4!}\,\varphi(q_1)\;\varphi(q_2)\;\varphi(q_3)\;\varphi(q_4) \tag{10.72}$$

Observe that there is no one-loop contribution to the

$$\int \frac{d^dq}{(2\pi)^d}\,q^2|\varphi(q)|^2 \tag{10.73}$$

term. Hence the correct rescaling is still

$$\varphi \to \varphi\, b^{-\frac{d+2}{2}} \tag{10.74}$$

As a result, we find the one-loop RG equations:

$$\begin{aligned}
r + dr = b^{-2}\Big(r &+ \frac{1}{2}\,u\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{d-2}\left(1-b^{d-2}\right) \\
&-\frac{1}{2}\,ur\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{d-4}\left(1-b^{d-4}\right)\Big)
\end{aligned}$$

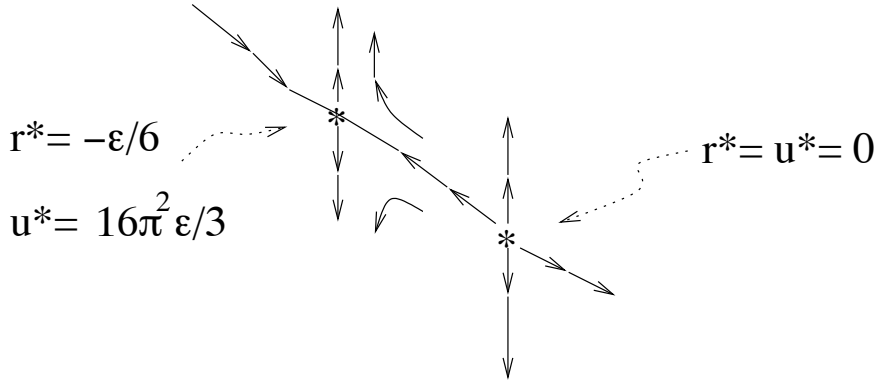


Figure 10.5: The flow diagram of a φ^4 theory in $4-\epsilon$ dimensions.

$$u + du = b^{d-4}(u - \frac{3}{2}\,u^2\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,\frac{1}{d-4}\left(1-b^{d-4}\right)) \qquad (10.75)$$

Writing $b = e^{-d\ell}$, and taking the limit of small $\epsilon = 4-d$ we have:

$$\begin{aligned}\frac{dr}{d\ell} &= 2r + \frac{1}{16\pi^2}\,u - \frac{1}{16\pi^2}\,ur + \ldots \\ \frac{du}{d\ell} &= \epsilon\,u - \frac{3}{16\pi^2}\,u^2 + \ldots\end{aligned} \qquad (10.76)$$

The corresponding flow diagram is shown in figure 10.5. These RG equations have a fixed point at $O(\epsilon)$:

$$\begin{aligned}r^* &= -\frac{1}{6}\,\epsilon \\ u^* &= \frac{16\pi^2}{3}\,\epsilon\end{aligned} \qquad (10.77)$$

At this fixed point, the eigenoperators are:

$$\begin{aligned}\frac{d}{d\ell}\,(r-r^*) &= \left(2 - \frac{1}{3}\epsilon\right)\,(r-r^*) \\ \frac{d}{d\ell}\,(u-u^*) &= -\,\epsilon\,(u-u^*)\end{aligned} \qquad (10.78)$$

There is only one relevant direction (corresponding to the temperature) with scaling dimension $1/\nu = 2 - \frac{1}{3}\epsilon$. The correlation length scales as:

$$\xi \sim \frac{1}{|T-T_c|^\nu} \qquad (10.79)$$

At the critical point, the correlation function has the power law decay

$$\langle \varphi(\vec{x})\, \varphi(0) \rangle \sim \frac{1}{x^{d-2+\eta}} \tag{10.80}$$

At order ϵ, $\eta = 0$, as we have seen. However, this is an artifact of the $O(\epsilon)$ calculation. At the next order, we find a non-vanishing contribution: $\eta = \epsilon^2/54$.

For ϵ small, our neglect of higher-loop contributions is justified. To compute in $d = 3$, however, we must go to higher loops or, equivalently, higher-order in ϵ.

Several remarks are in order:

- As we showed in chapter 8, the Ising model can be mapped onto a φ^4 theory if higher powers of φ are neglected. We can now justify their neglect: these terms are irrelevant.

- There are many different ways of implementing the cutoff, i.e. *regularizing* a theory: by putting the theory on a lattice (as in the $1D$ Ising model above), by introducing a hard cutoff (as in φ^4 theory above), or by introducing a soft cutoff – e.g. by multiplying all momentum integrals by e^{-q^2/Λ^2} – just to name a few.

- Corresponding to these different regularization schemes, there are different renormalization group transformations, such as real-space decimation on the lattice or momentum shell-integration for a momentum-space cutoff. Since all of these different cutoff theories will flow, ultimately, into the same fixed point under the different RG transformations, it is a matter of convenience which scheme we choose.

- Both RG equations and the fixed point values of the couplings are *scheme dependent*. The universal properties, such as exponents, are *scheme independent*. e.g. in the RG equation

$$\frac{dg}{d\ell} = \lambda_g\, (g - g^*) + \dots \tag{10.81}$$

  λ_g is *scheme independent* and independent of all microscopic details, but g^* is scheme-dependent, and can depend on microscopic details such as the cutoff.

- Integrals which are logarithmically divergent at large q are proportional to $\ln b$ and are independent of the cutoff Λ. Consequently they are scheme independent. Integrals which are more strongly ultra-violet divergent are scheme-dependent.

- The term in the RG equation for r which is independent of r determines only the fixed point value of r, i.e. r^*. It does not affect the scaling exponents. In fact, it is renormalization scheme dependent; in some schemes, it vanishes, so it can be dropped and we can work with:

$$\begin{aligned} \frac{dr}{d\ell} &= 2r - \frac{1}{16\pi^2}\, ur + \dots \\ \frac{du}{d\ell} &= \epsilon\, u - \frac{3}{16\pi^2}\, u^2 + \dots \end{aligned} \tag{10.82}$$

10.13 The $O(3)$ NLσM

In order to study the phase diagram of a quantum system we would like to consider both $T = 0$ and finite T in the same phase diagram. In particular, we would like to consider even the high temperatures at which classical phase transitions can occur. Finite-size scaling – which is useful for asymptotically low temperatures – is not appropriate for such an analysis. Instead, we carry out the renormalization group transformation directly on the finite-temperature quantum-mechanical functional integral. We will show how this is done for an antiferromagnet:

$$\int \mathcal{D}\vec{n}\, e^{-\frac{1}{g}\int d^d\vec{x}\int_0^\beta d\tau\, (\partial_\mu \vec{n})^2} \tag{10.83}$$

The requirement of $O(3)$ symmetry together with the constraint $\vec{n}^2 = 1$ implies that we can add to this action irrelevant terms such as

$$\int d^d\vec{x}\, d\tau \int_0^\beta d\tau\, (\partial_\mu \vec{n} \cdot \partial_\mu \vec{n})^2 \tag{10.84}$$

but no relevant terms for $d \geq 1$.

As usual, we rewrite the action as:

$$S = \int d^d\vec{x} \int_0^\beta d\tau\, \left((\partial_\mu n_i)^2 + g\, \frac{n_i \partial_\mu n_i\; n_j \partial_\mu n_j}{1 - g n_i n_i} \right) \tag{10.85}$$

We now define an RG transformation in which we integrate out $n_i(\vec{q}, \omega_n)$ with wavevectors $e^{-\ell}\Lambda < |q| < \Lambda$ but *arbitrary* Matsubara frequency ω_n. This is different from the RG which we defined earlier, but it is still perfectly well-defined. In the evaluation of diagrams, the internal momenta are restricted to the shell $e^{-\ell}\Lambda q < \Lambda$, but the Matsubara frequencies can run

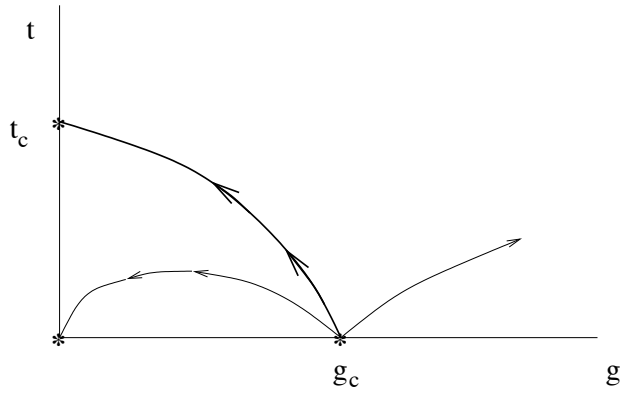


Figure 10.6: The flow diagram of an antiferromagnet in $d > 2$.

from $n = -\infty$ to $n = \infty$. However, in the rescaling step, we rescale both momenta and frequencies:

$$\begin{aligned} \vec{q} &\to \vec{q} e^{-\ell} \\ \omega_n &\to \omega_n e^{-\ell} \end{aligned} \tag{10.86}$$

The second equation means that the temperature is rescaled:

$$\beta \to \beta e^{\ell} \tag{10.87}$$

n_i must also be rescaled:

$$n_i \to n_i e^{-\ell\zeta} \tag{10.88}$$

In the problem set, you will compute the one-loop RG equation for g. It is:

$$\frac{dg}{d\ell} = (1-d)\, g \,+\, \frac{1}{2}\left(\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)}\right) g^2 \,\coth\frac{\beta}{2} \tag{10.89}$$

β changes trivially since it is only affected by the rescaling.

$$\frac{d\beta}{d\ell} = -\beta \tag{10.90}$$

Hence, if we define the parameter $t = g/\beta$, we can write a scaling equation for t:

$$\frac{d}{d\ell}\left(\frac{g}{\beta}\right) = \frac{1}{\beta}\frac{dg}{d\ell} + g\,\frac{d}{d\ell}\left(\frac{1}{\beta}\right)$$

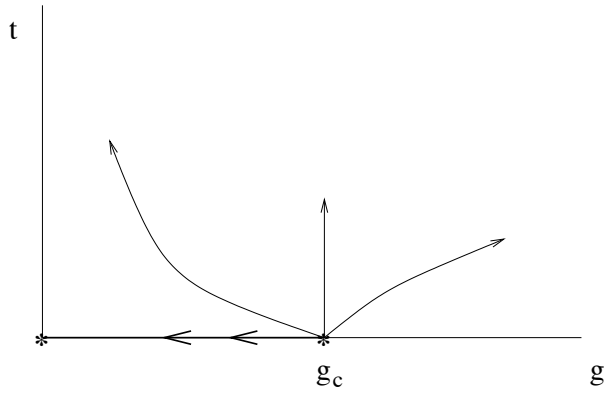


Figure 10.7: The flow diagram of an antiferromagnet in $d = 2$.

$$= \frac{1}{\beta}\left((1-d)\,g \;+\; \frac{1}{2}\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,g^2\,\coth\frac{\beta}{2}\right) + \frac{g}{\beta}$$
$$= (2-d)\,\frac{g}{\beta} \;+\; \frac{1}{2}\,\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\,g\,\frac{g}{\beta}\,\coth\frac{\beta}{2} \tag{10.91}$$

In other words, we can rewrite the RG equation for g and the trivial rescaling for β as the two equations:

$$\frac{dg}{d\ell} = (1-d)\,g \;+\; \frac{1}{2}\left(\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\right)\,g^2\,\coth\frac{g}{2t}$$
$$\frac{dt}{d\ell} = (2-d)\,t \;+\; \frac{1}{2}\left(\frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d\Gamma\left(\frac{d}{2}\right)}\right)\,g\,t\,\coth\frac{g}{2t} \tag{10.92}$$

At zero temperature, $t = 0$, the first equation shows that there is a stable fixed point at $g^* = 0$ for $d > 1$. This is the antiferromagnetically ordered phase. For t, g small, the system flows into the $g^* = t^* = 0$ fixed point. We will discuss the basin of attraction of this fixed point below.

There is an unstable fixed point at

$$g_c = \frac{(d-1)(2\pi)^d\Gamma\left(\frac{d}{2}\right)}{\pi^{\frac{d}{2}}} \tag{10.93}$$

For $g > g_c$, g flows to $g = \infty$. At this fixed point, the antiferromagnet is disordered by quantum fluctuations. Such fixed points are called *quantum*

critical points. g can be varied by introducing a next-neighbor coupling J' which frustrates the nearest-neighbor coupling J. Increasing J' increases g.

At finite temperature, $t > 0$, there is a fixed point for $d > 2$ at,

$$\begin{aligned} g^* &= 0 \\ t^* &= t_c = \frac{(d-2)(2\pi)^d \Gamma\left(\frac{d}{2}\right)}{2\,\pi^{\frac{d}{2}}} \end{aligned} \tag{10.94}$$

For $d \leq 2$, there is no fixed point at finite temperature; all flows go to $t = \infty$. The flow diagrams are shown in figures 10.6 and 10.7.

The region underneath the dark line in figure 10.6 is the antiferromagnetically ordered phase controlled by the $g^* = 0$, $t^* = 0$ fixed point. For given g – i.e. for a given system – there is a range of t for which the system is antiferromagnetically ordered. This range of t translates into a range of temperatures, $0 < T < T_c$. For $g \to 0$, $T_c \to \infty$.

At both the zero and finite-temperature fixed points, the correlation functions exhibit power-law decay. As these fixed points are approached, the correlation length diverges. In the zero-temperature case, the divergence is:

$$\xi \sim |g - g_c|^{-\nu_{d+1}} \tag{10.95}$$

while, at finite temperture, it is:

$$\xi \sim |t - t_c|^{-\nu_d} \tag{10.96}$$

In the problem set, you will calculate ν_d.

At the finite-temperature critical point, the correlation functions have power-law decay:

$$\langle n_i(\vec{x})\, n_j(0)\rangle \sim \frac{1}{x^{d-2+\eta_d}}\, \delta_{ij} \tag{10.97}$$

while at the zero-temperature critical point, they decay as:

$$\langle n_i(\vec{x},\tau)\, n_j(0)\rangle \sim \frac{1}{(x^2+\tau^2)^{(d-1+\eta_{d+1})/2}}\, \delta_{ij} \tag{10.98}$$

In the problem set, you will calculate η_d.

To summarize, in $d > 2$, an antiferromagnet described by the $O(3)$ NLσM exhibits the following physics:

- An antiferromagnetic phase controlled by the $g^* = t^* = 0$ fixed point. This phase is characterized by an $O(3)$ symmetry which is spontaneously broken to $U(1)$,

  $$\langle \vec{n} \rangle \neq 0 \tag{10.99}$$

  yielding two Goldstone modes.

- A zero-temperature *quantum critical point* $g^* = g_c$, $t^* = 0$ characterized by power-law correlation functions:

$$\langle n_i(\vec{x},\tau)\, n_j(0)\rangle \sim \frac{1}{\left(x^2+\tau^2\right)^{(d-1+\eta_{d+1})/2}}\,\delta_{ij} \tag{10.100}$$

- A zero-temperature paramagnetic phase controlled by a fixed point at $g^* = \infty$, $t^* = 0$ and characterized by exponentially-decaying correlation functions:

$$\langle n_i(\vec{x},\tau)\, n_j(0)\rangle \sim \frac{e^{-|x|/\xi}}{\left(x^2+\tau^2\right)^{(d-1)/2}}\,\delta_{ij} \tag{10.101}$$

  As g_c is approached at $t = 0$, the correlation length diverges as:

$$\xi \sim |g - g_c|^{-\nu_{d+1}} \tag{10.102}$$

- A finite-temperature critical point at $t^* = t_c$, $g^* = 0$ characterized by power-law correlation functions:

$$\langle n_i(\vec{x})\, n_j(0)\rangle \sim \frac{1}{x^{d-2+\eta_d}}\,\delta_{ij} \tag{10.103}$$

  Near 4 dimensions, this critical point can can be studied with an $O(3)$ $(\varphi_a\varphi_a)^2$ theory using a an $\epsilon = 4 - d$ expansion.

- A finite-temperature paramagnetic phase controlled by a fixed point at $t^* = \infty$, $g^* = 0$ and characterized by exponentially-decaying correlation functions:

$$\langle n_i(\vec{x})\, n_j(0)\rangle \sim \frac{e^{-|x|/\xi}}{x^{d-2}}\,\delta_{ij} \tag{10.104}$$

  As $t \to t_c$, the correlation length diverges as;

$$\xi \sim |t - t_c|^{-\nu_d} \tag{10.105}$$

In $d = 2$, we have an antiferromagnetic phase only at $T = 0$ for $0 < g < g_c$. The system is paramagnetic in the rest of the phase diagram.

The calculations of this section and the problem set are all to lowest order in $d - 1$ at zero-temperature and $d - 2$ at finite temperature. In the next section, we will generalize the $O(3)$ NLσM to the $O(N)$ NLσM and derive the RG equations to lowest order in $1/N$.

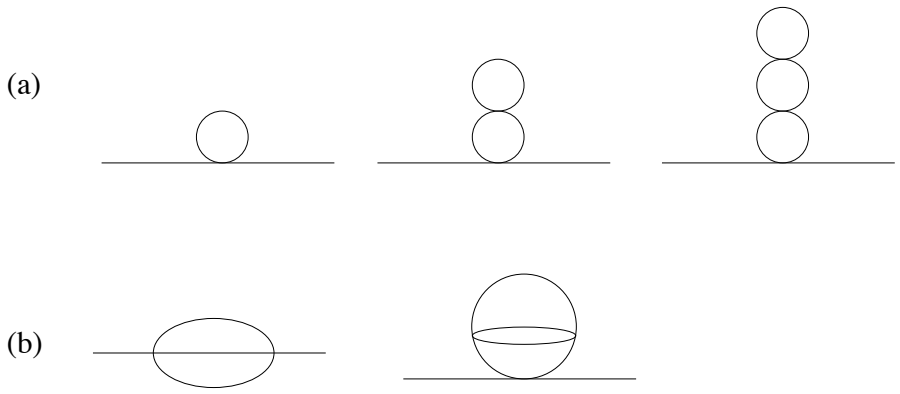


Figure 10.8: (a) $O(1)$ diagrams and (b) $O(1/N)$ diagrams with two external legs.

10.14 Large N

Suppose we generalize φ^4 theory to

$$S = \int d^d x \left(\frac{1}{2} \nabla\varphi_a \nabla\varphi_a + \frac{1}{2} r\, \varphi_a \varphi_a + \frac{1}{N} \frac{u}{8} (\varphi_a \varphi_a)^2 \right) \qquad (10.106)$$

where $a = 1, 2, \ldots, N$. For $N = 1$, this theory has the Z_2 symmetry of the Ising model. For $N = 2$, the theory has the $O(2) = U(1)$ symmetry of ^{4}He or an XY magnet. For arbitrary N, the action has $O(N)$ symmetry. The RG equations simplify for $N \to \infty$ as we now show.

Let's classify diagrams according to powers of N. Each vertex gets a factor of $1/N$. Every time we sum over an index a, we get a factor of N. First, let's consider the diagrams with two external legs. These diagrams renormalize r (and, possibly, ζ). Figure 10.8a contains some $O(1)$ two-leg diagrams.

Let's now turn to the diagrams with 4 external legs. Figure 10.9a contains some $O(1/N)$ diagrams with 4 external legs. Other diagrams, such as that of figure 10.9b are down by powers of $1/N$.

To organize the diagrams in powers of $1/N$, it is useful to perform a *Hubbard-Stratonovich transformation*. We introduce a field σ and modify the action by adding a term:

$$S \to S - \frac{N}{2u} \left(\sigma - \frac{u}{2N} \varphi_a \varphi_a \right)^2 \qquad (10.107)$$

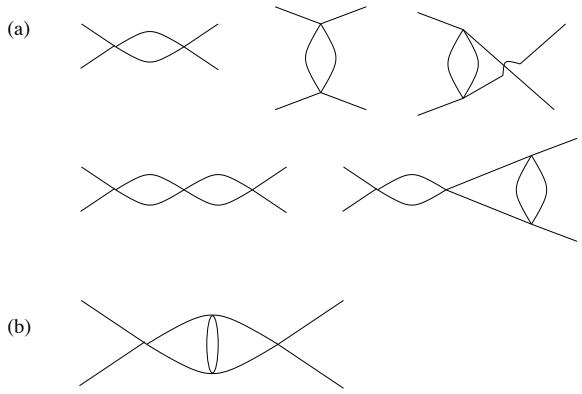


Figure 10.9: (a) Some $O(1/N)$ diagrams and (b) an $O(1/N^2)$ diagram with four external legs.

Since the action is quadratic in σ, we could integrate out σ without affecting the functional integral for φ_a. However, it is also possible to expand the square, which leads to the action:

$$S = \int d^d x \left(\frac{1}{2} \nabla\varphi_a \nabla\varphi_a + \frac{1}{2} r \, \varphi_a \varphi_a - \frac{N}{2u} \sigma^2 + \frac{1}{2} \sigma \varphi_a \varphi_a \right) \quad (10.108)$$

Notice that integrating out σ restores the $(\varphi_a \varphi_a)^2$ term.

This is now a quadratic action for φ_a. Hence, we can integrate out φ_a:

$$S_{\text{eff}}[\sigma] = \int \frac{d^d x}{(2\pi)^d} \left(-\frac{N}{2u} \sigma^2 + NTr \ln \left(\nabla^2 + r + \sigma \right) \right) \quad (10.109)$$

Dropping a constant, the logarithm can be expanded to give:

$$\begin{aligned} S_{\text{eff}}[\sigma] = &- \int \frac{d^d p}{2\pi} \frac{N}{2u} \sigma(p) \, \sigma(-p) \\ &+ N \sum_n \frac{(-1)^n}{n} \int \left(\prod_{i=1}^{n} \frac{d^d p_i}{2\pi} \, \frac{d^d k_i}{2\pi} \frac{1}{k_i^2 + r} \right) \sigma(p_1) \ldots \sigma(p_n) \prod \delta \left(p_i + k_i - k_{i+1} \right) \end{aligned} \quad (10.110)$$

Since there is a factor of N in front of $S_{\text{eff}}[\sigma]$, each σ propagator carries a $1/N$, while each vertex carries an N. Hence, a diagram goes as $N^{V-I-E} = N^{-E+1-L}$. The lowest order in $1/N$ for the E-point σ correlation function is a tree-level in $S_{\text{eff}}[\sigma]$ diagram. To compute the φ_a k-point correlation

function, we need to compute a diagram with $k/2$ external σ legs. Hence, the lowest order in $1/N$ contribution to the φ_a two-point correlation function is obtained from the σ one-point correlation function (which is determined by the diagram obtained by joining the one-point function to one end of the two-point function). It is $O(1)$ and it is given by the graph of figure **??**a. The lowest order in $1/N$ contribution to the φ_a four-point correlation function is obtained from the σ two-point correlation function. It is $O(1/N)$ and it is given by the graphs of figure **??**b.

Since the σ one-point function is:

$$N \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r} \tag{10.111}$$

while the σ two-point function at zero momentum is

$$\left[\frac{N}{2u} - N \int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + r)^2} \right]^{-1} \tag{10.112}$$

we have:

$$\begin{aligned} r + dr &= b^{-2} \left(r + \frac{N \int_b^1 \frac{d^d q}{(2\pi)^d} \frac{1}{q^2 + r}}{\frac{N}{2u} - N \int_b^1 \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + r)^2}} \right) \\ u + du &= b^{d-4} \left(\frac{N}{2u} - 3N \int_b^1 \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + r)^2} \right)^{-1} \end{aligned} \tag{10.113}$$

Differentiating these equations, we obtain the RG equations:

$$\begin{aligned} \frac{dr}{d\ell} &= 2r - \frac{1}{16\pi^2} ur + \dots \\ \frac{du}{d\ell} &= \epsilon\, u - \frac{3}{16\pi^2} u^2 + \dots \end{aligned} \tag{10.114}$$

In other words, the one-loop RG equations contain the same information as the geometric series of $O(1)$ and $O(1/N)$ diagrams! In the $N \to \infty$ limit, the one-loop RG equations are valid even when ϵ is not small.

We can also consider the $O(N)$ generalization of the NLσM:

$$\int \mathcal{D}\vec{n}\, e^{-\frac{1}{g} \int d^d \vec{x} \int_0^\beta d\tau\, (\partial_\mu \vec{n})^2} \tag{10.115}$$

where $\vec{n}$ is an N component vector. Imposing the constraint $\vec{n}^2 = 1$ with a Lagrange multiplier, we have:

$$S = \frac{1}{2g} \int d^d \vec{x} \int_0^\beta d\tau \, \Big((\nabla \vec{n})^2 + \lambda \left(n^2 - 1 \right) \Big) \tag{10.116}$$

Integrating out $\vec{n}$, we have:

$$S = \frac{1}{2g} \int d^d\vec{x} \int_0^\beta d\tau \left(\frac{1}{2g}\lambda \left(n^2 - 1\right) + \frac{1}{2} N \, Tr \ln \left(-\nabla^2 + \lambda(\vec{x})\right)\right) \quad (10.117)$$

In the $N \to \infty$ limit, the saddle-point approximation becomes exact, so:

$$N \sum_n \int \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega_n^2 + q^2 + \lambda} = \frac{1}{g} \quad (10.118)$$

Let's specialize to the case $T = 0$:

$$N \int^\Lambda \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda} = \frac{1}{g} \quad (10.119)$$

This integral equation can be solved using the RG transformation. First, we integrate out momenta $b\Lambda < |q| < \Lambda$, assuming that $\lambda << \Lambda^2$:

$$N \int^b \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda} + N \int_b^1 \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda} = \frac{1}{g} \quad (10.120)$$

or,

$$N \int^b \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda} + N \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1} \left(1 - b^{d-1}\right) = \frac{1}{g} \quad (10.121)$$

If we bring the second term on the left-hand-side to the right-hand-side, we have:

$$N \int^b \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda} = \frac{1}{g} - N \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1} \left(1 - b^{d-1}\right) \quad (10.122)$$

Rescaling the momenta in the integral, $q \to qb$, $\omega \to \omega b$ we have:

$$N \int^1 \frac{d\omega}{2\pi} \frac{d^d\vec{q}}{(2\pi)^d} \frac{1}{\omega^2 + q^2 + \lambda b^2} = b^{d-1} \left(\frac{1}{g} - N \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1} \left(1 - b^{d-1}\right) \right) \quad (10.123)$$

In other words,

$$\frac{1}{g} + d\left(\frac{1}{g}\right) = b^{d-1} \left(\frac{1}{g} - N \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \frac{1}{d-1} \left(1 - b^{d-1}\right) \right) \quad (10.124)$$

writing $b = e^{-d\ell}$, this gives:

$$\frac{dg}{d\ell} = (d-1)\, g - N\, \frac{2\,\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)}\, g^2 \tag{10.125}$$

Again, the large-N RG equation is essentially a one-loop RG equation.

As we will see again in the context of interacting fermions, the large-N limit is one in which RG equations can be calculated with minimum fuss.

10.15 The Kosterlitz-Thouless Transition

We turn now to the RG analysis of an XY magnet or, equivalently, ^{4}He at zero-temperature in 1D

$$S = \frac{1}{2}\, \rho_s \int d\tau\, dx\, (\partial_\mu \theta)^2 \tag{10.126}$$

or at finite temperature in 2D,

$$S = \frac{\rho_s}{2T} \int d^2x\, (\nabla \theta)^2 \tag{10.127}$$

We will use the notation

$$S = \frac{1}{2}\, K \int d^2x\, (\partial_\mu \theta)^2 \tag{10.128}$$

to encompass both cases. This is an $O(2)$ non-linear sigma model with $K = 1/g$. Clearly, $dK/d\ell = 0$ to all orders in K.

The two-point correlation function of the order parameter may be calculated using:

$$\left\langle e^{i\theta(x)}\, e^{-i\theta(0)} \right\rangle = \int \mathcal{D}\theta\, e^{\int d^2x \left(\frac{1}{2} K (\partial_\mu \theta)^2 + J(x)\theta(x)\right)} \tag{10.129}$$

where

$$J(y) = i\delta(y-x) - i\delta(y) \tag{10.130}$$

Hence it is given by,

$$\begin{aligned} \left\langle e^{i\theta(x)}\, e^{-i\theta(0)} \right\rangle &= e^{\frac{1}{2}\int d^2x\, d^2x'\, J(x) G(x-x') J(x')} \\ &= e^{G(x) - G(0)} \end{aligned} \tag{10.131}$$

Now,

$$\begin{aligned} G(x) - G(0) &= \langle \theta(x)\,\theta(0)\rangle - \langle \theta(0)\,\theta(0)\rangle \\ &= \int \frac{d^2q}{(2\pi)^2}\,\left(e^{iq\cdot x} - 1\right)\,\frac{1}{K}\,\frac{1}{q^2} \\ &= -\frac{1}{K}\int_{\frac{1}{|x|}}^{\Lambda} \frac{d^2q}{(2\pi)^2}\,\frac{1}{q^2} \\ &= -\frac{1}{2\pi K}\ln|x/a| \end{aligned} \tag{10.132}$$

where $a = 1/\Lambda$ is a short-distance cutoff. Hence,

$$\left\langle e^{i\theta(x)}\,e^{-i\theta(0)}\right\rangle = \frac{1}{|x|^{\frac{1}{2\pi K}}} \tag{10.133}$$

Similarly if $\sum_i n_i = 0$,

$$\left\langle e^{i\sum_i n_i\,\theta(x_i)}\right\rangle = e^{-\frac{1}{2\pi K}\sum_{i,j} n_i n_j \ln|x_i - x_j|} \tag{10.134}$$

In other words, the correlation function has the form of the Boltzmann weight for a Coulomb gas.

Thus far, we have neglected the periodicity of θ, i.e. the fact that $0 < \theta < 2\pi$. However, for $|x|$ large,

$$e^{-\frac{1}{2}\left\langle(\theta(x)-\theta(0))^2\right\rangle} = \frac{1}{|x|^{\frac{1}{2\pi K}}} \tag{10.135}$$

tells us that $(\theta(x) - \theta(0))^2$ becomes large for $|x|$ large. This means that θ must wind around 2π, i.e. that there are vortices.

A vortex is a singular configuration of the field $\theta(x)$ such that the vector field $\partial_\mu\theta(x)$ twists around an integer number, n, times as the vortex is encircled. In the context of ^{4}He, a vortex is a swirl of current. In an XY magnet, it is a point about which the spins rotate. In other words,

$$\oint_P \partial_\mu\theta\,dx_\mu = 2\pi n \tag{10.136}$$

for any path, P, which encloses the vortex. n is the winding number of the vortex.

We can understand this qualitatively by calculating the contribution of a vortex configuration to the functional integral. If there is a vortex at the origin with winding number n, then (10.136) implies that

$$\partial_\mu\theta \sim \frac{2\pi n}{r} \tag{10.137}$$

So the action of a vortex at the origin is:

$$\begin{aligned} A_v &= \frac{K}{2}\int\left(\frac{2\pi n}{r}\right)^2 d^2r \\ &= \pi K n^2 \ln\frac{R}{a} + E_c \end{aligned} \tag{10.138}$$

where R is the size of the system, a is the size of the core of the vortex and E_c is the core energy. Meanwhile a vortex-anti-vortex pair separated by a distance r has energy

$$A_{\text{pair}} = \pi K \ln\frac{r}{a} + 2E_c \tag{10.139}$$

To calculate the contribution of a vortex to the functional integral, we must take into account the fact that the vortex can be placed anywhere in the system. Hence, the contribution to the functional integral is proportional to the area of the system:

$$\begin{aligned} Z_v &\sim \left(\frac{R}{a}\right)^2 e^{-A_v} \\ &\sim e^{(2-\pi K)\ln\frac{R}{a}} \end{aligned} \tag{10.140}$$

For $K < 2/\pi$, this is a large contribution, so vortices can proliferate. The proliferation of vortices destroys the power-law correlation functions.

Let us now study this transition more systematically. We break θ into a smooth piece, θ_s, and a piece that contains the vortices θ_V,

$$\theta_V(x) = \sum_i n_i \arctan\left(\frac{(x-x_i)_2}{(x-x_i)_1}\right) \tag{10.141}$$

where the i^{th} vortex has winding number n_i and position x_i. Using

$$\begin{aligned} \partial_\mu\theta_V(x) &= \sum_i n_i\epsilon_{\mu\nu}\frac{(x-x_i)_\mu}{(x-x_i)^2} \\ &= \sum_i n_i\epsilon_{\mu\nu}\partial_\mu \ln|x-x_i| \end{aligned} \tag{10.142}$$

we can rewrite the action as (the cross term between θ_s and θ_V vanishes upon integration by parts)

$$S = \frac{1}{2}K\int d^2x\,(\partial_\mu\theta)^2$$

$$
\begin{aligned}
&= \frac{1}{2} K \int d^2x \left(\partial_\mu \left(\theta_s + \theta_V\right)\right)^2 \\
&= \frac{1}{2} K \int d^2x \left(\partial_\mu \theta_s\right)^2 + \frac{1}{2} K \int d^2x \left(\partial_\mu \theta_V\right)^2 \\
&= \frac{1}{2} K \int d^2x \left(\partial_\mu \theta_s\right)^2 + \frac{1}{2} K \int d^2x \sum_{i,j} n_i n_j \partial_\mu \ln|x - x_i| \, \partial_\mu \ln|x - x_j| \\
&= \frac{1}{2} K \int d^2x \left(\partial_\mu \theta_s\right)^2 - 2\pi K \sum_{i,j} n_i n_j \ln|x_i - x_j| + n_v E_c
\end{aligned}
\tag{10.143}
$$

In the last line, we have restored the core energies of the n_v vortices.

Hence, the partition function is:

$$
\mathcal{Z} = \int \mathcal{D}\theta_s \, e^{-\frac{1}{2} K \int d^2x \left(\partial_\mu \theta_s\right)^2} \times \sum_{n_v=0}^{\infty} e^{-n_v E_c} \sum_{i=1}^{n_v} \sum_{n_i = 0, \pm 1, \pm 2, \ldots} \int \prod_i \frac{1}{N_1! \, N_{-1}! N_2! \, N_{-2}! \ldots} d^2x_i e^{-\pi K \sum_{i,j} n_i n_j \ln|x_i - x_j|}
\tag{10.144}
$$

where N_k is the number of vortices of strength k in a given configuration.

Observe that (10.134) implies that this can be rewritten as:

$$
\mathcal{Z}[\phi] = \int \mathcal{D}\theta_s \, \mathcal{D}\phi e^{-\int d^2x \left(\frac{1}{8\pi^2 K} \left(\partial_\mu \phi\right)^2 + \sum_m y_m \cos m\phi\right)} e^{-\frac{1}{2} K \int d^2x \left(\partial_\mu \theta_s\right)^2}
\tag{10.145}
$$

where $y_m = e^{-n_v E_c}$ is the vortex fugacity. The perturbative expansion of $\mathcal{Z}[\phi]$ function is the sum over all vortex configurations of $\mathcal{Z}[\theta]$. Expanding perturbatively in the y_i's and using (10.134), we have:

$$
\mathcal{Z}[\phi] = \sum_{n_v=0}^{\infty} e^{-n_v E_c} \sum_{i=1}^{n_v} \sum_{n_i = 0, \pm 1, \pm 2, \ldots} \int \prod_i \frac{y_1^{N_1 + N_{-1}} y_2^{N_2 + N_{-2}} \cdots}{N_1! \, N_{-1}! N_2! \, N_{-2}! \ldots} d^2x_i e^{-\pi K \sum_{i,j} n_i n_j \ln|x_i - x_j|}
\tag{10.146}
$$

Integrating out θ_s, we are left with:

$$
\mathcal{Z} = \int \mathcal{D}\phi \, e^{-\int d^2x \left(\frac{1}{8\pi^2 K} \left(\partial_\mu \phi\right)^2 + \sum_m y_m \cos m\phi\right)}
\tag{10.147}
$$

Notice that we have transformed the partition function for the vortices in the field θ into the partition function for another scalar field, ϕ. This is an example of a *duality transformation*. The action for ϕ is a *sine-Gordon model*. Let us consider the $\cos m\phi$ term in the action:

$$
S = \int d^2x \left(\frac{1}{8\pi^2 K} \left(\partial_\mu \phi\right)^2 + \sum_m y_m \cos m\phi\right)
\tag{10.148}
$$

Is this term relevant or irrelavant? At $y_m = 0$, we can determine the first term in the RG equation for y_m from its scaling dimension. This can be determined from the correlation function:

$$\langle \cos\theta(x)\, \cos\theta(0)\rangle \sim \frac{1}{|x|^{2\pi m K}} \tag{10.149}$$

which tells us that $\cos m\phi$ has dimension $\pi m K$. Hence, the RG equation for y is:

$$\frac{dy_m}{d\ell} = (2 - \pi m K) y_m + \dots \tag{10.150}$$

Consequently, $y \equiv y_1$ is the most relevant operator. As K is decreased, y_1 becomes relevant first – i.e. at $K = 2/\pi$. Let us, therefore, focus on the action with only the $m = 1$ term:

$$S = \int d^2x \left(\frac{1}{8\pi^2 K} (\partial_\mu \phi)^2 + y \cos\phi \right) \tag{10.151}$$

In order to study the flow of K resulting from the presence of y, let us expand the functional integral perturbatively in y.

$$\begin{aligned}
\mathcal{Z} &= \int \mathcal{D}\phi\, e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2 + y\cos\phi\right)} \\
&= \dots + \int \mathcal{D}\phi \left(\frac{y^2}{2} \int d^2x\, \cos\phi(x) \int d^2y\, \cos\phi(y) \right) e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2\right)} + \dots \\
&= \dots + \int \mathcal{D}\phi [\left(\frac{y^2}{2} \int_{|x-y|>1/b\Lambda} d^2x\, \cos\phi(x) \int d^2y\, \cos\phi(y) \right) + \\
&\qquad \left(y^2 \int_{1/\Lambda<|x-y|<1/b\Lambda} d^2x \int d^2y\, \frac{1}{2}\, e^{i\phi(x)} e^{-i\phi(y)} \right)] e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2\right)} + \dots \\
&= \dots + \int \mathcal{D}\phi \left(y^2 \int_{1/\Lambda<|x-y|<1/b\Lambda} d^2x \int d^2y\, \frac{1}{2}\, \frac{1}{(x-y)^{2\pi K}} e^{i\phi(x) - i\phi(y)} \right) \\
&\qquad \times e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2 + y\cos\phi\right)} + \dots \\
&= \dots + \int \mathcal{D}\phi (y^2 \int_{1/\Lambda<|x-y|<1/b\Lambda} d^2x \int d^2y\, \frac{1}{2}\, \frac{1}{(x-y)^{2\pi K}} \\
&\qquad \times \left(1 + i(x-y)\partial\phi(y) - \frac{1}{2}(x-y)^2 (\partial\phi(y))^2 \right)) \, e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2 + y\cos\phi\right)} + \dots \\
&= \dots + \int \mathcal{D}\phi (-\frac{1}{4}\, y^2 \left(\int_{1/\Lambda<|x-y|<1/b\Lambda} d^2x\, \frac{1}{(x-y)^{2\pi K - 2}} \right) \\
&\qquad \times \left(\int d^2y\, (\partial\phi(y))^2 \right)) e^{-\int d^2x \left(\frac{1}{8\pi^2 K}(\partial_\mu\phi)^2 + y\cos\phi\right)} + \dots
\end{aligned}$$

$$= \ldots + \int \mathcal{D}\phi e^{-\int d^2x \left[\left(\frac{1}{8\pi^2 K} + \frac{\pi}{2} y^2 \ln b\right)(\partial_\mu \phi)^2 + y\cos\phi\right]} \tag{10.152}$$

In the last line, we have done the integral at $\pi K = 2$ (since we are interested in the vicinity of the transition) where it is logarithmic, and re-exponentiated the result. Hence, K^{-1} flows as a result of y:

$$\frac{d}{d\ell} K^{-1} = 4\pi^3 y^2 + O(y^4) \tag{10.153}$$

Together with the flow equation for y,

$$\frac{dy}{d\ell} = (2 - \pi K) y + O(y^3) \tag{10.154}$$

these RG equations determine the physics of an XY model in 2 dimensions at finite temperature or in 1 dimension at zero temperature. These equations may be analyzed by defining $u = \pi K - 2$ and $v = 4\pi y$, in terms of which the RG equations are:

$$\begin{aligned} \frac{du}{d\ell} &= -v^2 + O(uv^2) \\ \frac{dv}{d\ell} &= -uv + O(v^3) \end{aligned} \tag{10.155}$$

Observe that $u^2 - v^2$ is an RG invariant to this order:

$$\frac{d}{d\ell}\left(u^2 - v^2\right) = 0 \tag{10.156}$$

Hence, the RG trajectories in the vicinity of $K = K_c = 2/\pi$ are hyperbolae which asymptote the lines $u = \pm v$. The resulting Kosterlitz-Thouless flow diagram is shown in figure 10.10.

These RG flows feature a line of fixed points – or a *fixed line* – $y^* = 0$, $K > K_c$. Any point below the the asymptote $u = v$ – or, equivalently, $\pi K - 2 = 4\pi y$ – flows into one of these fixed points. Correlation functions exhibit power-law falloff at these fixed points:

$$\left\langle e^{i\theta(x)}\, e^{-i\theta(0)} \right\rangle = \frac{1}{|x|^{\frac{1}{2\pi K}}} \tag{10.157}$$

The line $\pi K - 2 = 4\pi y$ which separates these power-law phases from the exponentially decaying phase is called the *Kosterlitz-Thouless separatrix*. At the critical point,

$$\left\langle e^{i\theta(x)}\, e^{-i\theta(0)} \right\rangle = \frac{1}{|x|^{\frac{1}{4}}} \tag{10.158}$$

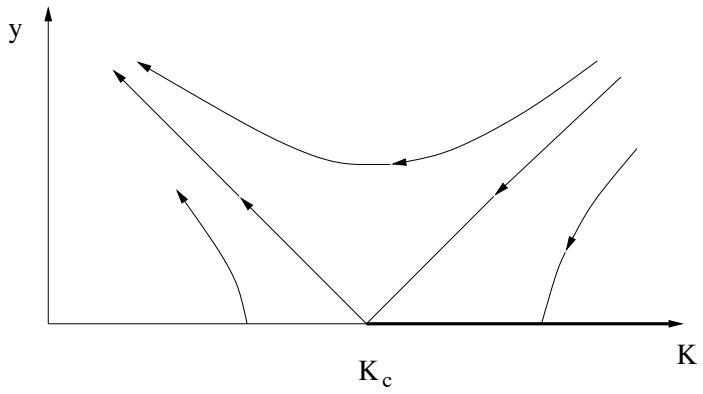


Figure 10.10: The Kosterlitz-Thouless flow diagram.

When the system is above the line $\pi K - 2 = 4\pi y$, it flows away to large y: the system is disordered by the proliferation of vortices and has exponentially decaying correlation functions. Since the $\cos\phi$ term is relevant, it bounds the fluctuations of ϕ, just as an $r\phi^2$ term would. In the problem set, you will show that as K_c is approached from below, the correlation length diverges as:

$$\xi \sim e^{\frac{1}{(K_c - K)^{1/2}}} \tag{10.159}$$

Hence, at finite temperature in $2D$ or at zero-temperature in $1D$, ^{4}He and XY magnets have a phase transition between a disordered phase and a power-law ordered 'phase'.

10.16 Inverse square models in one dimension**

10.17 Numerical renormalization group**

10.18 Hamiltonian methods**

CHAPTER 11

Fermions

11.1 Canonical Anticommutation Relations

In the remainder of this course, we will be applying the field-theoretic techniques which we have developed to systems of interacting electrons. In order to do this, we will have to make a detour into formalism so that we can handle systems of fermions.

Let us first consider a system of non-interacting spinless fermions at chemical potential μ. As in the case of ^{4}He, we must modify the Hamiltonian by $H \to H - \mu N$. The action is the same as for a system of free bosons:

$$S = \int d\tau \, d^3\vec{x} \, \psi^\dagger \left(\frac{\partial}{\partial \tau} + \frac{1}{2m} \nabla^2 - \mu \right) \psi \tag{11.1}$$

The difference is that we want the associated Fock space to be fermionic, i.e. we would like the Pauli exclusion principle to hold. This can be accomplished by imposing the *canonical anticommutation relations*.

$$\left\{ \psi\left(\vec{x}, t\right), \psi^\dagger\left(x', t\right) \right\} = \delta(\vec{x} - \vec{x}') \tag{11.2}$$

$$\left\{ \psi\left(\vec{x}, t\right), \psi\left(\vec{x}', t\right) \right\} = \left\{ \psi^\dagger\left(\vec{x}, t\right), \psi^\dagger\left(\vec{x}', t\right) \right\} = 0 \tag{11.3}$$

Performing a mode expansion

$$\psi(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \, c_{\vec{k}} \, e^{\xi_k \tau + i\vec{k}\cdot\vec{x}}$$

$$\psi^\dagger(x) = \int \frac{d^3\vec{k}}{(2\pi)^{3/2}}\, c^\dagger_{\vec{k}}\, e^{-\xi_k \tau - i\vec{k}\cdot\vec{x}} \tag{11.4}$$

where $\xi_k = \epsilon_k - \mu = k^2/2m - \mu$, we see that the creation and annihilation operators satisfy:

$$\begin{aligned} \left\{c_{\vec{k}}, c^\dagger_{\vec{k}'}\right\} &= \delta(\vec{k} - \vec{k}') \\ \left\{c_{\vec{k}}, c_{\vec{k}'}\right\} = \left\{c^\dagger_{\vec{k}}, c^\dagger_{\vec{k}'}\right\} &= 0 \end{aligned} \tag{11.5}$$

Hence, $(c^\dagger_k)^2 = c^2_k = 0$, i.e. no state can be doubly occupied.

The Green function is:

$$\begin{aligned} \mathcal{G}(\vec{x},\tau) = \theta(\tau)\, Tr\left\{e^{-\beta(H_0-\mu N)}\psi^\dagger(\vec{x},\tau)\psi(0,0)\right\} \\ -\theta(-\tau)\, Tr\left\{e^{-\beta(H_0-\mu N)}\psi(0,0)\psi^\dagger(\vec{x},\tau)\right\} \end{aligned} \tag{11.6}$$

Note the $-$sign in the definition of the Green function. It is necessary because the fermions satisfy canonical anticommutation relations. You may verify that $\mathcal{G}$ as defined above satisfies:

$$\left(\frac{\partial}{\partial\tau} + \frac{1}{2m}\nabla^2 - \mu\right)\mathcal{G}(\vec{x},\tau) = \delta(\tau)\,\delta(\vec{x}) \tag{11.7}$$

As in the bosonic case, we find a further condition which follows from the cyclic property of the trace. Since $0 < \tau, \tau' < \beta$, it follows that $-\beta < \tau - \tau' < \beta$. Now suppose that $\tau < \tau'$. Then,

$$\begin{aligned} \mathcal{G}(\tau - \tau' < 0) &= -Tr\left\{e^{-\beta(H-\mu N)}\, e^{\tau'(H-\mu N)}\psi(\vec{x}')e^{-\tau'(H-\mu N)}\, e^{\tau(H-\mu N)}\psi^\dagger(\vec{x})e^{-\tau(H-\mu N)}\right\} \\ &= -Tr\left\{e^{\tau(H-\mu N)}\psi^\dagger(\vec{x})e^{-\tau(H-\mu N)}\, e^{-\beta(H-\mu N)}\, e^{\tau'(H-\mu N)}\psi(\vec{x}')e^{-\tau'(H-\mu N)}\right\} \\ &= -Tr\{e^{-\beta(H-\mu N)}\, e^{\beta(H-\mu N)}\, e^{\tau(H-\mu N)}\psi^\dagger(\vec{x})e^{-\tau(H-\mu N)}\, e^{-\beta(H-\mu N)} \\ &\qquad e^{\tau'(H-\mu N)}\psi(\vec{x}')e^{-\tau'(H-\mu N)}\} \\ &= -\mathcal{G}(\tau - \tau' + \beta) \end{aligned} \tag{11.8}$$

The first equality follows from the cyclic property of the trace. The final equality follows from the fact that $\tau - \tau' + \beta > 0$. Hence, a fermion Green function is anti-periodic in imaginary time.

As a result of antiperiodicity in imaginary-time, we can take the Fourier transform over the interval $[0, \beta]$:

$$\mathcal{G}(i\epsilon_n) = \int_0^\beta d\tau\, e^{-i\epsilon_n\tau}\mathcal{G}(\tau) \tag{11.9}$$

where the *Matsubara frequencies* ϵ_n, are given by:

$$\epsilon_n = \frac{(2n+1)\pi}{\beta} \tag{11.10}$$

Inverting the Fourier transform, we have:

$$\mathcal{G}(\tau) = \frac{1}{\beta}\sum_n \mathcal{G}(i\epsilon_n)\, e^{i\epsilon_n \tau} \tag{11.11}$$

Using the mode expansion and the Fermi-Dirac distribution,

$$Tr\left\{e^{-\beta(H_0-\mu N)} c_k^\dagger c_k\right\} = n_F(\xi_k) = \frac{1}{e^{\beta\xi_k}+1} \tag{11.12}$$

we can compute the propagator:

$$\begin{aligned}\mathcal{G}(\vec{x},\tau) &= \theta(\tau)\, Tr\left\{e^{-\beta(H_0-\mu N)}\psi^\dagger(\vec{x},\tau)\psi(0,0)\right\} \\ &\quad -\theta(-\tau)\, Tr\left\{e^{-\beta(H_0-\mu N)}\psi(0,0)\psi^\dagger(\vec{x},\tau)\right\} \\ &= \int \frac{d^3\vec{k}}{(2\pi)^3 2\omega_k}\, e^{-i\vec{k}\cdot\vec{x}+\xi_k\tau}\left(\theta(\tau)\, n_F(\xi_k) - \theta(-\tau)\,(1-n_F(\xi_k))\right)\end{aligned} \tag{11.13}$$

We can now compute the Fourier representation of the Green function:

$$\begin{aligned}\mathcal{G}(\vec{p}, i\epsilon_n) &= \int d^3\vec{x} e^{i\vec{p}\cdot\vec{x}} \int_0^\beta d\tau e^{-i\epsilon_n\tau}\mathcal{G}(\vec{x},\tau) \\ &= -\frac{n_F(\xi_k)\left(e^{\beta(-i\epsilon_n+\xi_k)}-1\right)}{-i\epsilon_n+\xi_k} \\ &= \frac{1}{i\epsilon_n - \xi_k}\end{aligned} \tag{11.14}$$

11.2 Grassmann Integrals

Fermionic systems can also be described by functional integrals. In order to do this, we will need the concept of a *Grassmann number*. Grassmann numbers are objects ψ_i which can be multiplied together and anticommute under multiplication:

$$\psi_i\,\psi_j = -\psi_i\,\psi_j \tag{11.15}$$

and

$$\psi_i^2 = 0 \tag{11.16}$$

Grassmann numbers can be multiplied by complex numbers; multiplication by a complex number is distributive:

$$a\left(\psi_1+\psi_2\right)=a\psi_1+b\psi_2 \tag{11.17}$$

$\psi=\psi_1+i\psi_2$ and $\overline{\psi}=\psi_1-i\psi_2$ can be treated as independent Grassmann variables,

$$\psi\,\overline{\psi}=-\,\overline{\psi}\,\psi \tag{11.18}$$

Since the square of a Grassmann number vanishes, the Taylor expansion of a function of Grassmann variables has only two terms. For instance,

$$e^{\psi}=1+\psi \tag{11.19}$$

Integration is defined for Grassmann numbers as follows:

$$\begin{aligned}\int d\psi &= 0\\ \int d\psi\,\psi &= 1\end{aligned} \tag{11.20}$$

Similarly,

$$\begin{aligned}\int d\psi\,d\overline{\psi} &= 0\\ \int d\psi\,d\overline{\psi}\,\psi &= 0\\ \int d\psi\,d\overline{\psi}\,\overline{\psi} &= 0\\ \int d\psi\,d\overline{\psi}\,\overline{\psi}\,\psi &= 1\end{aligned} \tag{11.21}$$

As a result of the anticommutation, the order is important in the last line:

$$\int d\psi\,d\overline{\psi}\,\psi\,\overline{\psi}=-1 \tag{11.22}$$

Since the square of a Grassmann number vanishes, these rules are sufficient to define integration.

With these definitions, we can do Grassmann integrals of Gaussians. Suppose θ_i and $\overline{\theta}_i$ are independent Grassmann variables. Then

$$\int d\theta_1 d\overline{\theta}_1\,\ldots\,d\theta_n d\overline{\theta}_n\,e^{\sum_{i,j}\overline{\theta}_i A_{ij}\theta_j}=\int d\theta_1 d\overline{\theta}_1\,\ldots\,d\theta_n d\overline{\theta}_n\prod_{i,j}e^{\overline{\theta}_i A_{ij}\theta_j}$$

$$
\begin{aligned}
&= \int d\theta_1 d\overline{\theta}_1 \, \ldots \, d\theta_n d\overline{\theta}_n \prod_{i,j} \left(1 + \overline{\theta}_i A_{ij} \theta_j\right) \\
&= \int d\theta_1 d\overline{\theta}_1 \, \ldots \, d\theta_n d\overline{\theta}_n \sum_{\sigma} \left(A_{1\sigma(1)} A_{2\sigma(2)} \, \ldots \, A_{n\sigma(n)}\right) \\
&\qquad\qquad \times \, \overline{\theta}_1 \theta_{\sigma(1)} \, \ldots \, \overline{\theta}_n \theta_{\sigma(n)} \\
&= \sum_{\sigma} (-1)^{\sigma} \left(A_{1\sigma(1)} A_{2\sigma(2)} \, \ldots \, A_{n\sigma(n)}\right) \\
&= \det(A)
\end{aligned}
\tag{11.23}
$$

We can prove Wick's theorem for Grassmann integrals:

$$
Z(\eta_i, \overline{\eta}_i) = \int \prod_i d\theta_i d\overline{\theta}_i \, e^{\sum_{i,j} \overline{\theta}_i A_{ij} \theta_j + \sum_i \left(\overline{\eta}_i \theta_i + \overline{\theta}_i \eta_i\right)} \tag{11.24}
$$

By making the change of variables,

$$
\begin{aligned}
\theta_i &= \theta_i{}' + \left(A^{-1}\right)_{ij} \eta_j \\
\overline{\theta}_i &= \overline{\theta}_i{}' + \overline{\eta}_j \left(A^{-1}\right)_{ji}
\end{aligned}
\tag{11.25}
$$

we get

$$
\begin{aligned}
Z(\eta_i, \overline{\eta}_i) &= \int \prod_i d\theta_i' d\overline{\theta}_i{}' \, e^{\sum_{i,j} \left(\overline{\theta}_i{}' A_{ij} \theta_j' + \overline{\eta}_i \left(A^{-1}\right)_{ij} \eta_j\right)} \\
&= \det(A) \, e^{\sum_{i,j} \overline{\eta}_i \left(A^{-1}\right)_{ij} \eta_j}
\end{aligned}
\tag{11.26}
$$

Hence,

$$
\begin{aligned}
\left\langle \overline{\theta}_{i_1} \theta_{j_1} \, \ldots \, \overline{\theta}_{i_k} \theta_{j_k} \right\rangle &= \frac{\int \prod_i d\theta_i d\overline{\theta}_i \, \overline{\theta}_{i_1} \theta_{j_1} \, \ldots \, \overline{\theta}_{i_k} \theta_{j_k} \, e^{\sum_{i,j} \overline{\theta}_i A_{ij} \theta_j}}{\int \prod_i d\theta_i d\overline{\theta}_i \, e^{\sum_{i,j} \overline{\theta}_i A_{ij} \theta_j}} \\
&= \frac{1}{\det(A)} \left(\frac{\partial}{\partial \eta_{i_1}} \frac{\partial}{\partial \overline{\eta}_{j_1}} \, \ldots \, \frac{\partial}{\partial \eta_{i_k}} \frac{\partial}{\partial \overline{\eta}_{j_k}} \, Z(\eta_i, \overline{\eta}_i) \right)_{\eta_i = \overline{\eta}_i = 0} \\
&= \sum_{\sigma} (-1)^{\sigma} \left(A_{j_{\sigma(1)} i_1} A_{j_{\sigma(2)} i_2} \, \ldots \, A_{j_{\sigma(k)} i_k} \right) \\
&= \sum_{\sigma} (-1)^{\sigma} \left\{ \left\langle \overline{\theta}_{i_1} \theta_{j_{\sigma(1)}} \right\rangle \, \ldots \, \left\langle \overline{\theta}_{i_k} \theta_{j_{\sigma(k)}} \right\rangle \right\}
\end{aligned}
\tag{11.27}
$$

In other words, we sum over all possible Wick contractions, multiplying by -1 every time the contraction necessitates a reordering of the fields by an odd permutation.

Thus far, we have considered finite-dimensional Grassmann integrals. However, the generalization to functional Grassmann integrals is straightforward.

11.3 Solution of the $2D$ Ising Model by Grassmann Integration

In this section, we will present the solution of the $2D$ Ising model as an application of Grassmann integration. Our strategy will be to represent a state of the Ising model in terms of the domain walls which separate up-spins from down-spins. We will then concoct a Grassmann integral which generates all allowed domain wall configurations.

We will assume that the Ising model is on a square lattice. The Grassmann variables will live on the sites of the dual lattice. The first thing to observe is that there are 8 possible configurations of the four spins surrounding a given site on the dual lattice, up to an overall flip of the four spins. They are depicted in figure **??**. These 8 configurations will give a graphical representation of the possible non-vanishing terms in the Grassmann integral. Thus, if we weight each of these terms in the Grassmann integral with the appropriate Boltzmann weight, the Grassmann integral will be equal to the partition function of the Ising model.

Consider the following Grassmann integral.

$$Z = \int \prod_{i,j} d\eta^{h,v}_{i,j}\, d\chi^{h,v}_{i,j}\; e^{\sum_{i,j} A_{i,j}} \tag{11.28}$$

where i, j labels a site on the square lattice, and $\eta^{h,v}_{i,j}$, $\chi^{h,v}_{i,j}$ are Grassmann variables. The h, v index stands for horizontal and vertical. $A_{i,j}$ is given by

$$\begin{aligned} A_{i,j} = {} & z\,\chi^h_{i,j}\eta^h_{i+1,j} + z\,\chi^v_{i,j}\eta^v_{i,j+1} \\ & +\chi^h_{i,j}\eta^v_{i,j} + \chi^v_{i,j}\eta^h_{i,j} + \chi^v_{i,j}\chi^h_{i,j} + \eta^v_{i,j}\eta^h_{i,j} \\ & +\chi^h_{i,j}\eta^h_{i,j} + \chi^v_{i,j}\eta^v_{i,j} \end{aligned} \tag{11.29}$$

where $z = e^{\beta J}$. It is straightforward to allow anisotropic coupling constants J_h, J_v by replacing the coefficient of the first term by z_h and the second by z_v. The Grassmann integral (11.29) is quadratic, so it can be performed using the definitions in this chapter. We will do this shortly, but first we must establish the *raison d'être* of this integral, namely that it reproduces $2D$ Ising model configurations.

When we expand the exponential of (11.29) we produce terms which are strings of $\chi^{h,v}$s and $\eta^{h,v}$s. The integral of such a term is non-zero if and only if every single $\chi^{h,v}$ and $\eta^{h,v}$ appears exactly once in the term. We can introduce the following graphical representation for the terms which are generated by expanding the exponential of (11.29). The first two terms are

represented by lines of domain wall connecting sites (i, j) and $(i + 1, j)$ or $(i, j + 1)$. They come with a coefficient z, which is the Boltzmann weight for such a segment of domain wall.

The next four terms place a corner at site (i, j). The four types of corners connect, respectively, an incoming line from the West to an outgoing line to the North (fig **??**); an incoming line from the South to an outgoing line to the East (fig **??**); an incoming line from the West to an incoming line from the South (fig **??**); and an outgoing line to the North to an outgoing line to the East (fig **??**). These terms implement these corners in the following way. Consider the first such term. When it appears in a string, $\chi^h_{i,j}\eta^v_{i,j}$ cannot appear a second time in the string. Therefore, it is impossible to have a segment of domain wall originating at (i, j) and heading to $(i + 1, j)$. It is also impossible for a segment of wall coming from $(i, j-1)$ to arrive at (i, j). It is also impossible for any of the final five terms in $A_{i,j}$ to appear in the string because either $\chi^h_{i,j}$ or $\eta^v_{i,j}$ would appear twice. However, $\chi^v_{i,j}$ or $\eta^h_{i,j}$ must appear once in the string. This can only happen if the second term in $A_{i,j}$ and the first term in $A_{i-1,j}$ appear. Hence, this term places a corner at site (i, j). Note that two such corners cannot occur at the same site. However, a horizontal domain wall can cross a vertical domain wall, which accounts for the configuration in fig **??**; this occurs when the first term in $A_{i-1,j}$ and $A_{i,j}$ appear *and* the second term in $A_{i,j-1}$ and $A_{i,j}$ also appear.

The last two terms simply prevent, respectively, a horizontal or vertical domain wall from passing through (i, j). In a configuration in which there is no domain wall passing through (i, j), both will appear. If there is a vertical domain wall passing through (i, j), then the first of these terms will appear (as will the second term of $A_{i,j}$, with its concomitant z).

Thus, we see that by expanding the exponential of bilinears in Grassmann variables we can reproduce all possible domain wall configurations. The only concern is that some of them may not come with the correct sign. The reader may check that they do, essentially because every loop of domain wall contains an even number of segments and and even number of corners.

Thus, we can, indeed, compute the partition function of the 2D Ising model by evaluating the Grassmann integral above. It is equal to the square root of the determinant of the quadratic form in $\sum_{i,j} A_{i,j}$. We can almost diagonalize it by going to momentum space:

$$\eta^{h,v}_{i,j} = \frac{1}{\sqrt{N}} \sum_{k_x,k_y} \eta^{h,v}_{k_x,k_y}\, e^{i(k_x i + k_y j)} \tag{11.30}$$

with a similar equation for χ. Now the exponent in the Grassmann integral

can be rewritten:

$$\begin{aligned}\sum_{i,j} A_{i,j} = \sum_{k_x,k_y} [&z\, e^{ik_x}\, \chi^h_{-k_x,-k_y}\eta^h_{k_x,k_y} + z\, e^{ik_y}\, \chi^v_{-k_x,-k_y}\eta^v_{k_x,k_y} \\ &+ \chi^h_{-k_x,-k_y}\eta^v_{k_x,k_y} + \chi^v_{-k_x,-k_y}\eta^h_{k_x,k_y} \\ &+ \chi^v_{-k_x,-k_y}\chi^h_{k_x,k_y} + \eta^v_{-k_x,-k_y}\eta^h_{k_x,k_y} \\ &+ \chi^h_{-k_x,-k_y}\eta^h_{k_x,k_y} + \chi^v_{-k_x,-k_y}\eta^v_{k_x,k_y}] \\ = \sum_{k_x,k_y} [&z\left(e^{ik_x}+1\right)\chi^h_{-k_x,-k_y}\eta^h_{k_x,k_y} + z\left(e^{ik_y}+1\right)\chi^v_{-k_x,-k_y}\eta^v_{k_x,k_y} \\ &+ \chi^h_{-k_x,-k_y}\eta^v_{k_x,k_y} + \chi^v_{-k_x,-k_y}\eta^h_{k_x,k_y} \\ &+ \chi^v_{-k_x,-k_y}\chi^h_{k_x,k_y} + \eta^v_{-k_x,-k_y}\eta^h_{k_x,k_y} \end{aligned} \tag{11.31}$$

Up to a normalization N, the Grassmann integral is given by the:

$$\begin{aligned} Z &= \mathcal{N} \prod_{k_x,k_y} \left(\det M(k_x,k_y)\right)^{1/2} \\ &= \mathcal{N}\, e^{\frac{1}{2}V \int \frac{d^2k}{(2\pi)^2}\, \ln\det M(k_x,k_y)} \end{aligned} \tag{11.32}$$

where $M(k_x,k_y)$ is the 4×4 matrix of coefficients in (11.31) and V is the area of the system. Computing the determinant of this 4×4 matrix, we find:

$$\ln Z = \frac{1}{2}V \int \frac{d^2k}{(2\pi)^2}\, \ln\left[\cosh^2 2\beta J - \sinh 2\beta J\,(\cos k_x + \cos k_y)\right] \tag{11.33}$$

The infrared behavior of this integral, where possible singularities lurk, can be obtained by studying small $\mathbf{k}$. In this limit, the argument of the logarithm is of the form Kp^2+r, where $r=(\sinh 2\beta J-1)^2$. This is precisely what we would expect for a system of particles with propagator Kp^2+r. Such a system has correlation length $\xi\sim 1/\sqrt{r}$. There is a critical point, and its concomitant singularities and power-law correlation function occur when $r=0$, i.e. $\sinh 2\beta_c J = 1$. Expanding about this critical point, we have $\xi \sim 1/|T-T_c|$, i.e. $\nu = 1$, which is rather different from the mean-field result $\nu = 1/2$. Performing the integral above, we obtain the free energy density:

$$F = F_{\text{non-sing.}} - \frac{T}{2\pi}\left(\sinh 2\beta J - 1\right)^2 \ln\left|\sinh 2\beta J - 1\right| \tag{11.34}$$

Hence, the specific heat diverges logarithmically, $\alpha = 0^-$.

11.4 Feynman Rules for Interacting Fermions

Let us now turn to a system of fermions with a δ-function interaction. The grand canonical partition function is given by:

$$\mathcal{Z} = \mathcal{N} \int \mathcal{D}\psi \, \mathcal{D}\psi^\dagger \, e^{-S} \tag{11.35}$$

where the functional integral is over all Grassmann-valued functions which are antiperiodic in the interval $[0, \beta]$ (so that $\epsilon_n = (2n+1)\pi/\beta$).

$$S = \int_0^\beta d\tau \int d^d x \, \left(\psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi + V \psi^\dagger \psi \psi^\dagger \psi \right) \tag{11.36}$$

This action has the $U(1)$ symmetry $\psi \to e^{i\theta}\psi$, $\psi^\dagger \to e^{-i\theta}\psi^\dagger$. According to Noether's theorem, there is a conserved density,

$$\rho = \psi^\dagger \psi \tag{11.37}$$

and current

$$\vec{j} = \psi^\dagger \frac{\nabla}{m} \psi - \left(\frac{\nabla}{m} \psi^\dagger \right) \psi \tag{11.38}$$

satisfying the conservation law

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{j} = 0 \tag{11.39}$$

For $V = 0$, this is the free fermion functional integral:

$$\int \mathcal{D}\psi \, \mathcal{D}\psi^\dagger \, e^{-\int_0^\beta d\tau \int d^d x \, \psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi} = \det \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \tag{11.40}$$

The Green function is:

$$\begin{aligned} \mathcal{G}(\vec{x}, \tau) &= \mathcal{N} \int \mathcal{D}\psi \, \mathcal{D}\psi^\dagger \, \psi^\dagger(\vec{x}, \tau) \psi(0,0) \; e^{-\int_0^\beta d\tau \int d^d x \, \psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi} \\ &= \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right)^{-1} \end{aligned} \tag{11.41}$$

The difference between this Green function and the Green function of a bosonic system with the same Hamiltonian is that this is the inverse of this operator on the space of functions with antiperiodic boundary conditions on $[0, \beta]$. The Fourier transform of the Green function is:

$$\mathcal{G}(\vec{k}, \epsilon_n) = \frac{1}{i\epsilon_n - \left(\frac{k^2}{2m} - \mu \right)} \tag{11.42}$$

Figure 11.1: The graphical representation of the fermion propagator and vertex.

In the presence of source fields $\eta, \bar{\eta}$,

$$\begin{aligned} Z_0[\eta,\bar{\eta}] &= \mathcal{N} \int \mathcal{D}\psi\, \mathcal{D}\psi^\dagger\, e^{-\int_0^\beta d\tau \int d^dx\, \psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu\right)\right)\psi + \eta\psi^\dagger + \psi\bar{\eta}} \\ &= e^{-\int_0^\beta d\tau\, d\tau' \int d^dx\, d^dx'\, \bar{\eta}(\vec{x},\tau)\mathcal{G}(\vec{x}-\vec{x}',\tau-\tau')\eta(\vec{x}',\tau')} \end{aligned} \tag{11.43}$$

For interacting fermions, it is straightforward to generalize (7.31) to Grassmann integrals, so that

$$Z[\eta,\bar{\eta}] = \frac{N}{N_0}\, e^{-\int \mathcal{L}_{int}\left(\frac{\delta}{\delta\eta},\frac{\delta}{\delta\bar{\eta}}\right)}\, Z_0[\eta,\bar{\eta}] \tag{11.44}$$

In applying this formula, we must remember that η and $\bar{\eta}$ are Grassmann numbers so a $-$ sign results every time they are anticommuted. As in the bosonic case, we can use (7.32) to rewrite this as:

$$Z[\eta,\bar{\eta}] = e^{-\int_0^\beta d\tau\, d\tau' \int d^dx\, d^dx' \frac{\delta}{\delta\psi^\dagger}\mathcal{G}(\vec{x}-\vec{x}',\tau-\tau')\frac{\delta}{\delta\psi}}\; e^{-\int \mathcal{L}_{int}\left(\psi^\dagger,\psi\right)+\eta\psi^\dagger+\psi\bar{\eta}} \tag{11.45}$$

By expanding the

$$e^{-\int \mathcal{L}_{int}\left(\psi^\dagger,\psi\right)+\eta\psi^\dagger+\psi\bar{\eta}} \tag{11.46}$$

we derive the following Feynman rules for fermions with δ-function interactions. The lines of these Feynman diagrams have a direction which we denote by an arrow. Each vertex has two lines directed into it and two lines directed out of it. Momenta and Matrsubara frequencies are directed in the direction of the arrows. The propagator and vertex are shown in figure 11.1

- To each line, we associate a momentum, $\vec{p}$ and a Matsubara frequency, ϵ_n.
- The propagator assigned to each internal line is:

$$-\frac{1}{\beta}\sum_n \int \frac{d^3\vec{p}}{(2\pi)^3} \frac{1}{i\epsilon_n - \left(\frac{k^2}{2m} - \mu\right)}$$

- For each vertex with momenta, Matsubara frequencies $(\vec{p}_1, \epsilon_{n_1}), (\vec{p}_2, \epsilon_{n_2})$ directed into the vertex and $(\vec{p}_3, \epsilon_{n_3}), (\vec{p}_4, \epsilon_{n_4})$ directed out of the vertex, we write

$$V\,(2\pi)^3 \delta(\vec{p}_1 + \vec{p}_2 - \vec{p}_3 - \vec{p}_4)\;\beta\,\delta_{n_1+n_2,n_3+n_4}$$

- Imagine labelling the vertices $1, 2, \ldots, n$. Vertex i will be connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$ *by directed lines*. Consider a permutation of these labels. Such a permutation leaves the diagram invariant if, for all vertices i, i is still connected to vertices $j_1, \ldots, j_m$ $(m \leq 4)$ and to external momenta $p_1, \ldots, p_{4-m}$ *by lines in the same direction*. If S is the number of permutations which leave the diagram invariant, we assign a factor $1/S$ to the diagram.

- If two vertices are connected by l lines in the same direction, we assign a factor $1/l!$ to the diagram.

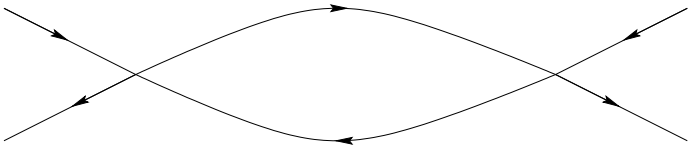
- To each closed loop, we assign a factor of -1.

The final rule follows from the necessity of performing an odd number of anticommutations in order to contract fermion fields around a closed loop.

For $\mu < 0$, (11.36) describes an insulating state. There is a gap to all excited states. For $V = 0$, the gap is simply $-\mu$. In the problem set, you will compute the gap for $V \neq 0$ perturbatively.

For $\mu > 0$, the ground state has a Fermi surface. For $V = 0$, this Fermi surface is at $k_F = \sqrt{2m\mu}$. In the problem set, you will compute the Fermi momentum for $V \neq 0$ perturbatively. Since this phase has gapless excitations, we must worry whether the interaction term is relevant. If the interactions are irrelevant, then we can perturbatively compute corrections to free fermion physics. If interactions are relevant, however, the system flows away from the free fermion fixed point, and we must look for other fixed points. Such an analysis is taken up in the next chapter, where we see the importance of the new feature that the low-energy excitations are not at $k = 0$, but, rather, at $k = k_F$. We construct the renormalization group which is appropriate to such a situation, thereby arriving at Fermi

a.



b.

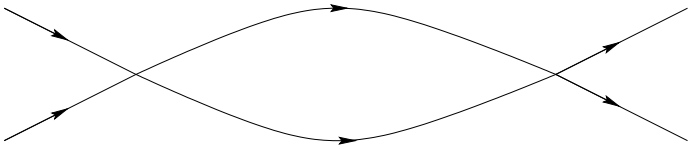


Figure 11.2: A one-loop diagram with an intermediate (a) particle-hole pair and (b) particle-particle pair.

liquid theory. First, however, we will investigate the two-point function of the interacting Fermi gas perturbatively.

Fermion lines with arrows that point to the right represent fermions above the Fermi surface. Those which point to the left represent holes below the Fermi surface. This is analogous to electrons and positrons in QED. However, unlike in QED, where a positron can have any momentum, fermions must have $k > k_F$ and holes must have $k < k_F$ at $T = 0$ (at finite-temperature, this is smeared out by the Fermi function). Hence, the diagram of figure 11.2a corresponds to the expression

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p} + \vec{q})\, \mathcal{G}(i\epsilon_n, \vec{q}) \tag{11.47}$$

When $q < k_f$, the second Green function represents the propagation of a hole at $\vec{q}$ while the first Green function represents the propagation of a fermion at $\vec{p}+\vec{q}$. If $\vec{p}+\vec{q}$ isn't above the Fermi surface (smeared by the Fermi function), then this expression vanishes, as we will see shortly. Similarly, when $\vec{q}$ is above the Fermi surface, $\vec{p} + \vec{q}$ must be a hole below the Fermi surface. Meanwhile, the diagram of figure 11.2b corresponds to the expression

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \mathcal{G}(i\Omega_m - i\epsilon_n, \vec{p} - \vec{q})\, \mathcal{G}(i\epsilon_n, \vec{q}) \tag{11.48}$$

where $\vec{q}$ and $\vec{p} - \vec{q}$ are now both fermions above the Fermi surface.

11.5 Fermion Spectral Function

Following our earlier derivation of the phonon spectral representation, we construct a spectral representation for the fermion two-point Green function. By inserting a complete set of intermediate states, $|m\rangle\langle m|$, we have,

$$\mathcal{G}(\vec{x}, \tau) = \int d^3\vec{p}\, d\epsilon [\sum_{n,m} \delta(\vec{p} - \vec{p}_m + \vec{p}_n) \delta(\epsilon - \epsilon_{nm}) (\theta(\tau) e^{-i\vec{p}\cdot\vec{x} + \epsilon\tau}\, e^{-\beta E_n} - \theta(-\tau)) e^{i\vec{p}\cdot\vec{x} - \epsilon\tau}\, e^{-\beta E_m}) \left| \left\langle m \left| \psi^\dagger(0,0) \right| n \right\rangle \right|^2 \tag{11.49}$$

The Fourier transform,

$$\mathcal{G}(\vec{p}, i\epsilon_j) = \int d^3\vec{x} \int_0^\beta d\tau\, \mathcal{G}(\vec{x}, \tau)\, e^{-i\epsilon_j \tau} \tag{11.50}$$

is given by:

$$\mathcal{G}(\vec{p}, i\epsilon_j) = [\sum_{n,m} \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \left|\left\langle m \left|\psi^\dagger(0,0)\right| n\right\rangle\right|^2 \times \delta(\vec{p} - \vec{p}_m + \vec{p}_n)\delta(E - E_m + E_n)] \frac{1}{E - i\epsilon_j} \quad (11.51)$$

Writing

$$A(\vec{p}, E) = \sum_{n,m} \left(e^{-\beta E_n} + e^{-\beta E_m}\right) \left|\left\langle m \left|\psi^\dagger(0,0)\right| n\right\rangle\right|^2 \delta(\vec{p}-\vec{p}_m+\vec{p}_n)\, \delta(E-E_{mn}) \quad (11.52)$$

we have the spectral representation of $\mathcal{G}$:

$$\mathcal{G}(\vec{p}, i\epsilon_n) = \int_{-\infty}^{\infty} dE\, \frac{A(\vec{p}, E)}{E - i\epsilon_j} \quad (11.53)$$

As usual, the spectral function $A(\vec{p}, E)$ is real and positive. It also satisfies the sum rule:

$$\int_{-\infty}^{\infty} \frac{dE}{2\pi}\, A(\vec{p}, E) = 1 \quad (11.54)$$

$\mathcal{G}$ is not analytic since it does not satisfy the Kramers-Kronig relations. However, the advanced and retarded correlation functions,

$$G_{\rm ret}(\vec{p}, \epsilon) = \int_{-\infty}^{\infty} dE\, \frac{A(\vec{p}, E)}{E - \epsilon - i\delta}$$
$$G_{\rm adv}(\vec{p}, \epsilon) = \int_{-\infty}^{\infty} dE\, \frac{A(\vec{p}, E)}{E - \epsilon + i\delta} \quad (11.55)$$

are analytic functions of ϵ in the upper- and lower-half-planes, respectively.

As usual, the spectral function is the difference between the retarded and advanced correlation functions.

$$G_{\rm ret}(\vec{p}, \epsilon) - G_{\rm adv}(\vec{p}, \epsilon) = 2\pi i A(\vec{p}, \epsilon) \quad (11.56)$$

The spectral function of a free Fermi gas is a δ-function:

$$A(\vec{p}, \epsilon) = \delta\left(\epsilon - \left(\frac{p^2}{2m} - \mu\right)\right) \quad (11.57)$$

In an interacting Fermi gas, the spectral weight is not concentrated in a δ function but spread out over a range of frequencies as in figure 11.3.

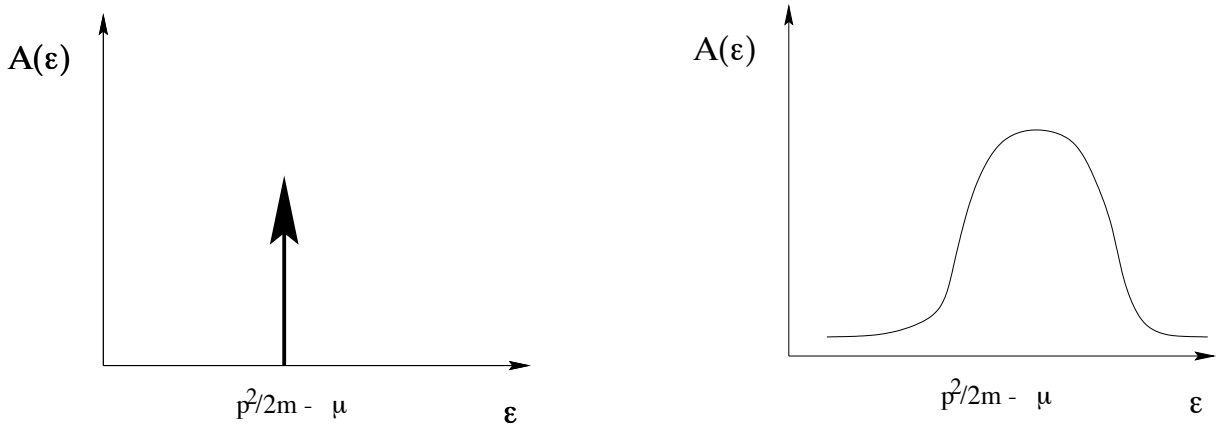


Figure 11.3: The spectral function in free and interacting Fermi systems.

11.6 Frequency Sums and Integrals for Fermions

To compute fermion Green functions perturbatively, we will need to do summations over Matsubara frequencies. Sums over fermion Matsubara frequencies can be done using contour integrals, as in the bosonic case. Consider the Matsubara sum:

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p}+\vec{q})\,\mathcal{G}(i\epsilon_n, \vec{q}) = \oint_C \frac{d\epsilon}{2\pi i}\, n_F(\epsilon)\,\mathcal{G}(i\Omega_m + \epsilon, \vec{p}+\vec{q})\,\mathcal{G}(\epsilon, \vec{q}) \tag{11.58}$$

where the contour avoids the singularties of the Green functions, as in chapter 6. ϵ_n and $\Omega_m + \epsilon_n$ are fermionic Matsubara frequencies, so Ω_m is a bosonic one. The contour integration is given by two contributions: ϵ real and $i\Omega_m + \epsilon$ real. Hence,

$$\begin{aligned}\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p}+\vec{q})\,\mathcal{G}(i\epsilon_n, \vec{q}) = &\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, n_F(E)\,\mathcal{G}(E + i\Omega_m)\,(\mathcal{G}(E+i\delta) - \mathcal{G}(E-i\delta)) \\ &+\frac{1}{2\pi i}\int_{-\infty}^{\infty} dE\, n_F(E - i\Omega_m)\,(\mathcal{G}(E+i\delta) - \mathcal{G}(E-i\delta))\,\mathcal{G}(E - i\Omega_m)\end{aligned}$$

Analytically continuing the imaginary-time Green functions, we have:

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p}+\vec{q})\,\mathcal{G}(i\epsilon_n, \vec{q}) = \int_{-\infty}^{\infty} dE\, n_F(E)\,\mathcal{G}(E + i\Omega_m, \vec{p}+\vec{q})\,A(E, \vec{q})$$

$$+\int_{-\infty}^{\infty} dE\, n_F(E)\, \mathcal{G}(E - i\Omega_m, \vec{q})\, A(E, \vec{p}+\vec{q}) \tag{11.59}$$

In the case of free fermions, the spectral function is a δ-function, so the dE integrals can be done:

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon_n, \vec{q}) = \frac{n_F(\xi_q) - n_F(\xi_{p+q})}{i\Omega_m + \xi_q - \xi_{p+q}} \tag{11.60}$$

At zero-temperature, the discrete frequency sum becomes a frequency integral,

$$\frac{1}{\beta}\sum_n \to \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \tag{11.61}$$

so

$$\frac{1}{\beta}\sum_n \mathcal{G}(i\Omega_m + i\epsilon_n, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon_n, \vec{q}) \to \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \mathcal{G}(i\Omega + i\epsilon, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon, \vec{q}) \tag{11.62}$$

Using the spectral representation of $\mathcal{G}$ we can rewrite this as:

$$\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \mathcal{G}(i\Omega + i\epsilon, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon, \vec{q}) = \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2\, \frac{A(\vec{p}+\vec{q}, E_1)}{E_1 - i\Omega - i\epsilon}\, \frac{A(\vec{q}, E_2)}{E_2 - i\epsilon} \tag{11.63}$$

The $d\epsilon$ integral can be done by closing the contour in the upper-half-plane. The pole at $\epsilon = -iE_1 - \Omega$ is enclosed by the contour when $E_1 < 0$; the pole at $\epsilon = -iE_2$ is enclosed when $E_2 < 0$. Hence,

$$\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \mathcal{G}(i\Omega + i\epsilon, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon, \vec{q}) = \int_{-\infty}^{\infty} dE_1 \int_{-\infty}^{\infty} dE_2\, \frac{\theta(-E_2) - \theta(-E_1)}{E_2 - E_1 + i\omega} A(\vec{p}+\vec{q}, E_1)\, A(\vec{q}, E_2) \tag{11.64}$$

In the case of free fermions, the dE_i integrals may be done:

$$\int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \mathcal{G}(i\Omega + i\epsilon, \vec{p}+\vec{q})\, \mathcal{G}(i\epsilon, \vec{q}) = \frac{\theta(-\xi_q) - \theta(-\xi_{p+q})}{i\Omega + \xi_q - \xi_{p+q}} \tag{11.65}$$

which is the zero-temperature limit of (11.60).

11.7 Fermion Self-Energy

We can begin to understand the role played by the Fermi surface when we start computing perturbative corrections to the behavior of free fermions. Let us look first at the fermion two-point Green function. As in the bosonic

case, we can define the self-energy, $\Sigma(\epsilon, k)$, as the 1PI two-point function and sum the geometric series to obtain:

$$\mathcal{G}(\vec{p}, i\epsilon_n) = \frac{1}{i\epsilon_n - \left(\frac{p^2}{2m} - \mu\right) - \Sigma(\epsilon_n, p)} \tag{11.66}$$

The retarded Green function is defined by analytic continuation:

$$G_{\text{ret}}(\vec{p}, \epsilon) = \frac{1}{\epsilon - \left(\frac{p^2}{2m} - \mu\right) - \Sigma_{\text{ret}}(\epsilon, p)} \tag{11.67}$$

The spectral function can be written as:

$$A(\vec{p}, \epsilon) = \frac{1}{\pi} \frac{-2\text{Im}\Sigma_{\text{ret}}(\epsilon, p)}{\left(\epsilon - \left(\frac{p^2}{2m} - \mu\right) - \text{Re}\Sigma_{\text{ret}}(\epsilon, p)\right)^2 + \left(\text{Im}\Sigma_{\text{ret}}(\epsilon, p)\right)^2} \tag{11.68}$$

When $\text{Im}\Sigma_{\text{ret}}(\epsilon, p) = 0$, the spectral function can be rewritten as:

$$A(\vec{p}, \epsilon) = Z(p)\, \delta\left(\epsilon - \xi_p\right) \tag{11.69}$$

where ξ_p is the location of the pole, defined by the implicit equation

$$\xi_p = \frac{p^2}{2m} - \mu - \text{Re}\Sigma_{\text{ret}}(\xi_p, p) \tag{11.70}$$

and $Z(p)$ is its residue

$$Z(p) = \left| 1 - \left(\frac{\partial}{\partial \epsilon} \text{Re}\Sigma_{\text{ret}}(\epsilon, p) \right)_{\epsilon = \xi_p} \right|^{-1} \tag{11.71}$$

ξ_p can be expanded about the Fermi surface:

$$\xi_p = v_F^* \left(p - p_F\right) + O\left(\left(p - p_F\right)^2\right) \tag{11.72}$$

where

$$v_F^* = \frac{p_F}{m^*} \tag{11.73}$$

and m^* is the *effective mass*. ξ_p and v_F^* define the one-particle density of states, $N(\epsilon_F)$, of the interacting problem.

$$\int \frac{d^d k}{(2\pi)^d} = \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \int k^{d-1} dk$$

$$\begin{aligned}
&\approx \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} k_F^{d-1} \int dk \\
&= \frac{2\pi^{\frac{d}{2}}}{(2\pi)^d \Gamma\left(\frac{d}{2}\right)} \frac{k_F^{d-1}}{v_F^*} \int d\xi_k \\
&\equiv N(\epsilon_F) \int d\xi_k
\end{aligned} \tag{11.74}$$

The lowest-order contribution to $\mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, p)$ comes from the diagram of figure **??**. We can do the zero-temperature calculation by contour integration:

$$\Sigma(i\epsilon, k) = V^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d p}{(2\pi)^d} \int \frac{d\omega}{2\pi} \int \frac{d\zeta}{2\pi} \mathcal{G}(i\zeta, p)\, \mathcal{G}(i\zeta + i\omega, p+q)\, \mathcal{G}(i\epsilon - i\omega, k-q) \tag{11.75}$$

The $d\zeta$ integral may be done by contour integration, as in (11.65):

$$\Sigma(i\epsilon, k) = V^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d p}{(2\pi)^d} \int \frac{d\omega}{2\pi}\, \mathcal{G}(i\epsilon - i\omega, k-q)\, \frac{\theta(-\xi_p) - \theta(-\xi_{p+q})}{i\omega + \xi_p - \xi_{p+q}} \tag{11.76}$$

The $d\omega$ integral may be done the same way:

$$\Sigma(i\epsilon, k) = V^2 \int \frac{d^d q}{(2\pi)^d} \int \frac{d^d p}{(2\pi)^d} \frac{\left(\theta(-\xi_p) - \theta(-\xi_{p+q})\right)\left(\theta(\xi_{k-q}) - \theta(\xi_p - \xi_{p+q})\right)}{i\omega + \xi_p - \xi_{p+q} - \xi_{k-q}} \tag{11.77}$$

Hence, the imaginary part of the self-energy at zero-temperature is:

$$\begin{aligned}
\mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, k) &= V^2 \frac{k_F^{2(d-1)}}{v_F^2} \int_{-\infty}^{0} d\xi_p \int_0^{\infty} d\xi_{k-q}\, \frac{d\Omega_p}{(2\pi)^d} \frac{d\Omega_{k-q}}{(2\pi)^d}\, \delta\left(\xi_p - \xi_{p+q} - \xi_{k-q} + \epsilon\right) \\
&\leq V^2 \frac{k_F^{2(d-1)}}{v_F^2}\, S_{d-1}^2 \int_{-\epsilon}^{0} d\xi_p \int_0^{\epsilon} d\xi_{k-q} \\
&= V^2 \frac{k_F^{2(d-1)}}{v_F^2}\, S_{d-1}^2\, \epsilon^2
\end{aligned} \tag{11.78}$$

Hence, we have seen that the phase space restrictions (imposed by the δ function above) due to the existence of a Fermi surface severely restricts $\mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, k)$. For $\epsilon \to 0$, $\mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, k) \sim \epsilon^2$. In other words, for ϵ small, the decay rate is much smaller than the energy: near the Fermi surface, single-fermion states are long-lived.

11.8 Luttinger's Theorem

Up until now, the Fermi surface has essentially been a tree-level, or free fermion, concept. However, the notion of a Fermi surface is not tied to perturbation theory. In fact, the existence and location of a Fermi surface is constrained by a non-perturbative theorem due to Luttinger, which we now discuss. Luttinger's theorem defines the Fermi surface as the surface in $\vec{k}$-space at which $G(0,\vec{k})$ changes sign. Inside the Fermi surface, $G(0,\vec{k})$ is positive; outside the Fermi surface, $G(0,\vec{k})$ is negative. In a free fermion system, $G(0,\vec{k})$ diverges at the Fermi surface. (In a superconductor, $G(0,\vec{k})$ vanishes at the Fermi surface, as we will see later.) According to Luttinger's theorem, the volume enclosed by the Fermi surface is equal to the electron density, N/V, so long as $\mathrm{Im}\Sigma(0,k) = 0$.

To prove this, we begin with

$$\begin{aligned}\frac{N}{V} &= \int \frac{d^dk}{(2\pi)^d} \left\langle \psi^\dagger(k,t)\psi(k,t)\right\rangle \\ &= -i \int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi} G(k,\epsilon)\end{aligned} \tag{11.79}$$

In the second line, we have the time-ordered Green function; the advanced and retarded Green functions vanish at equal times. If we write

$$G(k,\epsilon) = \frac{1}{\epsilon - \left(\frac{k^2}{2m} - \mu\right) - \Sigma(\epsilon,k)} \tag{11.80}$$

then

$$\begin{aligned}\frac{N}{V} &= -i \int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi} G(k,\epsilon) \\ &= -i \int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi} G(k,\epsilon)\left[\frac{\partial}{\partial\epsilon}\left(\epsilon - \left(\frac{k^2}{2m}-\mu\right)\right)\right] \\ &= -i \int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi} G(k,\epsilon)\left[\frac{\partial}{\partial\epsilon}\left(G^{-1}(k,\epsilon) + \Sigma(\epsilon,k)\right)\right] \\ &= i \int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi}\left[\frac{\partial}{\partial\epsilon}\ln G(k,\epsilon) \; - G(k,\epsilon)\frac{\partial}{\partial\epsilon}\Sigma(\epsilon,k)\right]\end{aligned} \tag{11.81}$$

We will now use the following 'lemma' which we will prove later:

$$\int \frac{d^dk}{(2\pi)^d}\frac{d\epsilon}{2\pi} G(k,\epsilon)\frac{\partial}{\partial\epsilon}\Sigma(\epsilon,k) = 0 \tag{11.82}$$

Then

$$\begin{aligned}\frac{N}{V} &= i\int \frac{d^dk}{(2\pi)^d}\,\frac{d\epsilon}{2\pi}\,\frac{\partial}{\partial\epsilon}\ln G(k,\epsilon)\\ &= i\int \frac{d^dk}{(2\pi)^d}\,\frac{d\epsilon}{2\pi}\,\frac{\partial}{\partial\epsilon}\ln G_{\rm ret}(k,\epsilon) + i\int \frac{d^dk}{(2\pi)^d}\,\frac{d\epsilon}{2\pi}\,\frac{\partial}{\partial\epsilon}\ln\frac{G(k,\epsilon)}{G_{\rm ret}(k,\epsilon)}\end{aligned} \tag{11.83}$$

Since $G_{\rm ret}$ is analytic in the upper-half-plane, the first integral vanishes. Also note that $G = G_{\rm ret}$ for $\epsilon > 0$, while $G^* = G_{\rm ret}$ for $\epsilon < 0$. Hence,

$$\begin{aligned}\frac{N}{V} &= i\int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{0}\frac{d\epsilon}{2\pi}\,\frac{\partial}{\partial\epsilon}\ln\frac{G(k,\epsilon)}{G_{\rm ret}(k,\epsilon)}\\ &= i\int \frac{d^dk}{(2\pi)^d}\left[\ln\frac{G(k,\epsilon)}{G_{\rm ret}(k,\epsilon)}\right]_{-\infty}^{0}\\ &= -\int \frac{d^dk}{(2\pi)^d}\,[\varphi(0,k)-\varphi(-\infty,k)]\end{aligned} \tag{11.84}$$

From the spectral representation,

$$G(\vec{p},\epsilon) = \int_{-\infty}^{\infty} dE\,\frac{A(\vec{p},E)}{\epsilon - E + i\delta\,\mathrm{sgn}(\epsilon)} \tag{11.85}$$

and the normalization property of the spectral function, we see that $\varphi(-\infty,k) = \pi$. Hence,

$$\frac{N}{V} = -\int \frac{d^dk}{(2\pi)^d}\,[\varphi(0,k)-\pi] \tag{11.86}$$

Since $\mathrm{Im}\Sigma(0,k) = 0$ by assumption, $\varphi(0,k)$ is equal to 0 or π. The integral only receives contributions from the former case:

$$\frac{N}{V} = \int_{\mathcal{R}}\frac{d^dk}{(2\pi)^d} \tag{11.87}$$

where $\mathcal{R} = \{\vec{k}|G(0,\vec{k}) > 0\}$. In other words, the volume enclosed by the Fermi surface is equal to the electron density.

To complete the proof of the theorem, we must prove that

$$\int \frac{d^dk}{(2\pi)^d}\,\frac{d\epsilon}{2\pi}\,G(k,\epsilon)\frac{\partial}{\partial\epsilon}\Sigma(\epsilon,k) = 0 \tag{11.88}$$

To do this, we prove that there exists a functional $X[G]$ defined by:

$$\delta X = \int\frac{d\omega}{2\pi}\int\frac{d^3k}{(2\pi)^3}\Sigma(\omega,k)\,\delta G(\omega,k) \tag{11.89}$$

According to this definition,

$$\begin{aligned}\delta X &= \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \Sigma(\omega, k)\, \delta G(\omega, k) \\ &= \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \Sigma(\omega + \epsilon, k)\, \delta G(\omega + \epsilon, k) \end{aligned} \tag{11.90}$$

Hence,

$$\begin{aligned}\frac{\delta X}{\delta \epsilon} &= \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \Sigma(\omega + \epsilon, k)\, \frac{\partial}{\partial \epsilon} \delta G(\omega + \epsilon, k) \\ &= \int \frac{d\omega}{2\pi} \int \frac{d^3k}{(2\pi)^3} \Sigma(\omega, k)\, \frac{\partial}{\partial \omega} \delta G(\omega, k) \end{aligned} \tag{11.91}$$

However, X is independent of ϵ, so $\delta X/\delta\epsilon = 0$ which proves (11.88).

To see that X actually exists, observe that

$$\frac{\delta X}{\delta G(p)} = \Sigma(p) \tag{11.92}$$

Hence,

$$\frac{\delta^2 X}{\delta G(p) \delta G(q)} = \frac{\delta \Sigma(p)}{\delta G(q)} \tag{11.93}$$

X exists if and only if the derivatives can be commuted:

$$\frac{\delta^2 X}{\delta G(p) \delta G(q)} = \frac{\delta^2 X}{\delta G(q) \delta G(p)} \tag{11.94}$$

Since

$$\frac{\delta \Sigma(p)}{\delta G(q)} = \Gamma(p, q) \tag{11.95}$$

where $\Gamma(p, q)$ is the irreducible 4-point function with external momenta p, p, q, q and $\Gamma(p, q) = \Gamma(q, p)$, the existence of X follows.

In the case of a free Fermi gas, $n_k = \langle c_k^\dagger c_k \rangle$ is a step function, $n_k = \theta(k_F - k)$ with $\int n_k = N/V$. One might imagine that, in an interacting Fermi gas, it would be possible to have $n_k = \lambda\theta(k_F - k\lambda)$ with $\lambda < 1$ which would preserve $\int n_k = N/V$ while moving the location of the singularity in n_k to k_F/λ. Luttinger's theorem tells us that this cannot happen. The singularity in $n_k = \int d\epsilon G(\epsilon, k)$ is fixed by the density to be at k_F.

CHAPTER 12

Interacting Neutral Fermions: Fermi Liquid Theory

12.1 Scaling to the Fermi Surface

We now consider a rotationally invariant system of interacting spinless fermions with $\mu > 0$ in $D \geq 2$. The RG analysis of such systems was pioneered by Shankar, Polchinski, ... The Fermi sea is filled up to some k_F. First, let us examine the free part of the action,

$$\int d\tau d^d x \, \psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi \tag{12.1}$$

or, in momentum space,

$$\int \frac{d\epsilon}{2\pi} \frac{d^d k}{(2\pi)^d} \, \psi^\dagger \left(i\epsilon - \left(\frac{k^2}{2m} - \mu \right) \right) \psi \tag{12.2}$$

If the kinetic energy is much larger than the potential energy, then it makes sense to focus first on it and use it to determine the scaling of ψ from the kinetic energy.

Since we will be interested in low-energy, i.e. the vicinity of the Fermi surface, we make the approximation

$$\begin{aligned} \frac{k^2}{2m} - \mu &\approx v_F \left(k - k_F \right) \\ &\equiv v_F l \end{aligned} \tag{12.3}$$

where $l = k - k_F$. We can also make the approximation

$$\begin{aligned} d^d k &= k_F^{d-1} dk\, d^d\Omega \\ &= k_F^{d-1} dl\, d^d\Omega \end{aligned} \tag{12.4}$$

Hence, the action can be written as:

$$\frac{k_F^{d-1}}{(2\pi)^d} \int dl\, d^d\Omega \frac{d\epsilon}{2\pi} \psi^\dagger \left(i\epsilon - v_F l\right) \psi \tag{12.5}$$

Restoring the cutoffs,

$$\frac{k_F^{d-1}}{(2\pi)^d} \int_{-\Lambda}^{\Lambda} dl\, d^d\Omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\, \psi^\dagger \left(i\epsilon - v_F l\right) \psi \tag{12.6}$$

The momentum integral is restricted to a shell of thickness 2Λ about the Fermi surface. We leave the frequency integral unrestricted (as we did in the case of the $O(3)$ NLσM). The angular integral has no cutoff, of course.

Our RG transformation now takes the following form:

- Integrate out $\psi(l, \vec{\Omega}, \epsilon), \psi^\dagger(l, \vec{\Omega}, \epsilon)$ for $b\Lambda < |l| < \Lambda$ and ϵ, $\vec{\Omega}$ arbitrary.
- Rescale:

$$\begin{aligned} \omega &\rightarrow b\omega \\ l &\rightarrow bl \\ \vec{\Omega} &\rightarrow \vec{\Omega} \\ \psi &\rightarrow b^{-\frac{3}{2}} \psi \end{aligned} \tag{12.7}$$

The principal difference between the renormalization group applied to a system with a Fermi surface and its application to more familiar contexts is that the low-energy degrees of freedom are located in the neighborhood of a *surface* in momentum space, rather than in the vicinity of a point. Hence, we do not scale to the origin of momentum space, $\vec{k} = 0$, but to the Fermi surface, $\vec{l} = 0$.

The free fermion action (12.5) is evidently a fixed point of this RG transformation. Thus, as we would expect, a free fermion system looks the same at any energy scale: it is always just a free theory.

To this action, we can add the following perturbation:

$$\frac{k_F^{d-1}}{(2\pi)^d} \int_{-\Lambda}^{\Lambda} dl\, d^d\Omega \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\, \delta\mu\, \psi^\dagger \psi \tag{12.8}$$

Under the scaling (12.7), $\delta\mu$ scales as:

$$\delta\mu \to b^{-1}\delta\mu \tag{12.9}$$

Since $\delta\mu$ is a relevant operator, we cannot study it perturbatively. Relevant operators typically bring about a fundamental change of the ground state, and $\delta\mu$ is no different. Changing the chemical potential shifts the Fermi surface. If we change coordinates to $\vec{l}' = \vec{l} + \frac{\delta\mu}{v_F}$, then we recover (12.5).

In a system which is not rotationally invariant, $\delta\mu$ can depend on the angle around the Fermi surface, $\delta\mu\left(\vec{\Omega}\right)$. $\delta\mu\left(\vec{\Omega}\right)$ is an example of a 'coupling function', which is a generalization of a coupling constant. Such a perturbation can change the shape of the Fermi surface.

A second perturbation is

$$\frac{k_F^{d-1}}{(2\pi)^d}\int_{-\Lambda}^{\Lambda} dl\, d^d\Omega \int_{-\infty}^{\infty}\frac{d\epsilon}{2\pi}\,\delta v_F \vec{l}\,\psi^\dagger\psi \tag{12.10}$$

δv_F shifts the Fermi velocity. It is clearly a marginal perturbation.

12.2 Marginal Perturbations: Landau Parameters

Let us now consider four-fermion interactions. Consider the term

$$S_4 = \int d\omega_1 d\omega_2 d\omega_3\, d^dk_1 d^dk_2 d^dk_3\; u(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)\times \\ \psi^\dagger(k_4,\omega_4)\psi^\dagger(k_3,\omega_3)\psi(k_2,\omega_2)\psi(k_1,\omega_1) \tag{12.11}$$

Rather than a single coupling constant, u, we have a coupling function, $u(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$. The RG equations for a coupling function are called *functional RG equations*. We will assume that $u(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4)$ is non-singular or, in other words, that the fermions have short-ranged interactions, as in ^{3}He. We will deal with the complications resulting from Coulomb interactions in the next section.

If $u(k_1,k_2,k_2,k_1)$ is a function only of the angular variables, $u(\vec{\Omega}_1,\vec{\Omega}_2,\vec{\Omega}_3,\vec{\Omega}_4)$, then it is marginal, *i.e.* it does not scale under (12.7). If it depends on the $|k_i| - k_F|$'s, then it is irrelevant, so we ignore this possibility. Similarly, six-fermion, eight-fermion, etc. interactions are neglected because they are highly irrelevant. Furthermore, momentum conservation implies that

$$u(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4) = u(\vec{k}_1,\vec{k}_2,\vec{k}_3)\,\delta\left(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4\right) \tag{12.12}$$

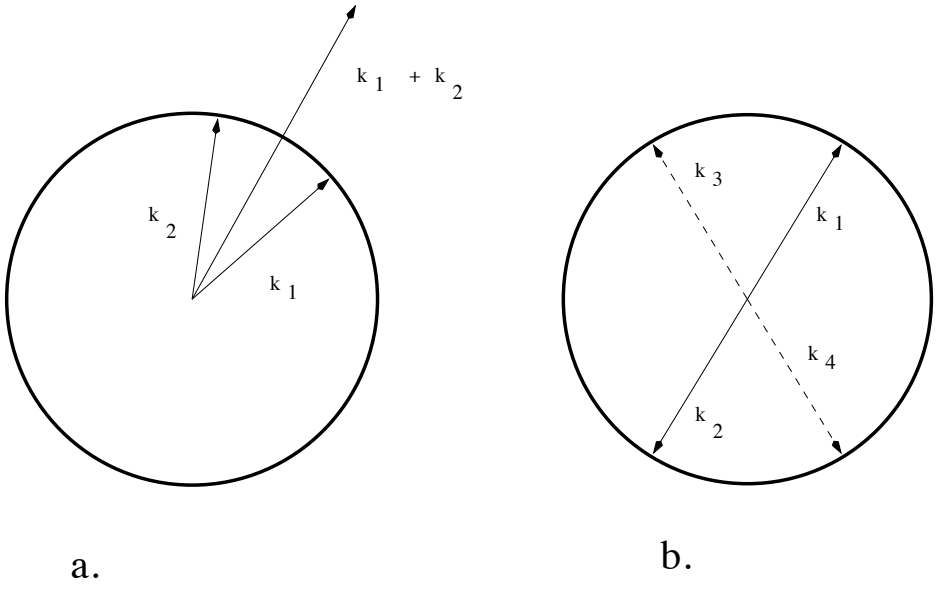


Figure 12.1: (a) If all of the momenta are constrained to lie on the Fermi surface, incoming momenta k_1, k_2 can only scatter into $k_3 = k_1, k_4 = k_2$ or $k_3 = k_2, k_4 = k_1$, unless (b) $k_1 = -k_2$.

In the last section, we considered the case of δ function interactions, for which $u(\vec{k}_1, \vec{k}_2, \vec{k}_3) = V$. Here, we are considering the more general case of arbitrary (non-singular) $u(\vec{k}_1, \vec{k}_2, \vec{k}_3)$.

The crucial observation underlying Fermi liquid theory, which is depicted in Figure 12.1, is the following. Consider, for simplicity, the case of $D = 2$. For $\Lambda \ll k_F$, $u(k_1, k_2, k_3) = 0$ for generic k_1, k_2, k_3 because k_4 typically does not lie within the cutoff. The constraint of momentum conservation, $\vec{k}_1 + \vec{k}_2 = \vec{k}_3 + \vec{k}_4$ together with the restriction that $\vec{k}_1, \vec{k}_2, \vec{k}_3, \vec{k}_4$ lie within Λ of the Fermi surface severely limits the phase space for scattering. As we scale to the $\Lambda \to 0$ limit, only forward scattering, $u(k_1, k_2, k_1, k_2)$ and exchange scattering, $u(k_1, k_2, k_2, k_1) = -u(k_1, k_2, k_1, k_2)$, can satisfy momntum conservation. At small but non-zero Λ, a small subset of the u's are non-zero. As Λ is decreased, some of these are set discontinuously to zero; the rest do not scale. As Λ becomes smaller, fewer non-zero u's remain until, finally, at $\Lambda = 0$, only the three mentioned above remain. It is this drastic simplification which makes Fermi liquid theory soluble.

In three dimensions, the angle between $\vec{k}_3$ and $\vec{k}_4$ is the same as the angle between $\vec{k}_1$ and $\vec{k}_2$

$$\theta(\vec{k}_1, \vec{k}_2) = \pm\theta(\vec{k}_3, \vec{k}_4) \tag{12.13}$$

but the plane of $\vec{k}_3$ and $\vec{k}_4$ can be rotated relative to the plane of $\vec{k}_1$ and $\vec{k}_2$ by and angle ϕ as in figure 12.2.

These phase space restrictions imply that in two dimensions, we should focus on

$$F(\theta_1 - \theta_2) \equiv u(\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_2, \vec{\Omega}_1) \tag{12.14}$$

Fermi statistics dictates that exchange scattering is related to forward scattering by:

$$\begin{aligned} u(\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_2, \vec{\Omega}_1) &= -u(\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_1, \vec{\Omega}_2) \\ &= -F(\theta_1 - \theta_2) \end{aligned} \tag{12.15}$$

In three dimensions, we should focus on:

$$F(\vec{\Omega}_1 \cdot \vec{\Omega}_2, \phi) \equiv u(\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_3, \vec{\Omega}_4) \tag{12.16}$$

in the $\Lambda \to 0$ limit. The Fourier components of F are called *Landau Parameters*.

There is one loophole in the preceeding analysis, depicted in figure 12.1b. If $\vec{k}_1 = -\vec{k}_2$, then $\vec{k}_3 = -\vec{k}_4$ is arbitrary. This is the Cooper pairing channel. We write:

$$V(\vec{\Omega}_1 \cdot \vec{\Omega}_3) \equiv u(k_1, -k_1, k_3, -k_3) \tag{12.17}$$

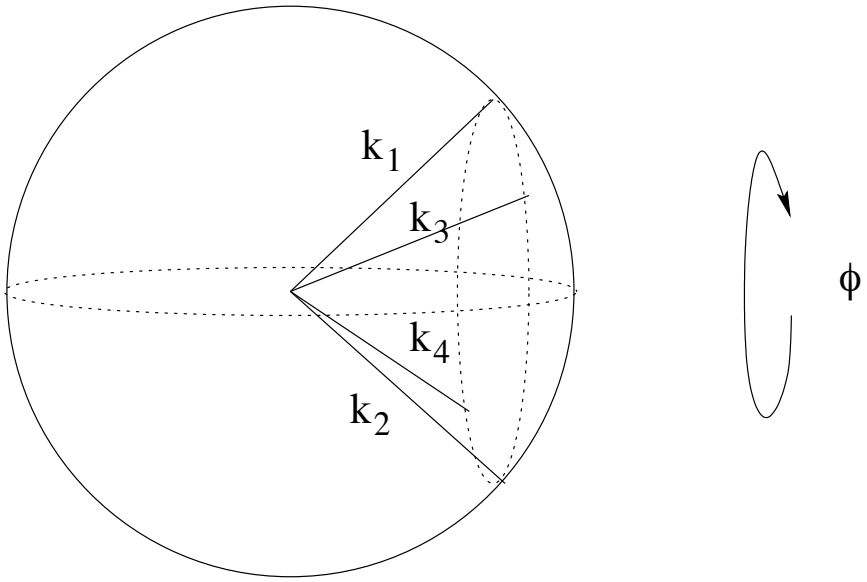


Figure 12.2: In three dimensions, the outgoing momenta can be rotated relative to the incoming momenta.

In $D = 2$, this can be written as:

$$V(\theta_1 - \theta_3) \tag{12.18}$$

Then, at tree-level, in the $\Lambda \to 0$ limit, we have the following action:

$$\begin{aligned} S =& \frac{k_F^{d-1}}{(2\pi)^d} \int dl\, d^d\Omega\, \frac{d\epsilon}{2\pi} \psi^\dagger \left(i\epsilon - v_F l\right) \psi \\ &+ \int d\epsilon_1 d\epsilon_2 d\epsilon_3\, dk_1 dk_2 dk_3\, d^d\Omega_1 d^d\Omega_2\, F(\vec{\Omega}_1 \cdot \vec{\Omega}_2, \phi) \times \\ &\qquad \psi^\dagger(k_4, \epsilon_4) \psi^\dagger(k_3, \epsilon_3) \psi(k_2, \epsilon_2) \psi(k_1, \epsilon_1) \\ &+ \int d\epsilon_1 d\epsilon_2 d\epsilon_3\, dk_1 dk_2 dk_3\, d^d\Omega_1 d^d\Omega_3\, V(\vec{\Omega}_1 \cdot \vec{\Omega}_3) \times \\ &\qquad \psi^\dagger(k_4, \epsilon_4) \psi^\dagger(k_3, \epsilon_3) \psi(k_2, \epsilon_2) \psi(k_1, \epsilon_1) \end{aligned} \tag{12.19}$$

For Λ finite, we have to keep the full coupling function $u(\vec{\Omega}_1, \vec{\Omega}_2, \vec{\Omega}_3, \vec{\Omega}_4)$ with

$$\left|\vec{\Omega}_1 - \vec{\Omega}_3\right| < \frac{\Lambda}{k_F} \tag{12.20}$$

or

$$\left|\vec{\Omega}_2 - \vec{\Omega}_3\right| < \frac{\Lambda}{k_F} \tag{12.21}$$

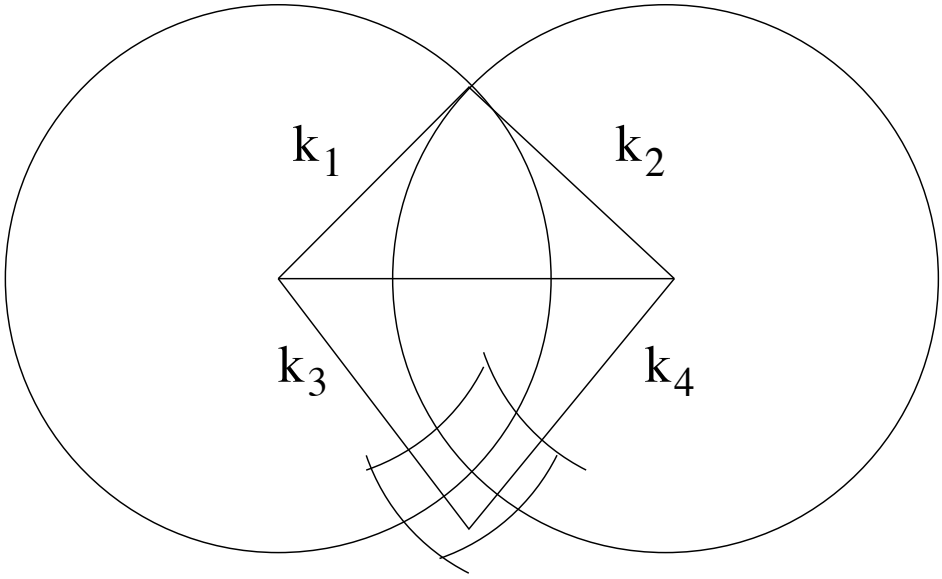


Figure 12.3: The phase space available to k_3 and k_4 when Λ is small but non-zero is the region bounded by the arcs which has area $\sim \Lambda^2$.

as in figure 12.3

12.3 One-Loop

At tree-level, $F(\vec{\Omega}_1 - \vec{\Omega}_2)$ and $V(\vec{\Omega}_1 - \vec{\Omega}_3)$ are marginal. We would now like to compute the one-loop RG equations for $F(\vec{\Omega}_1 - \vec{\Omega}_2)$ and $V(\vec{\Omega}_1 - \vec{\Omega}_3)$.

First, consider the renormalization of F. The one-loop diagrams are in figure 12.4. Since F is independent of the frequencies and the l_i, we can set the external frequencies to zero and put the external momenta on the Fermi surface. The first diagram gives a contribution

$$dF(\vec{\Omega}_1 - \vec{\Omega}_2) = \int dl\, d\Omega\, d\epsilon\, F(\vec{\Omega}) F(\vec{\Omega} + \vec{\Omega}_1 - \vec{\Omega}_2) \frac{1}{i\epsilon - v_F l} \frac{1}{i\epsilon - v_F l} \quad (12.22)$$

which vanishes since both poles in ϵ are on the same side of the axis.

The internal momenta in these diagrams must lie in thin shells at the cutoff, $\Lambda - d\Lambda < |p| - k_F < \Lambda$. In the second diagram, $\vec{p}$ and $\vec{p} + \vec{k}_1 - \vec{k}_2$ must both satisfy this condition. The condition on $|p|$ restricts its magnitude; the condition on $|\vec{p} + \vec{k}_1 - \vec{k}_2|$ restricts the direction of $\vec{p}$. The kinematic restriction

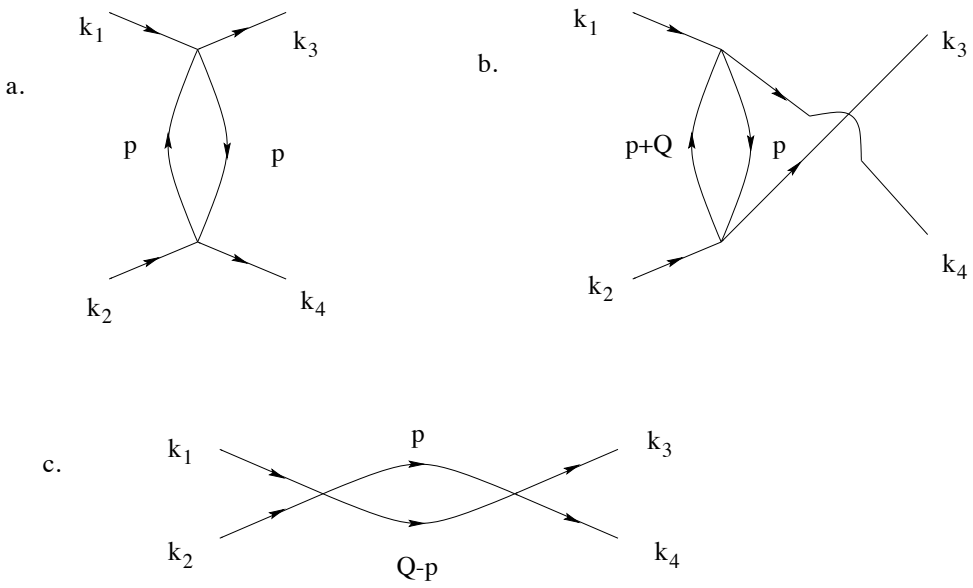


Figure 12.4: The one-loop diagrams which can contribute to the renormalization of F.

is essentially the same as that depicted in figure 12.3. As a result, the dl and $d\Omega$ integrals each give a contribution proportional to $d\Lambda$, and therefore

$$dF \sim (d\Lambda)^2 \tag{12.23}$$

in the $d\Lambda \to 0$ limit, this gives a vanishing contribution to $dF/d\ell$. The third diagram gives a vanishing contribution for the same reason. Hence, at one-loop,

$$\frac{d}{d\ell}F(\vec{\Omega}_1 - \vec{\Omega}_2) = 0 \tag{12.24}$$

The Landau parameters are strictly marginal; they remain constant as we scale to lower energies.

We now turn to the one-loop RG equations for V. The relevant diagrams are analogous to those of 12.4. The first two diagrams are proportional to $(d\Lambda)^2$ and, therefore, do not contribute to the RG equation. However, the third diagram gives the contribution

$$dV(\vec{\Omega}_1 - \vec{\Omega}_3) = -\int \frac{d\epsilon}{2\pi}\frac{dl}{2\pi}\frac{d^d\Omega}{(2\pi)^{d-1}}V(\vec{\Omega}_1 - \vec{\Omega})V(\vec{\Omega} - \vec{\Omega}_3)\frac{1}{i\epsilon - v_F l}\frac{1}{-i\epsilon - v_F l}$$

$$= \int \frac{dl}{2\pi} \frac{d^d\Omega}{(2\pi)^{d-1}} V(\vec{\Omega}_1 - \vec{\Omega}) V(\vec{\Omega} - \vec{\Omega}_3) \frac{1}{2v_F p}$$
$$= \frac{1}{2\pi v_F} d\ell \int \frac{d^d\Omega}{(2\pi)^{d-1}} V(\vec{\Omega}_1 - \vec{\Omega}) V(\vec{\Omega} - \vec{\Omega}_3) \tag{12.25}$$

In two dimensions, we write

$$V_m = \int_0^{2\pi} \frac{d\theta}{2\pi} e^{im\theta} V(\theta) \tag{12.26}$$

The renormalization group flow equation for V_l is:

$$\frac{dV_l}{d\ln\Lambda} = -\frac{1}{2\pi v_F} V_l^2 \tag{12.27}$$

$$V_l(\Lambda) = \frac{V_l(\Lambda_0)}{1 + \frac{1}{2\pi v_F} V_l(\Lambda_0) \ln(\Lambda_0/\Lambda)} \tag{12.28}$$

Therefore, repulsive BCS interactions are marginally irrelevant, while attractive BCS interactions are marginally relevant. From (12.28), we see that an attractive BCS interaction will grow as we go to lower scales, until it reaches the scale:

$$\Lambda \sim \Lambda_0 \, e^{-2\pi v_F/|V_l(\Lambda_0)|} \tag{12.29}$$

As we will discuss later, at this scale, pairing takes place. In the BCS theory of a phonon-mediated superconductor, this leads to a critical temperature or zero-temperature gap given by

$$T_c \sim \Delta(T=0) \sim \omega_D \, e^{-2\pi v_F/|V_0|} \tag{12.30}$$

Bardeen, Cooper, and Schrieffer found the paired ground state using a variational ansatz. A more general formalism, which can be used when the interactions are retarded was pioneered by Nambu, Gor'kov, and Eliashberg. Both of these approaches owe their success to the kinematic constraints of the problem. There are no relevant interactions other than $F(\Omega_1, \Omega_2)$ and $V(\Omega_1, \Omega_3)$. $F(\Omega_1, \Omega_2)$ does not contribute to the running of $V(\Omega_1, \Omega_3)$, so the diagram of Figure **??** is essentially the only diagram which must be taken into account. BCS theory and its refinement by Nambu, Gor'kov, and Eliashberg are mean-field theories which evaluate this diagram self-consistently. These theories will be discussed in chapter 16.

12.4 $1/N$ and All Loops

The one-loop structure of a system of interacting fermions is actually stable to all orders in perturbation theory. The essential reason for this (which was first recognized in this language by Shankar) is that Λ/k_F is a small parameter like $1/N$. To see this, consider the case $D = 2$. Break the angular integration into pieces, $\Delta\theta = \Lambda/k_F$ with $\theta_j = 2\pi j(\Lambda/k_F)$ and $j = 0, 1, \ldots, k_F/\Lambda$.

$$\int_0^{2\pi} d\theta \to \sum_i \int_{\theta_i}^{\theta_{i+1}} d\theta \tag{12.31}$$

Then, we can write

$$\begin{aligned} S = &\sum_i \int \frac{d^2k}{(2\pi)^2}\,\frac{d\epsilon}{2\pi}\,\psi_i^\dagger\,(i\epsilon - v_F l)\,\psi_i \\ &+\sum_{i.j}\int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\,\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi}\times \\ &\qquad F_{ij}\,\psi_j^\dagger(k_3,\epsilon_3)\psi_j(k_2,\epsilon_2)\,\psi_i^\dagger(k_4,\epsilon_4)\psi_i(k_1,\epsilon_1)\,\delta\left(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4\right) \\ &+\sum_{i,j}\int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\,\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi}\times \\ &\qquad V_{ij}\,\psi_j^\dagger(k_3,\epsilon_3)\psi^\dagger_{j+\frac{k_F}{2\Lambda}}(k_4,\epsilon_4)\,\psi_i(k_2,\epsilon_2)\psi_{i+\frac{k_F}{2\Lambda}}(k_1,\epsilon_1)\,\delta\left(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4\right) \end{aligned} \tag{12.32}$$

We have broken the angular integral into a summation over Fermi surface 'patches' and an integral over each patch. Hence, $F(\theta_i - \theta_j)$ has been replaced by F_{ij}. By restricting to F_{ij} rather than allowing u_{ijkl}, we have automatically restricted to nearly forward scattering – i.e. to scattering from one point to another within the same patch. Furthermore, the δ-function,

$$\delta\left(\vec{k}_1+\vec{k}_2-\vec{k}_3-\vec{k}_4\right) \tag{12.33}$$

does not contain any momenta of $O(k_F)$; the $\vec{k}_i$'s live within patches and, therefore, are all less than the cutoff.

In this expression,

$$\begin{aligned} \int d^2k &= \int_{-\Lambda}^{\Lambda} dl \int_0^{2\pi\Lambda/k_f} d\theta \\ &= \int_{-\Lambda}^{\Lambda} dk_\perp \int_{-\pi\Lambda}^{\pi\Lambda} dk_\parallel \end{aligned} \tag{12.34}$$

so both momenta have cutoff $\sim \Lambda$.

Hence, if we rescale all momenta by Λ and the field, ψ, as well:

$$\begin{aligned} \omega &\to \frac{\omega}{\Lambda} \\ k_\perp &\to \frac{k_\perp}{\Lambda} \\ k_\perp &\to \frac{k_\parallel}{\Lambda} \\ \psi &\to \Lambda^2 \psi \end{aligned} \tag{12.35}$$

we can rewrite the action as:

$$\begin{aligned} S = &\sum_{i=0}^{k_F/\Lambda} \int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \psi_i^\dagger \left(i\epsilon - v_F l\right) \psi_i \\ &+ \sum_{i,j=0}^{k_F/\Lambda} \int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi} \times \\ &\qquad \frac{\Lambda}{k_F} F_{ij}\, \psi_j^\dagger(k_3,\epsilon_3)\psi_j(k_2,\epsilon_2)\, \psi_i^\dagger(k_4,\epsilon_4)\psi_i(k_1,\epsilon_1)\, \delta\left(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4\right) \\ &+ \sum_{i,j=0}^{k_F/\Lambda} \int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi} \times \\ &\qquad \frac{\Lambda}{k_F} V_{ij}\, \psi_j^\dagger(k_3,\epsilon_3)\psi^\dagger_{j+\frac{k_F}{2\Lambda}}(k_4,\epsilon_4)\, \psi_i(k_2,\epsilon_2)\psi_{i+\frac{k_F}{2\Lambda}}(k_1,\epsilon_1)\, \delta\left(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4\right) \end{aligned} \tag{12.36}$$

In other words, if we write $N = k_F/\Lambda$,

$$\begin{aligned} S = &\sum_{i=0}^{N} \int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \psi_i^\dagger \left(i\epsilon - v_F l\right) \psi_i \\ &+ \sum_{i,j=0}^{N} \int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi} \times \\ &\qquad \frac{1}{N} F_{ij}\, \psi_j^\dagger(k_3,\epsilon_3)\psi_j(k_2,\epsilon_2)\, \psi_i^\dagger(k_4,\epsilon_4)\psi_i(k_1,\epsilon_1)\, \delta\left(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4\right) \\ &+ \sum_{i,j=0}^{N} \int \frac{d\epsilon_1}{2\pi}\frac{d\epsilon_2}{2\pi}\frac{d\epsilon_3}{2\pi}\frac{d^2k_1}{2\pi}\frac{d^2k_2}{2\pi}\frac{d^2k_3}{2\pi}\frac{d^2k_4}{2\pi} \times \\ &\qquad \frac{1}{N} V_{ij}\, \psi_j^\dagger(k_3,\epsilon_3)\psi^\dagger_{j+\frac{k_F}{2\Lambda}}(k_4,\epsilon_4)\, \psi_i(k_2,\epsilon_2)\psi_{i+\frac{k_F}{2\Lambda}}(k_1,\epsilon_1)\, \delta\left(\vec{k}_1 + \vec{k}_2 - \vec{k}_3 - \vec{k}_4\right) \end{aligned} \tag{12.37}$$

then we see that we have a model in the large N limit.

Recall the analysis of the $O(N)$ model in the large N limit. The only $O(1)$ corrections to the two-point function are diagrams of the form of figure

??a. These shift the chemical potential. The non-trivial diagrams such as **??**b are $O(1/N)$. Consider now the correction to the four-point function. Only diagrams such as those of **??**a,b are $O(1/N)$. In the case of forward scattering, **??**a vanishes because both poles are on the same side of the axis. In the case of Cooper scattering, **??**b, gives a non-trivial contribution. The other corrections, such as those of **??**c are $O(1/N^2)$.

As we learned earlier in the context of $O(N)$ models, the large-N limit introduces the following simplifications. The only diagrams which need to be considered are the bubble diagrams. The one-loop RG is the full story. Consequently, the action (12.37) is a stable fixed point if $V_ij > 0$. A system of fermions which is controlled by this fixed point is called a *Fermi liquid*.

12.5 Quartic Interactions for Λ Finite

The scaling of generic four-fermi interactions is quite awkward for calculations at a finite frequency or temperature scale because the u's don't scale continuously. Thus, the scaling of a physical quantity which depends on the u's is determined not by the scaling of the u's, which is marginal, but on the *number of non-zero u's*, which is scale dependent (except in the important case where the quantity is determined by forward, exchange, or Cooper scattering – which do scale continuously). For such calculations, a different scaling transformation is useful. Suppose Λ is small but finite. Then, in two dimensions, we can consider nearly forward scatering, from $\vec{k}_1$, $\vec{k}_2$ to $\vec{k}_3$, $\vec{k}_4$ with $|\vec{k}_1 - \vec{k}_3| < \Lambda$, $|\vec{k}_2 - \vec{k}_4| < \Lambda$. Since all of the action is taking place in the neighborhods of $\vec{\Omega}_1$, $\vec{\Omega}_2$, we focus on these points. We construct cartesian coordinates, k_x (tangent to the Fermi surface) and k_y (perpendicular to the Fermi surface), at these two points on the Fermi surface. In the vicinity of these points,

$$\epsilon(k_x, k_y) = v_F \left(k_y + \frac{k_x^2}{2k_F} \right) \tag{12.38}$$

We now scale to $\vec{\Omega}_1$, $\vec{\Omega}_2$, using the scaling $k_y \to sk_y$, $k_x \to s^{1/2}k_x$, $\omega \to s\omega$. (We have assumed $d = 2$; in $d > 2$, there are $d - 1$ momenta which scale as k_x.) The same answers are obtained with either scaling transformation; it's just that some calculations are easier with this one. On the other hand, it's a less natural renormalization group transformation because it involves selecting preferred points on the Fermi surface and scaling differently at different points on the Fermi surface. Let's briefly see how this

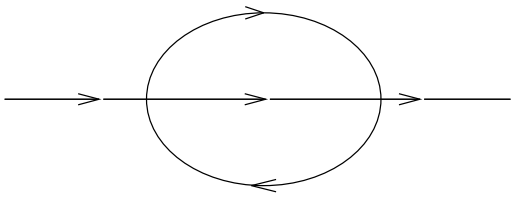


Figure 12.5: A one-loop self-energy correction.

works. The quadratic part of the Lagrangian is of the form:

$$S_0 = \int d\omega\, dk_y dk_x \{\psi^\dagger \left(i\omega - v_F \left(k_y + \frac{k_x^2}{2k_F} \right) \right) \psi\} \tag{12.39}$$

Hence, the field now scales as $\psi \to s^{-7/4}\psi$, so four-fermi interactions,

$$S_4 = \int d\omega_1 d\omega_2 d\omega_3\, d^2k_1 d^2k_2 d^2k_3\, u(k_1,k_2,k_3)\, \psi^\dagger(k_4,\omega_4)\psi^\dagger(k_3,\omega_3)\psi(k_2,\omega_2)\psi(k_1,\omega_1) \tag{12.40}$$

scale as $s^{1/2}$.The scaling is perfectly continuous. If $k_1, k_2, k_3, k_4 = k_1+k_2-k_3$ lie within the cutoff Λ, then they continue to do so under this renormalization group transformation. If we insert a $\delta(k_{1x} - k_{3x})$ or $\delta(k_{1x} - k_{4x})$ into the integrand, then we get a marginal interaction, namely forward scattering, as before.[5] To see why this is a useful scaling, consider the diagram in figure 12.5. It has a real part, proportional to $F_f^2\omega$ which comes from the marginal forward scattering interaction, and an imaginary part, proportional to $F_{nf}^2\omega^2$ coming from irrelevant non-forward processes, in agreement with the explicit calculation which we did in chapter 13. The above scaling immediately yields the suppression of F_{nf}^2 with respect to F_f^2 by one power of ω, a result which is more cumbersome to derive with the other RG transformation.

12.6 Zero Sound, Compressibility, Effective Mass

As a result of the preceeding analysis, the density-density correlation function can be computed by summing bubble diagrams. Other diagrams are down by powers of $1/N$. The approximation which consists of neglecting these other diagrams is called the RPA, or random-phase approximation (for historical reasons). The bubble diagrams form a geometric series which may be summed. Let us consider the simplest case, in which F is a constant,

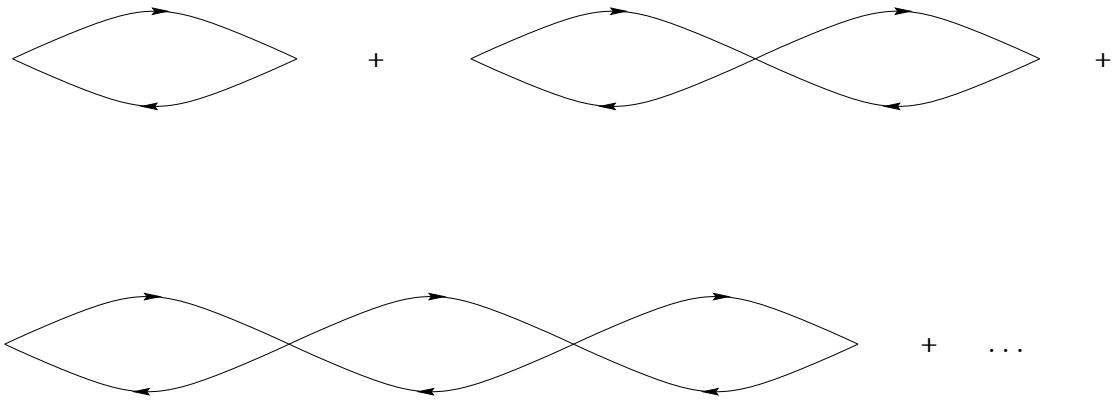


Figure 12.6: The geometric series of bubble diagrams which determine $\langle \rho(q,i\omega)\,\rho(-q,-i\omega)\rangle$ to $O(1)$.

$F(\Omega_1 \cdot \Omega_2) = F_0$:

$$\begin{aligned}\langle \rho(q,i\omega)\,\rho(-q,-i\omega)\rangle &= I(q,i\omega) + (I(q,i\omega))^2 F_0 + \ldots + (I(q,i\omega))^{n+1} F_0^n + \ldots \\ &= \frac{I(q,i\omega)}{1 - I(q,i\omega)F_0} \end{aligned} \tag{12.41}$$

where I is the value of a single particle-hole bubble. In the limit of $q \ll k_F$, this is:

$$\begin{aligned} I(q,i\omega) &= \int \frac{d\epsilon}{2\pi}\,\frac{d^3k}{(2\pi)^3}\,\mathcal{G}(i\epsilon,k)\,\mathcal{G}(i\epsilon+i\omega,k+q) \\ &= k_F^2 \int \frac{d\epsilon}{2\pi}\,\frac{dl\,d\varphi}{(2\pi)^3}\,d(\cos\theta)\,\frac{1}{i\epsilon - v_F l}\,\frac{1}{i\epsilon + i\omega - v_F l - v_F q\cos\theta} \\ &= k_F^2 \int \frac{dl}{(2\pi)^2}\,d(\cos\theta)\,\frac{\theta(l) - \theta(l + q\cos\theta)}{i\omega - v_F q\cos\theta} \\ &= \frac{k_F^2}{(2\pi)^2}\int d(\cos\theta)\,\frac{q\cos\theta}{i\omega - v_F q\cos\theta} \\ &= \frac{k_F^2}{4\pi^2 v_F}\int_{-1}^{1} dx\,\frac{x}{\frac{i\omega}{v_F q} - x} \\ &= \frac{k_F^2}{2\pi^2 v_F}\left[\frac{1}{2}\,\frac{i\omega}{v_F q}\,\ln\left(\frac{i\omega + v_F q}{i\omega - v_F q}\right) - 1\right] \end{aligned} \tag{12.42}$$

The retarded density-density correlation function is a response function, $-\chi_{\rho\rho}$, of the type which we discussed in chapter 7. If we imagine changing the chemical potential by a frequency- and wavevector-dependent amount,

$\delta\mu(\omega, q)$, then the action changes by

$$S \to S - \int \frac{d\omega}{2\pi} \frac{d^d q}{(2\pi)^d} \, \delta\mu(\omega, q) \, \rho(\omega, q) \tag{12.43}$$

Hence, following the steps of chapter 7, we have

$$\langle \delta\rho(\omega, q) \rangle = \chi_{\rho\rho}(\omega, q) \, \delta\mu(\omega, q) \tag{12.44}$$

Since it reflects the density change resulting from a variation of the chemical potential, it is called the *compressibility*. As usual, a pole of $\chi_{\rho\rho}(q, \omega)$ on the real axis is a propagating mode. According to (12.41), there is such a pole when:

$$\frac{1}{F_0} = \frac{k_F^2}{2\pi^2 v_F} \left[\frac{1}{2} \frac{\omega}{v_F q} \ln \left(\frac{\omega + v_F q}{\omega - v_F q} \right) - 1 \right] \tag{12.45}$$

or,

$$\frac{1}{F_0} = \frac{k_F^2}{2\pi^2 v_F} \left[\frac{1}{2} s \ln \left(\frac{s+1}{s-1} \right) - 1 \right] \tag{12.46}$$

where $s = \omega / v_F q$ The solution of this equation occurs for $s > 1$, i.e. there is a mode with $\omega = s v_F q$. In other words, this mode, called *zero sound*, has a velocity of propagation, $s v_F$, which is greater than the Fermi velocity. Figure 12.7 shows the allowed phase space for a particle-hole pair. The continuum of states composed of a particle-hole pair lies beneath the line $\omega = v_F q$ for q small. Since the zero-sound mode lies outside this continuum for q small, it cannot decay; this explains why it is a propagating mode. Energy and momentum conservation *do* allow it to decay into multiple particle-hole pairs, but the interactions which would allow such decay are six-fermion and higher interactions which are highly irrelevant.

According to (12.41),

$$\chi_{\rho\rho}(q, \omega) = \frac{\chi^0_{\rho\rho}(q, \omega)}{1 + F_0 \chi^0_{\rho\rho}(q, \omega)} \tag{12.47}$$

where $\chi^0_{\rho\rho}$ is the compressibility in the absence of interactions. Let us consider the static compressibility, $\chi_{\rho\rho}(q \to 0, 0)$, for both the interacting and non-interacting systems. From (12.42),

$$\begin{aligned} \chi^0_{\rho\rho}(q \to 0, 0) &= \frac{k_F^2}{2\pi^2 v_F} \\ &= \frac{m^* k_F}{2\pi^2} \end{aligned} \tag{12.48}$$

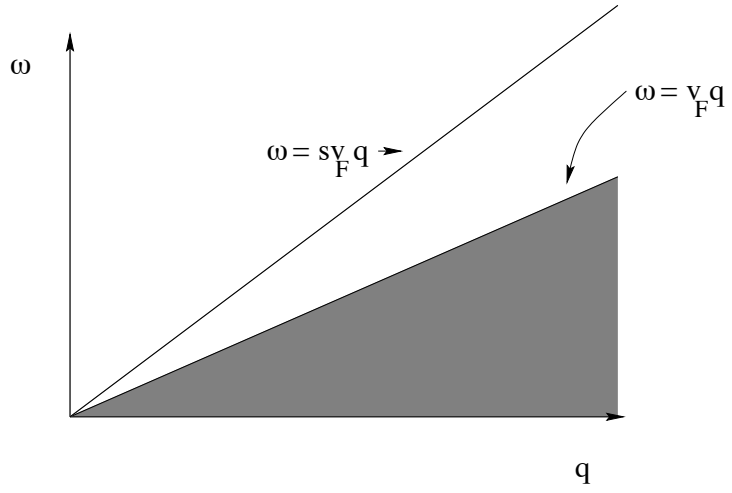


Figure 12.7: The allowed ω, q values of the the zero sound mode and the continuum of particle-hole excitations.

where $m^* \equiv k_F/v_F$. Hence,

$$\frac{\chi_{\rho\rho}(q \to 0, 0)}{\chi^0_{\rho\rho}(q \to 0, 0)} = \frac{1}{1 + \frac{k_F^2}{2\pi^2 v_F} F_0} \tag{12.49}$$

The compressibility is decreased by interactions if we assume that m^* is the same in both the interacting and non-interacting systems. However, this is usually not the case.

Consider the behavior of a Fermi liquid under a Galilean boost by $\delta\vec{v} = \delta\vec{p}/m$. The kinetic term in the action transforms, but the potential energy is invariant.

$$S = \int d\tau\, d^d x \left(\psi^\dagger \left(\partial_\tau - \left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi + \psi^\dagger(\vec{x})\psi(\vec{x})\, V(\vec{x} - \vec{x}')\, \psi^\dagger(\vec{x}')\psi(\vec{x}') \right) \tag{12.50}$$

Similarly for the Hamiltonian,

$$H = \int d^d x \left(\psi^\dagger \left(\left(\frac{\nabla^2}{2m} - \mu \right) \right) \psi + \psi^\dagger(\vec{x})\psi(\vec{x})\, V(\vec{x} - \vec{x}')\, \psi^\dagger(\vec{x}')\psi(\vec{x}') \right) \tag{12.51}$$

so the energy transforms as:

$$\delta E = \vec{P} \cdot \delta\vec{p}/m \tag{12.52}$$

where $\vec{P}$ is the total momentum. If we consider a state with a filled Fermi sea – which has momentum zero – and a quasiparticle at the Fermi energy, and we boost the system in the direction of the quasiparticle's momentum, then

$$\delta E = k_F \delta p/m \tag{12.53}$$

On the other hand, we can compute the energy change using Fermi liquid theory. The boost shifts the quasiparticle momentum by δp and also moves the Fermi sea by this amount. This doesn't affect its momentum to lowest order in δp, but it does change $\psi_i^\dagger \psi_i$ by $\delta p \cos\theta_i$. Hence, the energy shift of this state is also

$$\begin{aligned} \delta E &= v_F \delta p + \delta p \int \frac{d^3\Omega}{(2\pi)^3} F(\theta,\phi)\, \cos\theta \\ &= v_F \delta p + \delta p \frac{1}{3} F_1 \end{aligned} \tag{12.54}$$

Hence, comparing these two expressions and using $m^* = k_F/v_F$, we have

$$\frac{m^*}{m} = 1 + \frac{1}{3}\left(\frac{F_1}{2\pi^2 v_F}\right) \tag{12.55}$$

Consequently, the ratio of the interacting and free compressibilities is:

$$\frac{\chi_{\rho\rho}(q \to 0, 0)}{\chi_{\rho\rho}^{\text{free}}(q \to 0, 0)} = \frac{1 + \frac{1}{3}\left(\frac{F_1}{2\pi^2 v_F}\right)}{1 + \frac{k_F^2}{2\pi^2 v_F} F_0} \tag{12.56}$$

a ratio which depends on the relative strengths of F_0 and F_1.

CHAPTER 13

Electrons and Coulomb Interactions

13.1 Ground State

Thus far, we have assumed that there are only short-range interactions between the fermions in our system. This assumption is appropriate for ^{3}He, but not for electrons in metals which interact through the Coulomb interaction, $V(r) = \kappa e^2/r$. When the Coulomb interaction energy is large compared to the kinetic energy, we expect the electrons to form a *Wigner crystal*. If, on the other hand, the Coulomb energy is small compared to the kinetic energy, we expect the electrons to form some kind of liquid state; later in this chapter, we will show that this liquid is a Fermi liquid.

A naive comparison of these energies estimates the kinetic energy by the kinetic energy of a free Fermi gas and the interaction energy from the average Coulomb energy of a system of electrons at that density:

$$\frac{E_{\text{Coulomb}}}{E_{\text{Kinetic}}} = (\text{const.})\, r_s \tag{13.1}$$

where r_s is ratio of the interparticle spacing to the effective Bohr radius in the metal

$$r_s = \left(\frac{3}{4\pi n}\right)^{\frac{1}{3}} a_0^{-1} \tag{13.2}$$

and $a_0 = 1/m\kappa e^2$ is the effective Bohr radius in the metal. Stated differently, charge e is enclosed within a sphere of radius $r_s a_0$. r_s is the controlling

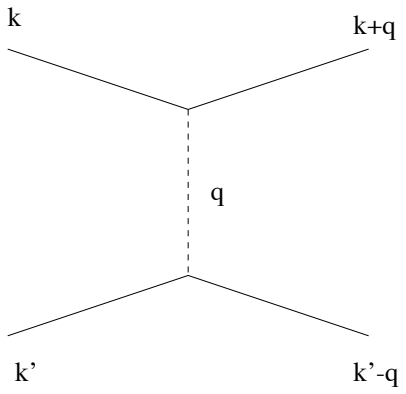


Figure 13.1: The graphical representation of Coulomb interactions.

parameter for many of the approximations which we make in this chapter. r_s is small in the high-density limit where we expect a Fermi liquid and large in the low-density limit where we expect Wigner crystallization.

Better estimates of the Wigner crystal and electron liquid energies can be obtained in the low- and high-density limits for a model of electrons in a fixed uniform background of positive charge (the *jellium* model). In the Wigner crystal state, this can be estimated to be:

$$\frac{E_g^{WC}}{n} \approx \frac{2.2099}{r_s^2} - \frac{1.7}{r_s} \tag{13.3}$$

in units of the Rydberg, $13.6eV$.

In the liquid state, this can be computed perturbatively from the Lagrangian:

$$\begin{aligned} S = & \int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{1}{2m} k^2 \, \psi^\dagger(k)\psi(k) \right) \\ & + \int d\tau \int d^3k \, d^3k' \, d^3q \left(\psi^\dagger(k+q)\psi(k) - n\delta(q) \right) \times \\ & \frac{4\pi\kappa e^2}{q^2} \left(\psi^\dagger(k'-q)\psi(k') - n\delta(q) \right) \end{aligned} \tag{13.4}$$

The Coulomb interaction is represented by a dotted line, as in figure 13.1

If we expand the ground state energy perturbatively, the zeroth-order term is the kinetic energy. The first-order terms come from diagrams of (a) and (b) of figure 13.2. The first – or Hartree – term vanishes as a result of

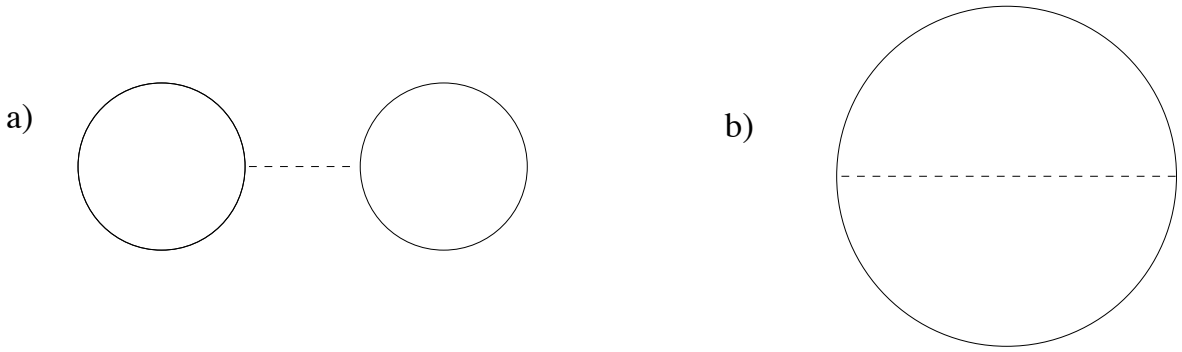


Figure 13.2: The (a) Hartree and (b) Fock contributions to the ground state energy of the electron gas.

the neutralizing background (i.e. the $n\delta(q)$). The second – or Fock – term is non-vanishing. In the *Hartree-Fock* approximation, the ground state energy is given by:

$$E_g^L = \frac{2.2099}{r_s^2} - \frac{0.9163}{r_s} \tag{13.5}$$

where the first term is the kinetic energy and the second term is the *exchange energy*.

The next terms in the expansion in r_s come from summing the diagrams of figure 13.3. The first term in this series is infrared divergent, but the sum is convergent:

$$E_g^L = \frac{2.2099}{r_s^2} - \frac{0.9163}{r_s} - 0.094 + 0.0622 \ln r_s \tag{13.6}$$

This is the sum over bubble diagrams – the *Random Phase Approximation* – which we encountered in the small Λ/k_F approximation for a Fermi liquid. In this context, it is justified for a calculation of the ground state energy in the small r_s limit since the neglected diagrams give contributions of $O(r_s)$.

For r_s large, the ground state of the electron gas is the Wigner crystal. For r_s small, it is the liquid state, the nature of which we discuss in this chapter.

13.2 Screening

In the presence of Coulomb interactions, naive perturbation theory is infrared divergent because the interaction $V(q) = 4\pi\kappa e^2/q^2$ (unless otherwise

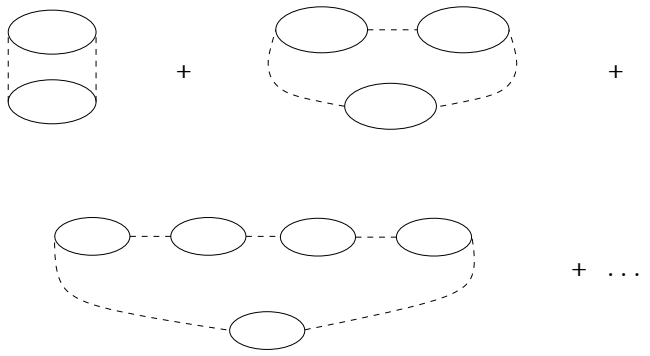


Figure 13.3: The RPA contributions to the ground state energy of the electron gas.

specified, we work in $d = 3$ in this chapter) is singular in the $q \to 0$ limit. In the language of the last chapter, we cannot divide the Fermi surface into N patches and justify Fermi liquid theory in the large N limit because the interaction $V(q) = 4\pi\kappa e^2/q^2$ is singular within a single patch when $q \to 0$.

However, the 'bare' Coulomb interaction, $V(q) = 4\pi\kappa e^2/q^2$, is not the actual interaction between two electrons. In fact, the interaction between any two electrons will be far weaker because all of the other electrons will act to *screen* the Coulomb interaction. The correct strategy for dealing with electrons with Coulomb interactions is to do perturbation theory in the *screened* Coulomb interaction. This can be done systematically, as we show in the next two sections.

First, however, we recall the Thomas-Fermi model, a simple model for screening which illustrates the basic physics in the low q, ω limit. To understand the physics at $q \to 2k_F$, we'll have to use more sophisticated approximations such as the RPA. Let us imagine that we have test charges described by ρ_{ext}. In a metal, they will induce a charge distribution ρ_{ind}. According to the Laplace equation

$$\frac{1}{\kappa}\nabla^2\phi = 4\pi\rho_{\text{ext}} + 4\pi\rho_{\text{ind}} \tag{13.7}$$

κ is the dielectric constant due to the ions and the core electrons. In the Thomas-Fermi aproximation, we assume that ϕ is very slowly-varying so that we can make the approximation that the local density is given by $n(\mu + e\phi(r))$:

$$\rho_{\text{ind}}(r) \approx -e\left(n(\mu + e\phi(r)) - n(\mu)\right)$$

$$\approx -e^2 \frac{\partial n}{\partial \mu}\, \phi(r) \tag{13.8}$$

Then,

$$\left(\frac{1}{\kappa} q^2 + 4\pi e^2 \frac{\partial n}{\partial \mu} \right) \phi = 4\pi \rho_{\rm ext} \tag{13.9}$$

In other words, the bare Coulomb interaction has been replaced by a screened Coulomb interaction:

$$\frac{4\pi\kappa e^2}{q^2} \to \frac{4\pi\kappa e^2}{q^2 + k_0^2} \tag{13.10}$$

or

$$\frac{1}{r} \to \frac{e^{-k_0 r}}{r} \tag{13.11}$$

where k_0 is the inverse of the Thomas-Fermi screening length,

$$k_0 = \left(4\pi\kappa e^2 \frac{\partial n}{\partial \mu} \right)^{\frac{1}{2}} \tag{13.12}$$

For a free Fermi gas, $\mu = (3\pi n)^{2/3}/2m$, so the screening length is

$$k_0^{-1} = \left(\frac{\pi}{4} \left(\frac{4}{9} \right)^{1/3} \right)^{\frac{1}{2}} a_0\, r_s^{\frac{1}{2}} \tag{13.13}$$

When r_s is small, i.e. when the density is large, the screening length is short and the Coulomb interaction is effectively screened. When this is true, we expect the potential to be slowly-varying and the Thomas-Fermi approximaton to be reasonable. When r_s is large, however, the screening length is large and we do not expect the Thomas-Fermi approximation to be valid.

A more refined result may be obtained by by replacing the bare Coulomb interaction of figure 13.1 by the sum of the diagrams of figure **??**. Restricting attention to the sum of bubble diagrams is, again, the RPA approximation. The effective interaction, $V_{\rm eff}^{\rm RPA}(q,\omega)$, is:

$$\begin{aligned} V_{\rm eff}^{\rm RPA}(q,\omega) &= V(q) + V(q)\, I(q,\omega)\, V(q) + V(q)\, I(q,\omega)\, V(q)\, I(q,\omega)\, V(q) + \ldots \\ &= \frac{V(q)}{1 - I(q,\omega) V(q)} \end{aligned} \tag{13.14}$$

where $I(q,\omega)$ is the particle-hole bubble which we evaluated in the last chapter. For q small,

$$\begin{aligned} V_{\rm eff}^{\rm RPA}(q,0) &= \frac{V(q)}{1+\frac{m^*k_F}{2\pi^2}V(q)} \\ &= \frac{4\pi\kappa e^2}{q^2} + 4\pi\kappa e^2\,\frac{m^*k_F}{2\pi^2} \\ &= \frac{4\pi\kappa e^2}{q^2+k_0^2} \end{aligned} \tag{13.15}$$

which is the same asthe Thomas-Fermi result. However, for $\omega \neq 0$, the RPA result contains additional information about the dynamics of the electrons. Also, for $q \to 2k_F$, the RPA result contains information about the Fermi surface. Of course, it is not clear why we can restrict attention to the sum of bubble diagrams. As we will see below, this sum gives the leading contribution in r_s *in the limit of small* ω, q. For $\omega \neq 0$ and $q \to 2k_F$, the RPA approximation can be called into question.

13.3 The Plasmon

Although Coulomb interactions are ultimately screened and therefore allow a Fermi liquid treatment, there are non-trivial differences with the case of short-range interactions. The zero-sound mode no longer has linear dispersion, $\omega = v_s q$. This may be seen at a classical level from Maxwell's equations together with the continuity equation.

$$\begin{aligned} \frac{1}{\kappa}\nabla^2\phi &= 4\pi\rho \\ m\,\frac{d\vec{j}}{dt} &= ne^2\,\vec{\nabla}\phi \\ \frac{d\rho}{dt} + \vec{\nabla}\cdot\vec{j} &= 0 \end{aligned} \tag{13.16}$$

Combining these equations for a longitudinal disturbance, $\vec{j} = |j|\vec{q}/|q|$, we have

$$\left(\omega^2 + \frac{4\pi\kappa ne^2}{m}\right)\rho(q,\omega) = 0 \tag{13.17}$$

Hence, the frequency of a longitudinal density modulation is the plasma frequency, ω_p

$$\omega_p = \left(\frac{4\pi\kappa ne^2}{m}\right)^{\frac{1}{2}} \tag{13.18}$$

rather than the gapless dispersion of zero sound in a neutral Fermi liquid.

The same result may be seen by, again, considering the RPA sum of bubble diagrams which determines the density-density correlation fucntion. The Landau parameter, F_0, is replaced in (12.41) by the Coulomb interaction, $V(q)$. Consequently, the pole in this correlation function now occurs at:

$$\frac{1}{V(q)} = \frac{k_F^2}{2\pi^2 v_F}\left[\frac{1}{2}\,\frac{\omega}{v_F q}\,\ln\left(\frac{\omega + v_F q}{\omega - v_F q}\right) - 1\right] \tag{13.19}$$

On the right-hand-side, take the $q \to 0$ limit with $q \ll \omega$.

$$\begin{aligned}\frac{q^2}{4\pi\kappa e^2} &= \frac{k_F^2}{2\pi^2 v_F}\left[\frac{1}{2}\,\frac{\omega}{v_F q}\,\left(2\,\frac{v_F q}{\omega} + \frac{2}{3}\left(\frac{v_F q}{\omega}\right)^3 + \ldots\right) - 1\right] \\ &= \frac{1}{3}\,\frac{k_F^2}{2\pi^2 v_F}\left(\frac{v_F q}{\omega}\right)^2\end{aligned} \tag{13.20}$$

or,

$$\omega^2 = 4\pi\kappa e^2\,\frac{v_F k_F^2}{6\pi^2} \tag{13.21}$$

which is the same as (13.18).

Since $V(q) \to \infty$ as $q \to 0$, $\langle \rho(q,\omega)\,\rho(-q,-\omega)\rangle \to 0$ in this limit. One might be tempted to conclude that the compressibility of the electron gas vanishes. However, the density-density correlation function gives the compressibility in response to the *applied field*:

$$\delta\,\langle \rho(q,\omega)\rangle = \langle \rho(q,\omega)\,\rho(-q,-\omega)\rangle\;\delta\phi_{\rm ext}(q,\omega) \tag{13.22}$$

In linear response, $\phi_{\rm ind}(q,\omega)$ is given by,

$$\phi_{\rm ind}(q,\omega) = -\,\frac{4\pi\kappa e^2}{q^2}\,\chi^0_{\rho\rho}(q,\omega)\,\phi(q,\omega) \tag{13.23}$$

Hence,

$$\delta\,\langle \rho(q,\omega)\rangle = \langle \rho(q,\omega)\,\rho(-q,-\omega)\rangle\;\left(1 + \frac{4\pi\kappa e^2}{q^2}\,\chi^0_{\rho\rho}(q,\omega)\right)\delta\phi(q,\omega) \tag{13.24}$$

so the compressibility is *finite* as $q \to 0$.

In this section, we will show, following Bohm and Pines, how to separate the plasma oscillation from the rest of the degrees of freedom of an electronic system. When this is done, the remaining electronic degrees of freedom interact through a short-ranged, screened Coulomb interaction. Essentially, gauge invariance tells us that longitudinal photons – whose exchange gives

rise to the Coulomb interaction – and density fluctuations are not distinct objects. When long-wavelength density fluctuations aquire a mass gap as a result of their self-interaction, they (and the longitudinal photons to which they are equivalent) can no longer propagate over long-distances. Consequently, the Coulomb interaction becomes short-ranged.

To exhibit this clearly, we make the following manipulations:

- *Electrons with Coulomb Interactions.* We begin with the action of a system of electrons with Coulomb interactions:

$$S = \int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{1}{2m} k^2 \, \psi^\dagger(k)\psi(k) \right) + \int d\tau \int d^3k \, d^3k' \, d^3q \, \psi^\dagger(k+q)\psi(k) \, \frac{4\pi\kappa e^2}{q^2} \, \psi^\dagger(k'-q)\psi(k') \tag{13.25}$$

- *Electrons interacting with Longitudinal Photons.* The long-range $1/q^2$ interaction results from integrating out the longitudinal part of the electromagnetic field. We could equivalently write this as

$$S = \int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + e\psi^\dagger A_0 \psi + \frac{1}{2m} \psi^\dagger(k) \left(\vec{k} + e\vec{A} \right)^2 \psi(k) \right) + \int d\tau \int d^3k \, \frac{1}{8\pi\kappa} E(k)E(-k) \tag{13.26}$$

The magnetic part of the electromagnetic action has been dropped since we assume that all velocities are much smaller than the speed of light; we keep only the longitudinal modes of the electromagnetic field. Equation (13.25) is obtained from (13.26) by integrating out the electromagnetic field. To do this, we will choose Coulomb gauge, $A_0 = 0$. In doing so we must, however, impose the Gauss' law constraint which is the A_0 equation of motion.

$$S = \int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{1}{2m} \psi^\dagger(k) \left(\vec{k} + e\vec{A} \right)^2 \psi(k) \right) + \int d\tau \int d^3k \, \frac{1}{8\pi\kappa} (\partial_\tau A(k))^2 + \frac{1}{4\pi\kappa} A_0 \left(\vec{\nabla} \cdot \vec{E} - 4\pi\kappa e \psi^\dagger \psi \right) \tag{13.27}$$

Note that $A(k)$ is a scalar field because it is only the longitudinal part of the electromagnetic field – which not independent of the density fluctuations of the electrons. The real dynamics of the electromagnetic field is in its transverse components, which do not enter here. If we were to integrate out $A(k)$, then, since $A(k)$ is gapless at tree-level, we would get Coulomb interactions between electrons. However, a tree-level analysis misses the fact that $A(k)$ is, in fact, not gapless.

- *Integrate out Short-Wavelength Photons.* Instead of integrating out $A(k)$ fully, let us instead only integrate out those modes of $A(k)$ with $k > Q$ for some Q. Bohm and Pines did this at the Hamiltonian level, by applying a canonical transformation of the form:

$$U = e^{-i\int_Q^\Lambda d^3q\, a(q)\, \frac{\sqrt{4\pi\kappa e^2}}{q}\, \int d^3k\, \psi^\dagger(k+q)\psi(k)} \tag{13.28}$$

to the Hamiltonian corresponding to (13.27). Λ is the upper cutoff and Q is a wavevector to be determined.

Then, we obtain an action of the form:

$$\begin{aligned} S = &\int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{1}{2m}\,\psi^\dagger(k)\left(\vec{k}+e\vec{A}\right)^2 \psi(k)\right) \\ &+ \int d\tau \int_0^Q d^3q \left(\frac{1}{8\pi\kappa}(\partial_\tau A(q))^2 + \frac{1}{4\pi\kappa}A_0\left(\vec{\nabla}\cdot\vec{E}(q) - 4\pi\kappa e\psi^\dagger\psi\right)\right) \\ &+ \int d\tau \int d^3k\, d^3k' \int_Q^\Lambda d^3q\, \psi^\dagger(k+q)\psi(k)\, \frac{4\pi\kappa e^2}{q^2}\, \psi^\dagger(k'-q)\psi(k') \end{aligned} \tag{13.29}$$

Notice that the four-fermion interaction is now short-ranged since it is restricted to $|q| > Q$.

- *Isolate the term which gives a gap to long-wavelength Photons.* We now expand $(\vec{k}+e\vec{A})^2$:

$$\begin{aligned} S = &\int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{k^2}{2m}\,\psi^\dagger(k)\psi(k)\right) \\ &+ \int d\tau \int d^3k\, d^3k' \int_Q^\Lambda d^3q\, \psi^\dagger(k+q)\psi(k)\, \frac{4\pi\kappa e^2}{q^2}\, \psi^\dagger(k'-q)\psi(k') \\ &+ \int d\tau \int d^3k \int_0^Q d^3q\, \frac{e}{m}\left(\vec{k}+\frac{\vec{q}}{2}\right)\cdot\vec{A}(-q)\,\psi^\dagger(k+q)\psi(k) \\ &+ \int d\tau \int d^3k \int_0^Q d^3q \int_0^Q d^3q'\, \frac{e^2}{2m}A(q)A(q')\,\psi^\dagger(k-q-q')\psi(k) \\ &+ \int d\tau \int_0^Q d^3q \left(\frac{1}{8\pi\kappa}|\partial_\tau A(q)|^2 + \frac{1}{4\pi\kappa}A_0\left(\vec{\nabla}\cdot\vec{E}(q) - 4\pi\kappa e\psi^\dagger\psi\right)\right) \end{aligned} \tag{13.30}$$

We split the third line into a part which comes from the average density, n, and a part resulting from fluctuations in the density:

$$\int d\tau \int d^3k \int_0^Q d^3q \int_0^Q d^3q'\, \frac{e^2}{2m}A(q)A(q')\,\psi^\dagger(k-q-q')\psi(k) =$$

$$\int d\tau \int_0^Q d^3q \, \frac{ne^2}{2m} A(q)A(-q)$$
$$+ \int d\tau \int_0^Q d^3q \int_0^Q d^3q' \, \frac{e^2}{2m} A(q)A(q') \left(\int d^3k \psi^\dagger(k-q-q')\psi(k) \; - \; n\delta(q+q') \right) \tag{13.31}$$

the first term on the right-hand-side can be combined with the $|\partial_\tau A(q)|^2$ term to give $S_{\rm P}$ in the action:

$$S = S_{\rm FL} + S_{\rm P} + S_{\rm Int} + S_{\rm C} \tag{13.32}$$

with

$$\begin{aligned}
S_{\rm FL} &= \int d\tau \int d^3k \, \left(\psi^\dagger \partial_\tau \psi + \frac{k^2}{2m} \, \psi^\dagger(k)\psi(k) \right) \\
&\quad + \int d\tau \int d^3k \, d^3k' \int_Q^\Lambda d^3q \, \psi^\dagger(k+q)\psi(k) \, \frac{4\pi\kappa e^2}{q^2} \, \psi^\dagger(k'-q)\psi(k') \\
S_{\rm P} &= \frac{1}{8\pi\kappa} \int d\tau \int_0^Q d^3q \, \left(|\partial_\tau A(q)|^2 + \omega_p^2 |A(q)|^2 \right) \\
S_{\rm Int} &= \int d\tau \int d^3k \int_0^Q d^3q \, \frac{e}{m} \, \left(\vec{k} + \frac{\vec{q}}{2} \right) \cdot \vec{A}(-q) \, \psi^\dagger(k+q)\psi(k) \\
&\quad + \int d\tau \int_0^Q d^3q \int_0^Q d^3q' \, \frac{e^2}{2m} A(q)A(q') \left(\int d^3k \psi^\dagger(k-q-q')\psi(k) \; - \; n\delta(q+q') \right) \\
S_{\rm C} &= \int d\tau \int_0^Q d^3q \, \left(\frac{1}{4\pi\kappa} A_0 \left(\vec{\nabla} \cdot \vec{E}(q) - 4\pi\kappa e \psi^\dagger \psi \right) \right)
\end{aligned} \tag{13.33}$$

$S_{\rm FL}$ is the action of electrons with short-range interactions. $S_{\rm P}$ is the action of plasmon modes $A(q)$ with $|q| < Q$; these modes have frequency ω_p. If we were to integrate them out now, they would mediate a *short-range interaction*, not the long-range Coulomb interaction. $S_{\rm Int}$ describes the interaction between electrons and plasmons. $S_{\rm C}$ imposes the constraints which eliminate the additional degrees of freedom introduced with the plasmons; these degrees of freedom are not gauge invariant and are, therefore, unphysical.

By separating the plasmon from the other electronic degrees of freedom, we have obtained a theory of electrons with short-range interactions. The basic physics is already clear from (13.32). However, we are not yet in a position to make quantitative predictions. The interaction depends on a free parameter, Q, and is not the Thomas-Fermi interaction in the $\omega, q \to 0$ limit. To understand the electron gas at a quantitative level, we must consider $S_{\rm Int}$ and $S_{\rm C}$.

13.4 RPA

The conclusions which we drew at the end of the previous section were based on a neglect of $S_{\rm Int}$ and $S_{\rm C}$. In this section, we consider $S_{\rm Int}$ and the RPA approximation which simplifies it. We have used the term RPA in several contexts. The definition of the RPA is the following. *We neglect the coupling between $\rho(\vec{q})$ and $\rho(\vec{q}')$ if $\vec{q} \neq \vec{q}'$. In the computation of a correlation function at $\vec{q}$, we only consider diagrams in which the dotted Coulomb interaction line carries momentum $\vec{q}$. In other words, $V(\vec{q}')$ does not appear in these diagrams unless $\vec{q} = \vec{q}'$.* The RPA is justified in the limit of small r_s and the limit $q \to 0$. For the density-density response function or the ground state energy, this amounts to keeping only the bubble diagrams and neglecting other diagrams.

The first step is to choose a Q which optimizes $S_{\rm FL} + S_{\rm P}$, thereby making the effect of $S_{\rm Int}$ as small as possible. Without proof, we state that we can minimize the energy of the ground state of $S_{\rm FL} + S_{\rm P}$ (computed to lowest order in the screened Coulomb interaction) by taking

$$Q \approx k_F \left(\frac{r_s}{4}\right)^{\frac{1}{4}} \tag{13.34}$$

Physicaly, Q must be finite since, for q large, the plasmon mixes with the particle-hole continuum and is no longer a well-defined mode.

We now make the *Random Phase Approximation*, or RPA and completely neglect the term:

$$\int d\tau \int_0^Q d^3q \int_0^Q d^3q' \, \frac{e^2}{2m} A(q)A(q') \left(\int d^3k \psi^\dagger(k-q-q')\psi(k) \; - \; n\delta(q+q') \right) \tag{13.35}$$

in $S_{\rm Int}$. To justify the RPA, consider the effect of this term on the ground state energy. It shifts this energy by

$$\begin{aligned} \Delta E &\sim \frac{Q^3}{2k_F^3} Q^3 \omega_p \\ &\sim \left(\frac{r_s}{4}\right)^{\frac{3}{4}} Q^3 \omega_p \end{aligned} \tag{13.36}$$

Hence, the random phase approximation is valid in the small r_s limit since the energy shift is small compared to the plasmon zero-point energy.

We are now left with the action

$$S_{\rm RPA} = \int d\tau \int d^3k \left(\psi^\dagger \partial_\tau \psi + \frac{k^2}{2m} \psi^\dagger(k)\psi(k) \right)$$

$$
\begin{aligned}
&+\int d\tau \int d^3k\, d^3k' \int_Q^\Lambda d^3q\, \psi^\dagger(k+q)\psi(k)\, \frac{4\pi\kappa e^2}{q^2}\, \psi^\dagger(k'-q)\psi(k') \\
&+\frac{1}{8\pi\kappa}\int d\tau \int_0^Q d^3q\, \left(|\partial_\tau A(q)|^2 + \omega_p^2 |A(q)|^2\right) \\
&+\int d\tau \int d^3k \int_0^Q d^3q\, \frac{e}{m}\, \left(\vec{k}+\frac{\vec{q}}{2}\right)\cdot \vec{A}(-q)\, \psi^\dagger(k+q)\psi(k) \\
&+\int d\tau \int_0^Q d^3q\, \left(\frac{1}{4\pi\kappa} A_0 \left(\vec{\nabla}\cdot\vec{E}(q) - 4\pi\kappa e \psi^\dagger\psi\right)\right)
\end{aligned}
\tag{13.37}
$$

If we could ignore the last line, we would have a theory of electrons with short-range interactions together with gapped plasmons. At frequencies or temperatures much less than ω_p, we can ignore the plasmons, so we would have a Fermi liquid. However, the constraint cannot be ignored. Treating the electrons and plasmons as fully independent would be a double-counting of the degrees of freedom of the system. What we can do, instead, is decouple the plasmon from the electrons. When this is done, the constraint will only involve the particles. If we ignore the constraint – which is now a constraint on the electrons alone – then we can apply Fermi liquid theory to the electronic action. Fermi liquid theory (as we saw in the last chapter) instructs us to compute only bubble diagrams to obtain the screened Coulomb interaction.

13.5 Fermi Liquid Theory for the Electron Gas

Following Bohm and Pines, we now perform a canonical transformation,

$$
\begin{aligned}
\mathcal{O} &\to e^{-i\mathcal{S}} \mathcal{O} e^{i\mathcal{S}} \\
|\chi\rangle &\to e^{-i\mathcal{S}} |\chi\rangle
\end{aligned}
\tag{13.38}
$$

generated by $\mathcal{S}$:

$$
\mathcal{S} = \frac{e}{m}\int_0^Q d^3q \int d^3k \frac{1}{\omega(q,\psi,\psi^\dagger) - \vec{q}\cdot\vec{k} + q^2/2m} \left\{ \left(\vec{k}+\frac{\vec{q}}{2}\right)\cdot\vec{A}(-q)\, \psi^\dagger(k+q)\psi(k) \right. \tag{13.39}
$$

where

$$
\omega(q,\psi,\psi^\dagger) = \omega_p \left(1 + \frac{q^2}{2nm^2\omega_p^2}\left(\int d^3k\, k^2 \psi^\dagger(k)\psi(k)\right) + \frac{q^4}{8m^2\omega_p^2}\right) \tag{13.40}
$$

The principal results of this canonical transformation in the limit $r_s \to 0$ are:

- The elimination of the $A\psi^\dagger\psi$ interaction between plasmons and electrons
- The modification of the plasmon action to:

$$S_P = \frac{1}{8\pi\kappa}\int d\tau \int_0^Q d^3q \, \left(|\partial_\tau A(q)|^2 + \omega(q,\psi,\psi^\dagger)|A(q)|^2\right) \qquad (13.41)$$

- The replacement of the cutoff Coulomb interaction by the RPA screened Coulomb interaction,

$$\int_Q^\Lambda d^3q\, \psi^\dagger(k+q)\psi(k)\,\frac{4\pi\kappa e^2}{q^2}\,\psi^\dagger(k'-q)\psi(k') \rightarrow \int d^3q\, \psi^\dagger(k+q)\psi(k)\, V_{\rm RPA}(q)\, \psi^\dagger(k'-q)\psi(k') \qquad (13.42)$$

- The elimination of the plasmons from the contraints. The constraints now read:

$$\int d^3k \, \frac{\omega(q,\psi,\psi^\dagger)}{\left(\omega(q,\psi,\psi^\dagger)\right)^2 - \left(\vec{q}\cdot\vec{k} - q^2/2m\right)^2}\,\psi^\dagger(k+q)\psi(k) \;=\; 0 \quad (13.43)$$

  for $|q| < Q$.

Hence, we now have a theory of weakly-coupled electrons and plasmons. The electrons interact through a short-ranged interaction (which can be obtained by summing bubble diagrams). The contraints reduce the number of degrees of freedom of the electrons. For Q small, this is assumed to have a small effect on the electronic degrees of freedom.

CHAPTER 14

Electron-Phonon Interaction

14.1 Electron-Phonon Hamiltonian

14.2 Feynman Rules

14.3 Phonon Green Function

14.4 Electron Green Function

Let us consider the electron-phonon interaction,

$$S_{\text{el}-\text{ph}} = g \int d\tau \, d^3\vec{x} \, \psi^\dagger \psi \, \partial_i u_i \tag{14.1}$$

which couples electrons to transverse phonons. What effect does this have on the electron Green function?

The one-loop electron self-energy is given by the diagrams of figure 14.1. The first diagram just shifts the chemical potential. At zero-temperature, the second diagram gives a contribution:

$$\begin{aligned} \Sigma(i\epsilon, k) &= g^2 \int \frac{d\omega}{2\pi} \frac{d^3q}{(2\pi)^3} \, \mathcal{G}(i\epsilon - i\omega, k-q) \, \mathcal{D}(i\omega, q) \\ &= g^2 \int \frac{d\omega}{2\pi} \frac{d^3q}{(2\pi)^3} \frac{1}{i\epsilon - i\omega - \xi_{k-q}} \frac{q^2}{-\omega^2 - v^2 q^2} \end{aligned} \tag{14.2}$$

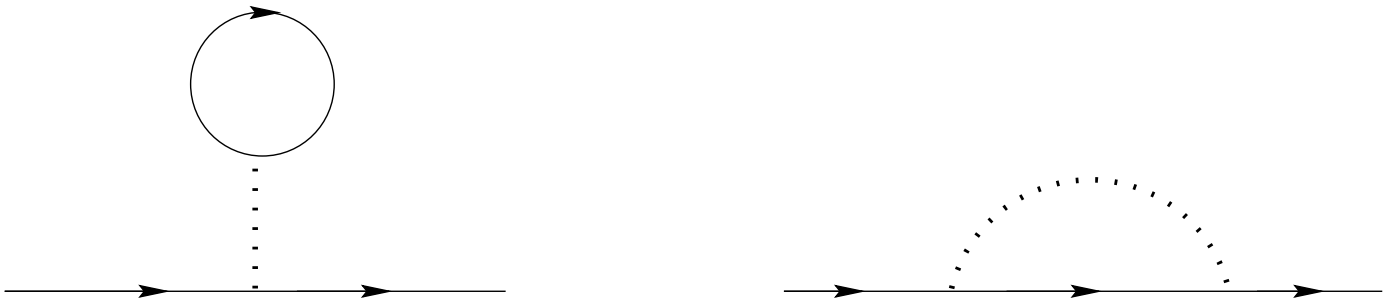


Figure 14.1: The one-loop diagrams contributing to the electron-self-energy.

Closing the contour in the upper-half-plane, we pick up the pole at $i\omega = i\epsilon - \xi_{k-q}$ if $\xi_{k-q} > 0$ and the pole at $i\omega = -vq$:

$$\Sigma(i\epsilon, k) = g^2 \int \frac{d^3q}{(2\pi)^3} \left[\frac{q^2}{-2qv} \frac{1}{i\epsilon + vq - \xi_{k-q}} + \frac{q^2\,\theta\left(\xi_{k-q}\right)}{\left(i\epsilon - \xi_{k-q}\right)^2 - v^2q^2} \right] \tag{14.3}$$

Analytically continuing $i\omega \to \omega + i\delta$ to obtain the retarded self-energy,

$$\mathrm{Re}\Sigma_{\mathrm{ret}}(\epsilon, k) = g^2 \int \frac{d^3q}{(2\pi)^3} \left[\frac{q}{-2v} \frac{1}{\epsilon + vq - \xi_{k-q}} + \frac{q^2\,\theta\left(\xi_{k-q}\right)}{\left(\epsilon - \xi_{k-q}\right)^2 - v^2q^2} \right] \tag{14.4}$$

At small ϵ, dropping a constant shift in the chemical potential, we have:

$$\mathrm{Re}\Sigma_{\mathrm{ret}}(\epsilon, k) = g^2\,\epsilon \int \frac{d^3q}{(2\pi)^3} \left[\frac{q}{2v} \frac{1}{\left(vq - \xi_{k-q}\right)^2} - \frac{2\xi_{k-q}\,q^2\,\theta\left(\xi_{k-q}\right)}{\left({\xi_{k-q}}^2 - v^2q^2\right)^2} \right] \tag{14.5}$$

We can take $k \to k_F$ in this integral to obtain the leading behavior:

$$\mathrm{Re}\Sigma_{\mathrm{ret}}(\epsilon, k) \sim g^2\,\epsilon \tag{14.6}$$

Meanwhile,

$$\begin{aligned} \mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, k) &= g^2 \int \frac{d^3q}{(2\pi)^3} \frac{q}{2v} \left[-\left(1 - \theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon + vq - \xi_{k-q}\right) - \theta\left(\xi_{k-q}\right)\,\delta\left(\epsilon - vq - \xi_{k-q}\right) \right] \\ &= \frac{g^2}{2(2\pi)^2} \int q^3 dq\, d(\cos\theta) \left[-\left(1 - \theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon + vq - \xi_{k-q}\right) \right. \\ &\qquad \left. - \theta\left(\xi_{k-q}\right)\,\delta\left(\epsilon - vq - \xi_{k-q}\right) \right] \end{aligned} \tag{14.7}$$

For ϵ small, $q \sim \epsilon$, and q^2 terms in the δ function can be dropped:

$$\mathrm{Im}\Sigma_{\mathrm{ret}}(\epsilon, k) = \frac{g^2}{2(2\pi)^2} \int q^3 dq\, d(\cos\theta) \left[-\left(1 - \theta\left(\xi_{k-q}\right)\right) \delta\left(\epsilon + \left(v + v_F\cos\theta\right) q\right) \right.$$

$$\begin{aligned} & - \theta\left(\xi_{k-q}\right)\ \delta\left(\epsilon - \left(v - v_F \cos\theta\right) q\right)] \\ \sim\ & g^2\, \epsilon^3 \end{aligned} \tag{14.8}$$

14.5 Polarons

CHAPTER 15

Rudiments of Conformal Field Theory

15.1 Introduction

In finite-temperature equilibrium statistical mechanical systems at their critical points or quantum systems at $T = 0$ with $z = 1$, scale invariance and rotational invariance (or pseudo-Lorentz invariance) give rise to the larger symmetry algebra of conformal transformations. In two or $1+1$ dimensions, this symmetry algebra is infinite-dimensional. Thus, it places strong constraints on the critical spectrum and correlation functions – much stronger than those due to, say, SU(2) symmetry. Some special conformal field theories – but, by no means, all – can be completely solved essentially through the representation theory of the conformal algebra (or, rather, its quantum version, the Virasoro algebra). Conformal field theory is a highly-developed subject which has taken on a life of its own, in part due to applications to string theory, so it is the subject of many full-length books and long review articles. Such an in-depth treatment would be out of place here. Rather, we would like to emphasize some of the key points, and use conformal field theory to illustrate some important concepts relating to critical points, on which this part of the book is focused. We also hope that our treatment of conformal field theory will help orient the reader who is interested in delving more deeply into the literature.

The techniques which we will use in this section, which are primarily algebraic, are the first truly non-perturbative ones which we have encountered

so far in this book. Once interesting feature, which will recur in the non-perturbative methods discussed in later parts of the book, is that the analysis is most easily done in *real space* rather than momentum space. The advantages of momentum space for perturbation theory – the ease with which differential operators are inverted and the simple expression of translational invariance as momentum conservation – are outweighed by the advantages of real space which we will discover in the following pages.

15.2 Conformal Invariance in 2D

A conformal transformation is any coordinate transformation which only changes the metric by a scale factor. We will be working in flat Euclidean space or Lorentzian spacetime, with metric $\eta_{\mu\nu} = diag(1, 1, \ldots, 1)$ or $\eta_{\mu\nu} = diag(-1, 1, \ldots, 1)$, so this means that $\eta_{\mu\nu}$ transforms as $\eta_{\mu\nu} \to \lambda(x)\,\eta_{\mu\nu}$.

$$x^\mu \to x^\mu + \epsilon^\mu \tag{15.1}$$

$$\begin{aligned} ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu &\to \eta_{\mu\nu} \left(dx^\mu + \partial_\alpha \epsilon^\mu dx^\alpha\right) \left(dx^\nu + \partial_\beta \epsilon^\nu dx^\beta\right) \\ &= \eta_{\mu\nu} dx^\mu dx^\nu + \left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) dx^\mu dx^\nu \end{aligned} \tag{15.2}$$

Such a transformation will be a conformal transformation if

$$\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) \propto \eta_{\mu\nu} \tag{15.3}$$

Comparing the traces of both sides, this can only be satisfied if

$$\left(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu\right) = \frac{2}{d} \left(\partial_\alpha \epsilon_\alpha\right) \eta_{\mu\nu} \tag{15.4}$$

For $d > 2$, there is a finite-dimensional group of such transformations comprised of translations, rotations, scale transformations, and special conformal transformations, which are inversion-translation-inversion combinations. In $d = 2$, this condition is simply the Cauchy-Riemann equations:

$$\begin{aligned} \partial_1 \epsilon_2 &= -\partial_2 \epsilon_1 \\ \partial_1 \epsilon_1 &= \partial_2 \epsilon_2 \end{aligned} \tag{15.5}$$

If we write $z, \overline{z} = x^1 \pm i x^2$, and $\epsilon, \overline{\epsilon} = \epsilon^1 \pm i \epsilon^2$, then ϵ is a holomorphic function of z, $\epsilon = \epsilon(z)$, while $\overline{\epsilon}$ is anti-holomorphic, $\overline{\epsilon} = \overline{\epsilon}(\overline{z})$.

Writing our coordinates and fields in terms of holomorphic and anti-holomorphic quantities will prove to be such a powerful tool, that all of the

subsequent development will be carried out in *real space* – or, rather, in the extension of it to the two complex dimensional space $(z, \overline{z}) \in C^2$, which contains real space as a section, $z = (\overline{z})^*$ – rather than in *momentum space* where most of our previous discussion of field theory has taken place.

This decoupling between holomorphic and anti-holomorphic – or right- and left-moving – degrees of freedom is a general feature of conformal field theory. Indeed, we can take this as a definition of a conformal field theory: *a 2D quantum field theory which has correlation functions which decouple in this way.* As we will see, this means that the correlation functions have simple scaling properties. Consider a free scalar field. Its equation of motion is $\partial_\mu \partial^\mu \varphi = 0$. We have been careful to write upper and lower indices because the metric tensor, which is used to lower indices, takes the form $\eta_{zz} = \eta_{\overline{z}\overline{z}} = 0$, $\eta_{z\overline{z}} = \eta_{\overline{z}z} = \frac{1}{2}$, in complex coordinates. Hence, the equation of motion takes the form $\partial_z \partial_{\overline{z}} \varphi \equiv \partial\overline{\partial}\varphi = 0$, where we have introduced the notation $\partial = \partial_z$, $\overline{\partial} = \partial_{\overline{z}}$. Thus, φ is the sum of an arbitrary holomorphic function and an arbitrary anti-holomorphic function.

The (classical) algebra of conformal transformations also decouples in this way. Consider the transformation $z \to z - \epsilon\, z^{n+1}$, where ϵ is infinitesimal. Such a transformation is generated by the linear operator $\ell_n = -z^{n+1}\partial$:

$$\begin{aligned} \delta f(z) &= f(z - \epsilon\, z^{n+1}) - f(z) \\ &= f(z) - \partial f \cdot \epsilon\, z^{n+1} - f(z) \\ &= -\epsilon\, z^{n+1} \partial f \\ &= \epsilon\, \ell_n f \end{aligned} \tag{15.6}$$

The analogous operator $\overline{\ell_n}$ generates transformations of the $\overline{z}$s.

By direct calculation, we see that the ℓ_ns and $\overline{\ell_n}$s generate two independent copies of the same algebra:

$$\begin{aligned} [\ell_m, \ell_n] &= (m-n)\ell_{m+n} \\ [\overline{\ell_m}, \overline{\ell_n}] &= (m-n)\overline{\ell_{m+n}} \\ [\ell_m, \overline{\ell_n}] &= 0 \end{aligned} \tag{15.7}$$

Note, however, that these transformations are not all globally well-defined. In particular, $\ell_n = -z^{n+1}\partial$ is non-singular as $z \to 0$ only for $n \geq -1$. Making the transformation $z = -1/w$ in order to study the $z \to \infty$ limit, we find that $\ell_n = -w^{1-n}\partial_w$ is non-singular as $w \to 0$ only for $n \leq 1$. Hence, $n = 0, \pm 1$ are the only globally well-defined transformations. $\ell_0 + \overline{\ell_0}$ generates scale transformations; $i\left(\ell_0 - \overline{\ell_0}\right)$ generates rotations; ℓ_{-1}, $\overline{\ell_{-1}}$ generate translations; and ℓ_1, $\overline{\ell_1}$ generate special conformal transformations.

The general form of a transformation generated by these operators is

$$z \to \frac{az+b}{cz+d} \tag{15.8}$$

with $ad - bc = 1$. This is the group $SL(2,C)/Z_2$ (the Z_2 is modded out because $a, b, c, d \to -a, -b, -c, -d$ leaves the transformation unchanged).

The special conformal transformations, $z \to z/(az+1)$ are somewhat less familiar than the others. Scale invariance is the defining property of a critical theory, and translational and rotational invariance are nice features which one would want most theories to have (at least in their continuum limits), but why special conformal transformations? As we will show in our discussion of the energy-momentum tensor, invariance under special conformal transformations is automatically a property of any theory which is scale and translationally invariant.

15.3 Constraints on Correlation Functions

Now, consider the transformation properties of the basic fields in a given theory. *Primary fields* generalize the notions of vectors and tensors under the rotation group to the conformal algebra. A primary field of weight $(h, \overline{h})$ transforms as

$$\Phi_{h,\overline{h}}(z,\overline{z}) \to \left(\frac{\partial f}{\partial z}\right)^h \left(\frac{\partial \overline{f}}{\partial \overline{z}}\right)^{\overline{h}} \Phi_{h,\overline{h}}\left(f(z), \overline{f}(\overline{z})\right) \tag{15.9}$$

under a conformal transformation $z \to f(z)$, $\overline{z} \to \overline{f}(\overline{z})$.

Under a rotation $z \to e^{i\theta}z$, $\overline{z} \to e^{-i\theta}\overline{z}$ a primary field of weight h transforms as $\Phi \to e^{i(h-\overline{h})\theta}\Phi$; under a scale transformation $z \to \lambda z$, $\overline{z} \to \lambda \overline{z}$ it transforms as $\Phi \to \lambda^{h+\overline{h}}\Phi$. Thus, a scalar under rotation has $h = \overline{h}$, while a vector has two components with $h = 1$, $\overline{h} = 0$ and $h = 0$, $\overline{h} = 1$.

Under an infinitesimal conformal transformation (on the holomorphic part of the theory; the anti-holomorphic part is analogous), a primary field transforms as

$$\begin{aligned} \delta_\epsilon \Phi_h(z) &= (1 + \partial\epsilon)^h \Phi_h(z + \epsilon(z)) - \Phi_h(z) \\ &= (h\partial\epsilon + \epsilon\partial)\Phi_h(z) \end{aligned} \tag{15.10}$$

Hence, the correlation function $\langle \Phi_1(z_1)\Phi_2(z_2)\rangle$ transforms as

$$\delta_\epsilon \langle \Phi_1(z_1\Phi_2(z_2)\rangle = [(h_1\partial_1\epsilon(z_1) + \epsilon(z_1)\partial_1) + (h_2\partial_2\epsilon(z_2) + \epsilon(z_2)\partial_2)] \langle \Phi_1(z_1)\Phi_2(z_2)\rangle \tag{15.11}$$

In a field theory which is invariant under the group of conformal (global) transformations, the vacuum state will be invariant under translations, rotations, and special conformal transformations. As a result, correlations functions will be invariant under these transformations. The implication of translational invariance may be seen by substituting $\epsilon(z) = 1$ in (15.11):

$$(\partial_1) + \partial_2) \langle \Phi_1(z_1)\Phi_2(z_2)\rangle = 0 \tag{15.12}$$

In other words, the correlation function is a function of $z_{12} = z_1 - z_2$. Similarly, its anti-holomorphic dependence is only on $\overline{z}_{12} = \overline{z}_1 - \overline{z}_2$ The implication of scale and rotational invariance may be seen by substituting $\epsilon(z) = z$ in (15.11):

$$[(z_1\partial_1) + h_1) + (z_2\partial_2 + h_2)] \langle \Phi_1(z_1)\Phi_2(z_2)\rangle = 0 \tag{15.13}$$

Combining this with the analogous equation for the anti-holomorphic dependence of the correlation function, we see that the correlation function vanishes unless $h_1 = h_2$ and

$$\langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\rangle = \frac{c_{12}}{z_{12}^{2h}\overline{z}_{12}^{2h}} \tag{15.14}$$

Thus, the two-point correlation function is reduced to single unknown constant. Since we can always redefine our fields in order to absorb this constant into their normalization, the two-point function is completely constrained.

We can also apply the constraint of conformal invariance to the three-point function. We find that it must take the form

$$\langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\Phi_3(z_3, \overline{z}_3)\rangle = \frac{c_{123}}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_1+h_3-h_2}} \times (\text{c.c.}) \tag{15.15}$$

Again, the correlation function is reduced to a single unknown constant (which remains unknown once the normalization is determined by the two-point function).

However, when we turn to the four-point correlation function, we see that there is real freedom here. the cross ratio (or anharmonic ratio) $x = z_{12}z_{34}/z_{13}z_{24}$ is left invariant by the group of global conformal transformations $SL(2, C)/Z_2$. Hence, the four-point function takes the form:

$$\langle \Phi_1(z_1, \overline{z}_1)\Phi_2(z_2, \overline{z}_2)\Phi_3(z_3, \overline{z}_3)\Phi_3(z_4, \overline{z}_4)\rangle = f(x, \overline{x}) \prod_{i>j} z_{ij}^{-h_i i h_j + h/3} \times (\text{c.c.}) \tag{15.16}$$

where $h = \sum_j h_j$. In order to solve a conformal field theory, we must determine the numbers c_{ijk}, the functions $f(x, \overline{x})$, and their counterparts for higher correlation functions.

15.4 Operator Product Expansion, Radial Quantization, Mode Expansions

As we have already discussed, the operator product expansion expresses the notion that two nearby particles will appear to be a single composite particle when viewed from a great distance. In a $2D$ conformal field theory, the operator product expansion is written in the form:

$$\phi_i(z,\overline{z})\ \phi_j(w,\overline{w}) = \sum_k c_{ijk}(z-w)^{h_k-h_i-h_j}(\overline{z}-\overline{w})^{\overline{h}_k-\overline{h}_i-\overline{h}_j}\ \phi_k(w,\overline{w}) \tag{15.17}$$

The coefficients c_{ijk} are precisely the same as appear in the three-point function. The leading term on the right-hand-side is the most singular one, i.e. the operator ϕ_k of lowest dimensions h_k, $\overline{h}_k$. The right-hand-side can be thought of as somewhat like a Taylor expansion. In addition to a given operator ϕ_k, all of its derivatives $\partial\phi_k$, $\partial^2\phi_k$ will appear. The new element, compared to an ordinary Taylor expansion, is that other operators which are formally unrelated to ϕ_k will also appear because they are generated under renormalization by ϕ_i, ϕ_j.

The operator product expansion is a very powerful tool when combined with contour integration. This is most nicely done if one considers a system with periodic boundary conditions, so that the spatial coordinate x is restricted to the interval $[0,2\pi]$. Then, if we perform a conformal transformation to coordinates $z = e^{\tau - ix}$, constant radius circle in the complex plane are constant time slices. Time-ordering is now radial ordering:

$$\mathcal{R}\big(A(z')B(z)\big) = \left\{ \begin{array}{ll} A(z')B(z) & \text{if } |z'|>|z| \\ B(z)A(z') & \text{if } |z|>|z'| \end{array} \right\} \tag{15.18}$$

Now, consider the transformation property of $B(z)$ under a transformation generated by $A = \oint A(z')dz'/2\pi i$. If $A(z')$ is purely holomorphic, this doesn't depend on the specific circle along which we do the integral. Hence, the variation in $B(z)$ is:

$$\begin{aligned} \delta_A B(z) &= \oint \frac{dz'}{2\pi i}\big(A(z')B(z) - B(z)A(z')\big) \\ &= \oint_{\mathcal{C}_1} \frac{dz'}{2\pi i}\mathcal{R}\big(A(z')B(z)\big) - \oint_{\mathcal{C}_2}\frac{dz'}{2\pi i}\mathcal{R}\big(A(z')B(z)\big) \\ &= \oint_{\mathcal{C}_z}\frac{dz'}{2\pi i}\mathcal{R}\big(A(z')B(z)\big) \end{aligned} \tag{15.19}$$

where $\mathcal{C}_1$ is a contour encircling the origin at radius larger than $|z|$, $\mathcal{C}_2$ is a contour encircling the origin at radius smaller than $|z|$, and $\mathcal{C}_z$ is a contour

encircling the point z. The latter contour integral is determined by the short-distance singularity of $\mathcal{R}(A(z')B(z))$, or, in other words, by the operator product expansion of $A(z')$ and $B(z)$. One customarily drops the radial ordering symbol, which is understood, and simply writes $A(z')B(z)$. Thus, the expansion which is useful to us is, in fact, and expansion of the radial product, not the ordinary product, but the notation hides this fact.

Let us consider, as a simple example, a free scalar field, with action

$$S = \frac{1}{2\pi} \int \partial\varphi\overline{\partial}\phi \, d^2x \tag{15.20}$$

According to the equation of motion,

$$\partial\overline{\partial}\phi = 0 \tag{15.21}$$

the field can be written as the sum of a holomorphic field and an antiholomorphic field, $\phi(z,\overline{z}) = \frac{1}{2}(\varphi(z) + \overline{\varphi}(\overline{z}))$. The correlation functions of these fields take the form

$$\langle \phi(z,\overline{z})\varphi(w,\overline{w})\rangle = -\frac{1}{2}\ln|z-w| \tag{15.22}$$

or, in other words,

$$\langle \varphi(z)\varphi(w)\rangle = -\ln(z-w) \tag{15.23}$$

with the analogous equation for $\overline{\varphi}(\overline{z})$. Hence, the correlation function of $\partial\varphi$ is

$$\langle \partial\varphi(z)\partial\varphi(w)\rangle = -\frac{1}{(z-w)^2} \tag{15.24}$$

φ is a primary field of dimension $h = 1$, $\overline{h} = 0$.

According to Wick's theorem,

$$\begin{aligned} \partial\varphi(z)\partial\varphi(w) &= \langle\partial\varphi(z)\partial\varphi(w)\rangle + :\partial\varphi(z)\partial\varphi(w): \\ &= -\frac{1}{(z-w)^2} + :\partial\varphi(z)\partial\varphi(w): \end{aligned} \tag{15.25}$$

(Recall that a radial ordering symbol is implicit.) Expanding $\partial\varphi(z) = \partial\varphi(w) + (z-w)\partial^2\varphi(w) + \ldots$, we have

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2} + :\partial\varphi(w)\partial\varphi(w): + (z-w):\partial^2\varphi(w)\partial\varphi(w): \tag{15.26}$$

As we will see in the next section, $:\partial\varphi(z)\partial\varphi(z): \, := -2T(z)$, where $T \equiv T_{zz}$ is a component of the energy-momentum tensor. Thus,

$$\partial\varphi(z)\partial\varphi(w) = -\frac{1}{(z-w)^2} - 2T(w) - (z-w)\partial T(w) + \ldots \tag{15.27}$$

Hence, the operator product of $\partial\varphi(z)$ with $\partial\varphi(w)$ contains the identity operator, the energy-momentum operator, and its derivatives. This operator plays a special role in conformal field theories, as we will see in the next section.

In free field theories, it is often useful to rewrite the field operators in terms of creation and annihilation operators, which are their Fourier modes. Indeed, this was the starting point of our development of quantum field theory way back in chapter 2. Very schematically, this looks like $\phi = \sum_k a_k e^{ikx-\omega t}$. In radial quantization, $z = e^{ix+\tau}$, so the expansion in modes is an expansion in powers of z. Since $x \in [0, 2\pi]$, the allowed momenta are simply the integers $k = 0, \pm 1, \pm 2, \ldots$, if we take periodic boundary conditions, i.e. only integer powers of z appear in the expansion. Consider the case of the primary field $\partial\varphi$:

$$\partial\varphi(z) = \sum_{n\in Z} \varphi_n z^{-n-1} \tag{15.28}$$

The -1 in the exponent is a matter of convention whereby we separate explicitly the weight of a primary field. The operator product expansion of two operators is equivalent to the commutation relation between their modes, as may be seen using contour integration. We will see an explicit example of this two sections hence.

15.5 Conservation Laws, Energy-Momentum Tensor, Ward Identities

According to Nöther's theorem (see chapter), to each symmetry of the action, there is an associated conserved quantity. Invariance under the coordinate transformation $x^\mu \to x^\mu + \epsilon^\mu$ is associated with the current $j_\mu = T_{\mu\nu}\epsilon^\mu$, satisfying the conservation law $\partial_\mu j^\mu = 0$, where $T_{\mu\nu}$ is the energy-momentum tensor. Translational invariance implies that

$$\partial^\mu T_{\mu\nu} = 0 \tag{15.29}$$

since ϵ^μ is constant for a translation. Scale invariance, for which $\epsilon^\mu = \lambda x^\mu$, implies that

$$\begin{aligned} 0 &= \partial^\mu \left(T_{\mu\nu} x^\nu\right) \\ &= \left(\partial^\mu T_{\mu\nu}\right) x^\nu + T_{\mu\nu} \left(\partial^\mu x^\nu\right) \\ &= T^\mu_\mu \end{aligned} \tag{15.30}$$

Hence, the energy-momentum tensor is both divergenceless and traceless.

As an aside, we note that invariance under special conformal transformations comes for free once we have scale and translational invariance. The associated conserved quantity is $K_{\mu\nu} = x^2 T_{\mu\nu} - x_\mu x^\alpha T_{\alpha\nu}$:

$$\begin{aligned}\partial^\nu K_{\mu\nu} &= \partial^\mu \left(x^2 T_{\mu\nu}\right) - \partial^\mu \left(x_\mu x^\alpha T_{\alpha\nu}\right) \\ &= 2x^\nu T_{\mu\nu} + x^2 \partial^\nu T_{\mu\nu} - \delta^\nu_\mu x^\alpha T_{\alpha\nu} - x_\mu g^{\alpha\nu} T_{\alpha\nu} - x_\mu x^\alpha \partial^\nu T_{\alpha\nu} \\ &= 2x^\nu T_{\mu\nu} - \delta^\nu_\mu x^\alpha T_{\alpha\nu} - x_\mu g^{\alpha\nu} T_{\alpha\nu} \\ &= 0 \end{aligned} \tag{15.31}$$

Let us rewrite the trace-free and divergence-free conditions in complex notation. $ds^2 = (dx^1)^2 + (dx^2)^2 = dz\, d\overline{z}$. Hence, $g_{zz} = g_{\overline{z}\overline{z}} = 0$, $g_{z\overline{z}} = g_{\overline{z}z} = \frac{1}{2}$ and, consequently, $g^{z\overline{z}} = g^{\overline{z}z} = 2$. Hence,

$$\begin{aligned} T^\mu_\mu &= 0 \\ g^{z\overline{z}} T_{z\overline{z}} + g^{\overline{z}z} T_{\overline{z}z} &= 0 \\ T_{z\overline{z}} &= 0 \end{aligned} \tag{15.32}$$

In the last line, we have used the fact that the energy-momentum tensor is symmetric, $T_{\overline{z}z} = T_{z\overline{z}}$. Thus, there are only two non-zero components, T_{zz}, $T_{\overline{z}\overline{z}}$. They are constrained by the divergencelessness of the energy-momentum tensor:

$$\begin{aligned} g^{\alpha\mu}\partial_\alpha T_{\mu\nu} &= 0 \\ g^{\overline{z}z}\overline{\partial} T_{zz} + g^{\overline{z}z}\partial T_{\overline{z}z} &= 0 \\ \overline{\partial} T_{zz} &= 0 \end{aligned} \tag{15.33}$$

Similarly, $\partial T_{\overline{z}\overline{z}} = 0$. Hence, $T(z) \equiv T_{zz}$ and $\overline{T}(\overline{z}) \equiv T_{\overline{z}\overline{z}}$ are, respectively, holomorphic and antiholomorphic.

At the quantum level, $T(z)$ and $\overline{T}(\overline{z})$ must implement conformal transformations according to (15.19). Under a transformation $z \to z + \epsilon(z)$, a field $\Phi(z)$ transforms as

$$\delta\Phi(w) = \oint \frac{dz}{2\pi i}\, \epsilon(z)\, T(z)\, \Phi(w) \tag{15.34}$$

(with radial ordering understood, as usual) At the same time, a primary field Φ of weight h must, by definition, transform as:

$$\delta\Phi = h\partial\epsilon\, \Phi \;+\; \epsilon\, \partial\Phi \tag{15.35}$$

Hence, such a primary field must have the following operator product expansion with the energy-momentum tensor:

$$T(z)\, \Phi(w) = \frac{h}{(z-w)^2}\, \Phi(w) + \frac{1}{z-w}\, \partial\Phi(w) + \ldots \tag{15.36}$$

We can take this to be the defining relation for a primary field of weight h.

Let's check that this does, indeed, hold for a free scalar field. The energy-momentum tensor for a free scalar field (with action normalized as in (15.20)) is:

$$T(z) = -\frac{1}{2} : \partial\varphi(z)\, \partial\varphi(z) : \tag{15.37}$$

This expression has been normal ordered in order to make it well-defined. There are other ways of defining the energy-momentum tensor; this is the usual type of ambiguity which is encountered in quantizing a classical theory. The operator product of $T(z)$ with $\partial\varphi(w)$ can be computed using Wick's theorem. Either of the $\partial\varphi(z)$ factors in $T(z)$ can be contracted with $\partial\varphi(w)$:

$$\begin{aligned} T(z)\, \partial\varphi(w) &= -\frac{1}{2} : \partial\varphi(z)\, \partial\varphi(z) : \ \partial\varphi(w) \\ &= 2 \left(-\frac{1}{2}\right) \left(-\frac{1}{(z-w)^2}\right) \partial\varphi(z) \\ &= \frac{1}{(z-w)^2}\, \partial\varphi(w) + \frac{1}{(z-w)}\, \partial^2\varphi(w) \end{aligned} \tag{15.38}$$

The factor of 2 in the second line comes from the two different choices for contraction. This is, indeed, of the form (15.36), with $h = 1$.

The quantum mechanical expression of the existence of a symmetry is the corresponding Ward identity which correlation functions satisfy. This can be derived by inserting the generator of the symmetry transformation into a correlation function and allowing it to act on all of the other fields in the correlation function. Consider the following contour integral

$$\left\langle \oint \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_1(w_1) \ldots \phi_n(w_n) \right\rangle \tag{15.39}$$

The contour is taken to encircle all of the w_is. This contour can be deformed into the sum of contour integrals along small circles encircling each of the w_is. Thus,

$$\begin{aligned} &\left\langle \oint_{\mathcal{C}} \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_1(w_1) \ldots \phi_n(w_n) \right\rangle = \\ &\qquad \sum_{i=1}^{n} \left\langle \phi_1(w_1) \ldots \oint_{\mathcal{C}_i} \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_i(w_i) \ldots \phi_n(w_n) \right\rangle \end{aligned} \tag{15.40}$$

From (15.34) and (15.35),

$$\oint_{\mathcal{C}_i} \frac{dz}{2\pi i} \epsilon(z) T(z) \phi_i(w_i) = \epsilon(w_i) \partial\phi_i(w_i) + h_i \partial\epsilon(w_i) \phi_i(w_i) \tag{15.41}$$

Hence,

$$\left\langle \oint_{\mathcal{C}} \frac{dz}{2\pi i}\epsilon(z)T(z)\phi_1(w_1)\ldots\phi_n(w_n)\right\rangle = \sum_{i=1}^{n}\oint_{\mathcal{C}_i}\frac{dz}{2\pi i}\epsilon(z)\left(\frac{h}{(z-w_i)^2}+\frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right)\langle\phi_1(w_1)\ldots\phi_n(w_n)\rangle \quad (15.42)$$

Since this holds for all $\epsilon(z)$, we can drop the integral and simply write

$$\langle T(z)\phi_1(w_1)\ldots\phi_n(w_n)\rangle = \sum_{i=1}^{n}\left(\frac{h}{(z-w_i)^2}+\frac{1}{z-w_i}\frac{\partial}{\partial w_i}\right)\langle\phi_1(w_1)\ldots\phi_n(w_n)\rangle \quad (15.43)$$

15.6 Virasoro Algebra, Central Charge

Let's now consider the OPE of the energy-momentum tensor with itself. If $T(z)$ is a primary field, then this OPE will take the form (15.36) with $h = 2$. The calculation can be done explicitly in the case of a free scalar field:

$$\begin{aligned} T(z)\,T(w) &= \left(-\frac{1}{2}\,:\partial\varphi(z)\,\partial\varphi(z):\right)\left(-\frac{1}{2}\,:\partial\varphi(w)\,\partial\varphi(w):\right) \\ &= 2\cdot\frac{1}{4}\left(-\frac{1}{(z-w)^2}\right)^2 + 4\cdot\frac{1}{4}\cdot\frac{1}{(z-w)^2}\,:\partial\varphi(z)\,\partial\varphi(w):+\ldots \\ &= \frac{1/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{(z-w)}\,\partial T(w) + \ldots \end{aligned} \quad (15.44)$$

The first term on the second line comes from contracting both $\partial\varphi$s in $T(z)$ with both of them in $T(w)$, which can be done in two different ways. The second term comes from contracting a single $\partial\varphi$ in $T(z)$ with a single one in $T(w)$, which can be done in four different ways.

This is almost what we would get if $T(z)$ were primary, but not quite. The leading term on the right-hand-side of the OPE prevents the energy-momentum tensor from being primary. A term of this form scales in the correct way, so it is allowed in general:

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w) + \ldots \quad (15.45)$$

Similarly,

$$\overline{T}(\overline{z})\,\overline{T}(\overline{w}) = \frac{\overline{c}/2}{(\overline{z}-\overline{w})^4} + \frac{2}{(\overline{z}-\overline{w})^2}\,\overline{T}(\overline{w}) + \frac{1}{\overline{z}-\overline{w}}\,\overline{\partial T}(\overline{w}) + \ldots \quad (15.46)$$

In the special case of a free scalar field, we have just shown that $c = 1$; a similar calculation shows that $\overline{c} = 1$. It is one of the numbers which characterizes any given conformal field theory. Later, we will consider conformal field theories with other values of c, $\overline{c}$. For now, let us continue to proceed with full generality and take c, $\overline{c}$ to be arbitrary.

Let us consider some of the consequences of the existence of c, which is usually called the *central charge*, for reasons which will become clear momentarily. By taking the contour integral of (15.45) with $\epsilon(z)$, we can find the transformation property of $T(z)$ under the conformal transformation $z \to z + \epsilon(z)$:

$$\begin{aligned}\delta_\epsilon T(w) &= \oint \frac{dz}{2\pi i}\,\epsilon(z)\,T(z)\,T(w) \\ &= \oint \frac{dz}{2\pi i}\,\epsilon(z)\,\left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w) + \ldots\right) \\ &= \frac{c}{12}\partial^3\epsilon(w) + 2\partial\epsilon(w)\,T(w) + \epsilon(w)\,\partial T(w) \qquad (15.47)\end{aligned}$$

For $\epsilon(z) = 1, z, z^2$, i.e. for the algebra of global conformal transformations, $T(z)$ transforms as a primary field. For a finite transformation, $z \to f(z)$,

$$T(z) \to \left(\frac{\partial f}{\partial z}\right)^2 T\left(f(z)\right) + \frac{c}{12}\,\frac{\partial f \partial^3 f - \frac{3}{2}\left(\partial^2 f\right)^2}{(\partial f)^2} \qquad (15.48)$$

The odd-looking second term is called the Schwartzian derivative. It is not so obvious that this is the finite transformation corresponding to the infinitesimal transformation (15.47), although the converse is clear. Some insight may be gained by checking that it vanishes for a global conformal transformation $z \to (az+b)/(cz+d)$. In fact, it is the unique such quantity, up to a constant coefficient. We can become a bit more comfortable with the Schwartzian derivative by considering the example of a free scalar field.

$$\begin{aligned}T(z) &= -\frac{1}{2}\,:\partial\varphi(z)\,\partial\varphi(z): \\ &= -\frac{1}{2}\lim_{a\to 0}\left(\partial\varphi\left(z+\frac{a}{2}\right)\,\partial\varphi\left(z-\frac{a}{2}\right) + \frac{1}{a^2}\right) \qquad (15.49)\end{aligned}$$

The transformation properties of $T(z)$ under a conformal transformation are determined by those of $\partial\varphi$, which is a primary field:

$$T(z) \to -\frac{1}{2}\lim_{a\to 0}\left(f'\Big(z+\frac{a}{2}\Big)\,f'\Big(z-\frac{a}{2}\Big)\,\partial\varphi\Big(f\Big(z+\frac{a}{2}\Big)\Big)\,\partial\varphi\Big(f\Big(z-\frac{a}{2}\Big)\Big) + \frac{1}{a^2}\right)$$

$$
\begin{aligned}
&= \lim_{a\to 0}\left(f'\Big(z+\frac{a}{2}\Big)\; f'\Big(z-\frac{a}{2}\Big)\; \left[T\left(f(z)\right) - \frac{1}{\big(f\big(z+\frac{a}{2}\big)-f\big(z-\frac{a}{2}\big)\big)^2}\right] + \frac{1}{a^2}\right) \\
&= \big(f'(z)\big)^2 T\left(f(z)\right) - \lim_{a\to 0}\left(\frac{f'\big(z+\frac{a}{2}\big)\; f'\big(z-\frac{a}{2}\big)}{\big(f\big(z+\frac{a}{2}\big)-f\big(z-\frac{a}{2}\big)\big)^2} - \frac{1}{a^2}\right) \\
&= \big(f'(z)\big)^2 T\left(f(z)\right) \;-\; \frac{1}{12}\frac{f'f''' - \frac{3}{2}(f'')^2}{(f')^2}
\end{aligned}
\tag{15.50}
$$

which is (15.48) with $c = 1$. Note that the real culprit here is the normal-ordering which must be done in order to define $T(z)$ in a quantum theory. Thus, the ordering ambiguities associated with the quantization of a classical theory are responsible for making $T(z)$ a non-primary field; classically, it is primary.

Let us rewrite (15.45) in terms of the commutator between the modes of $T(z)$,

$$T(z) = \sum_n L_n\, z^{-n-2} \tag{15.51}$$

This expansion may be inverted to give the modes, L_n,

$$\sum_n L_n = \oint \frac{dz}{2\pi i}\, z^{n+1}\, T(z) \tag{15.52}$$

If we take the integral of (15.45) over z and w after multiplying by z^{n+1} and w^{n+1}, we find

$$
\begin{aligned}
&\oint \frac{dz}{2\pi i}\, z^{n+1} \oint \frac{dw}{2\pi i}\, w^{m+1} T(z)\, T(w) = \\
&\oint \frac{dz}{2\pi i}\, z^{n+1} \oint \frac{dw}{2\pi i}\, w^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\, T(w) + \frac{1}{z-w}\, \partial T(w)\right) \\
&\qquad [L_n, L_m] = \frac{c}{12}\left(n^3 - n\right)\delta_{m+n,0} + 2(n+1)\, L_{m+n} - (m+n+2)\, L_{m+n} \\
&\qquad\quad [L_n, L_m] = (n-m)\, L_{m+n} \;+\; \frac{c}{12}\left(n^3 - n\right)\delta_{m+n,0}
\end{aligned}
\tag{15.53}
$$

Thus, we see that the classical algebra of infinitesimal conformal transformations has been modified. It is extended by an additional term, which is proportional to the identity operator – hence, a central extension.[6]

15.7 Interpretation of the Central Charge

Before taking up the representation theory of the Virasoro algebra, let's think about the physical interpretation of the central charge, c.

The first obersvation is that the central charge is additive. If we take two field theories and simply add together their actions without coupling them, then their energy-momentum tensors add, and the central charges add (since there won't be any cross terms in their OPEs). N free scalar fields have $c = N$. Thus, there is a sense in which the central charge measures the number of gapless modes which a system has. As we will see, it weighs such modes differently. Fermions have different central charges from bosons, etc. At the end of this section, we will see that the particular form of accounting which is done by the central charge is the 'right' one for the purposes of giving insight into RG flows.

15.7.1 Finite-Size Scaling of the Free Energy

Further insight into the central charge, and a way of calculating it for an arbitrary system, is given by the following interpretation in terms of the finite-size scaling of the free energy. In a large but finite system of linear scale L, the free energy scales as:

$$\beta F = A_2\, L^2 + A_1\, L + A_0\, \ln L + \ldots \tag{15.54}$$

The bulk free energy density and boundary free energy are clearly non-universal since they are dimensional quantities. However, A_0 can be – and is – universal. This means that it depends on the geometrical shape of the finite system in way which is independent of the particular theory under consideration and is completely independent of the short-distance cutoff and other microscopic details.

At first glance, it is somewhat surprising that the third term in this expansion is proportional to $\ln L$ rather than a constant. However, it can be seen that such a term is present by considering the effect on the free energy of an infinitesimal scale transformation, $x^\mu \to (1+\epsilon)x^\mu$. Under such a transformation, the action varies by

$$\delta S = -\frac{\epsilon}{2\pi} \int d^2x T_{z\overline{z}} \tag{15.55}$$

This follows from the definition of the energy-momentum tensor. Under such a rescaling, the size of the system changes by $L \to (1+\epsilon)L$. Hence, the free energy changes by

$$\begin{aligned} F(L+dL) - F(L) &= dL\, \frac{\partial F}{\partial L} \\ &= \epsilon L\, \frac{\partial F}{\partial L} \end{aligned} \tag{15.56}$$

However, since the free energy is the logarithm of the functional integral of the action, the expectation value of the change in the action is equal to the change in the free energy, to lowest order in ϵ:

$$\begin{aligned} e^{-\beta F(L+dL)} &= \int e^{-S-dS} \\ &= \int e^{-S}\left(1 - dS + O(\epsilon^2)\right) \\ &= e^{-\beta F(L)}\left(1 - \langle dS\rangle + O(\epsilon^2)\right) \\ &= e^{-\beta F(L) - \langle dS\rangle} \end{aligned} \tag{15.57}$$

Hence,

$$L\,\frac{\partial(\beta F)}{\partial L} = -\,\frac{1}{2\pi}\int d^2x\,\langle T_{z\overline{z}}\rangle \tag{15.58}$$

Hence, if the integral over the entire finite-size system of the right-hand-side of this equation is a non-zero, finite constant, then the free energy has a $\ln L$ term in its expansion.

In an infinite system in flat space, $T_{z\overline{z}} = 0$ in a critical theory. However, on a curved space or in a finite region of flat space, it need not vanish. The simplest example of this is a wedge of the plane of angle γ. We can compute $T_{z\overline{z}}$ in such a wedge using the conformal mapping $w = z^{\gamma/\pi}$ from the upper-half-plane to the wedge and the transformation law for $T(z)$.

$$T_{\rm uhp}(z) = \left(\frac{\gamma}{\pi}\,z^{\frac{\gamma}{\pi}-1}\right)^2 T_{\rm wedge}(w) - \frac{c}{24z^2}\left(\left(\frac{\gamma}{\pi}\right)^2 - 1\right) \tag{15.59}$$

Since $\Big\langle T_{\rm uhp}(z)\Big\rangle = 0$,

$$T_{\rm wedge}(w) = \frac{c}{24w^2}\left(1 - \left(\frac{\pi}{\gamma}\right)^2\right) \tag{15.60}$$

At the corner of the wedge, $w = 0$, $T(w)$ is singular. As a result of this singularity of $T(w)$, it is not true that $\overline{\partial}T = 0$‘, so the conservation law will require that $T_{z\overline{z}}$ be a delta-function situated at the corner.

$$\begin{aligned} \overline{\partial}\left(\frac{c}{24w^2}\left(1 - \left(\frac{\pi}{\gamma}\right)^2\right)\right) + \partial T_{z\overline{z}} &= 0 \\ \overline{\partial}\left(\frac{c}{24w^2}\left(1 - \left(\frac{\pi}{\gamma}\right)^2\right)\right) &= -\partial T_{z\overline{z}} \end{aligned}$$

$$\left(\frac{c}{24}\left(1-\left(\frac{\pi}{\gamma}\right)^2\right)\right)\delta^{(2)}(w) = -T_{z\overline{z}} \tag{15.61}$$

This gives a finite contribution to $\partial F/\partial(\ln L)$:

$$F = \ldots - \left(\frac{c\gamma}{24}\left(1-\left(\frac{\pi}{\gamma}\right)^2\right)\right)\ln L + \ldots \tag{15.62}$$

By measuring how large a $\ln L$ contribution there is to the free energy, we get a measure of c, and, hence, of how many degrees of freedom there are in the system. If one has a lattice model, but doesn't know what low-energy conformal field theories are associated with its critical points, then one could compute the free energy numerically and see how it scales with system size. Once, c is obtained, one can hope to compute all of the correlation functions of the theory using the techniques which we will discuss in the following sections – which would be a much more involved numerical calculation. If the theory is soluble by the Bethe ansatz, the the free energy can be computed analytically. The resulting central charge might enable one to compute the correlation functions of the theory, something which the Bethe ansatz solution does not give.

15.7.2 Zamolodchikov's c-theorem

The interpretation of the central charge as a measure of the number of degrees of freedom in a theory has an interesting corollary, due to Zamolodchikov. Consider an arbitrary two-dimensional field theory. Let $\Theta \equiv T_{z\overline{z}}$. Then, we define:

$$\begin{aligned}
\langle T(z,\overline{z})\ T(0,0)\rangle &= \frac{F\left(z\overline{z}\right)}{z^4} \\
\langle T(z,\overline{z})\ \Theta(0,0)\rangle &= \frac{G\left(z\overline{z}\right)}{z^3\overline{z}} \\
\langle \Theta(z,\overline{z})\ \Theta(0,0)\rangle &= \frac{H\left(z\overline{z}\right)}{z^2\overline{z}^2}
\end{aligned} \tag{15.63}$$

The forms of the right-hand-sides are dictated by rotational invariance. By unitarity, $F\left(z\overline{z}\right) \geq 0$, $H\left(z\overline{z}\right) \geq 0$.

Meanwhile, translational invariance implies that:

$$\overline{\partial}T + \partial\Theta = 0 \tag{15.64}$$

Multiplying the left-hand-side of this conservation law by $T(0,0)$ and taking the correlation function, we find

$$\begin{aligned} 0 &= \overline{\partial}\left(\frac{F\,(z\overline{z})}{z^4}\right) + \partial\left(\frac{G\,(z\overline{z})}{z^3\overline{z}}\right) \\ &= z\overline{z}\,F' + z\overline{z}G' - 3G \\ &= \dot{F} + \dot{G} - 3G \end{aligned} \tag{15.65}$$

where $\dot{F} \equiv z\overline{z}\,F'$, i.e. $\dot{F} = dF/dt$, where $t = \ln(z\overline{z})$. Similarly, we multiply the conservation law by $\Theta(0,0)$ and take the correlation function,

$$\begin{aligned} 0 &= \overline{\partial}\left(\frac{G\,(z\overline{z})}{z^3\overline{z}}\right) + \partial\left(\frac{H\,(z\overline{z})}{z^2\overline{z}^2}\right) \\ &= \dot{G} - G + \dot{H} - 2H \end{aligned} \tag{15.66}$$

If we define the quantity $C(t) = 2F(t) - 4G(t) - 6H(t)$, then we can immediately make two observations about $C(t)$:

- $C(t) = c$ when the theory is critical, i.e. $\Theta = 0$.
- $C(t)$ is a monotonically *decreasing* function of t since:

$$\frac{d}{dt}C(t) = -\,12\,H(t) < 0 \tag{15.67}$$

Hence, renormalization group flows always go from fixed points of large central charge to those of smaller central charge, *and* $C(t)$ decreases along the flows. One consequence is that RG flows are necessarily gradient flows in unitary $2D$ theories; there can't be limit cycles or other exotic flows.

This is sometimes justified by saying that the renormalization group procedure involves course graining a system, so 'information' should be lost in this procedure, which would imply that RG flows should flow 'downhill' according to some measure of 'information' or entropy. However, this would imply that some version of the c- theorem should hold in higher dimensions. No such theorem has been found. A little further thought suggest that another hole in this intuitive interpretation of the c-theorem. An RG transformation takes a system with cutoff Λ and transforms it into another system with the *same* cutoff Λ. (The two systems are, of course, related by a rescaling.) Thus, there are precisely the same number of degrees of freedom before and after an RG transformation. Thus, the interpretation of the c-theorem as a statement about the information loss (or entropy gain) associated with course-graining is probably a little too naive. The following

is a more accurate interpretation. The function $C(t)$ is the number of low-energy degrees of freedom, as measured by their contribution to the $T-T$ correlation function or, loosely speaking, by their thermal conductivity or specific heat. The c-theorem may be nothing more than the statement that systems like to form gaps by allowing the gapless degrees of freedom to interact and form gaps. Thereby, their number is reduced and $C(t)$ decreases. In more than two dimensions, it is possible to generate new gapless degrees of freedom by breaking a continuous symmetry, which results in Goldstone modes. Thus, it is not so clear in $D > 2$ that a low-energy theory will have fewer gapless modes than its high-energy parent. One could imagine having N species of fermions with $SU(N)$ symmetry which form bilinear order parameters which condense, thereby completely breaking the $SU(N)$ symmetry. In such a case, there would be N^2-1 Goldstone modes, so the number of gapless modes – by any measure – would have to increase at low-energies for N sufficiently large.

15.8 Representation Theory of the Virasoro Algebra

$L_0 + \overline{L}_0$ is the Hamiltonian. Since the right- and left-handed parts of the energy-momentum tensor decouple, we can separately consider L_0 which is $E+P$, which is plays the role of the Hamiltonian for the right-handed part of the theory. The L_ns are raising and lowering operators, as may be seen from the commutation relation:

$$[L_0, L_n] = -nL_n \tag{15.68}$$

For $n > 0$, L_n is a lowering operator. For $n < 0$, L_n is a raising operator. Hermiticity implies that $L_n^\dagger = L_{-n}$ since $T(z) = \sum L_n z^{-n-2}$ in radial quantization implies that $T(x,t) = \sum L_n e^{-in(x-t)}$ in the original Minkowski space. We will use these raising and lowering operators to build up representations of the Virasoro algebra.

Let us consider some representation. Since the energy is bounded below, there must be some state $|h\rangle$ in the representation such that

$$L_n|h\rangle = h|h\rangle \tag{15.69}$$

$$L_n|h\rangle = 0 \qquad \forall\, n > 0 \tag{15.70}$$

Such a state is called a highest weight state or a primary state. The latter term is suggestive of a primary field, to which a primary state correspond.

In conformal field theory, there is a simple correspondence between states and operators. We act on the vacuum with with the operator $\phi(z)$ at the origin $z = 0$

$$|\phi\rangle \equiv \phi(0)|0\rangle \tag{15.71}$$

In radial quantization, $z = 0$ is actually $t = -\infty$, so this state should be thought of as a state with a single ϕ quanta in the distant past. Suppose that $\Phi_h(z)$ is a primary field of weight h. We define

$$|h\rangle \equiv \Phi(0)|0\rangle \tag{15.72}$$

Then, according to (15.36)

$$T(z)\,\Phi_h(0) = \frac{h}{z^2}\,\Phi_h(0) + \frac{1}{z}\,\partial\Phi_h(0) + \text{non-singular terms} \tag{15.73}$$

Acting on the vacuum state and expanding $T(z)$, we have

$$\begin{aligned} T(z)\,\Phi_h(0)|0\rangle &= \frac{h}{z^2}\,\Phi_h(0)|0\rangle + \frac{1}{z}\,\partial\Phi_h(0)|0\rangle + \text{non-sing.} \\ T(z)|h\rangle &= \frac{h}{z^2}\,|h\rangle + \frac{1}{z}\,\partial\Phi_h(0)|0\rangle + \text{non-sing.} \\ \sum_{n>0} L_n z^{-n-2}|h\rangle \;\; &+ \\ \frac{h}{z^2}|h\rangle \;+\; \sum_{n<0} L_n z^{-n-2}|h\rangle &= \frac{h}{z^2}\,|h\rangle + \frac{1}{z}\,\partial\Phi_h(0)|0\rangle + \text{non-sing.} \end{aligned} \tag{15.74}$$

Comparing the right- and left-hand sides of the last line, we see that

$$L_n\Phi_h(0)|0\rangle = L_n|h\rangle = 0 \qquad \forall\, n > 0 \tag{15.75}$$

and

$$L_{-1}\Phi_h(z) = \partial\Phi_h(z) \tag{15.76}$$

Thus, as advertised, primary states are created by primary fields.

Implicit in this derivation, was the notion of a vacuum state $|0\rangle$. As usual, we expect the vacuum state to be invariant under the global symmetries of the theory. In this case, this means $L_0|0\rangle = L_{\pm1}|0\rangle = 0$. Another perspective on this condition on the vacuum state may be gained by considering the trivial OPE:

$$T(z)\,1 = T(0) + z\partial T(0) + \frac{z^2}{2}\partial^2 T(0) + \ldots \tag{15.77}$$

where the identity operator is taken to be nominally at the origin. From this OPE, we see that $L_n\,1\,|0\rangle = L_n|0\rangle = 0$ for all $n > 0$ since the identity

is a primary field. We further see that $L_0|0\rangle = L_{-1}|0\rangle = 0$, as required by global conformal invariance. Finally, we see that $L_{-2}|0\rangle = T(0)|0\rangle$ and, more generally, $L_{-n}|0\rangle = \partial^{n-2}T(0)|0\rangle$. For some purposes, it is useful to think of the primary states as 'vacuum' states of sectors – each sector corresponding to a different primary field – of the Hilbert space. The actual vacuum is then simply one of these states, the one corresponding to the identity operator $|0\rangle = 1\,|0\rangle$.

Any given conformal field theory will be characterized by a central charge c. It will contain some number of primary fields, Φ_{h_1}, Φ_{h_2}, ... If the theory has a fnite number of primary fields, it is called a *rational* conformal field theory. For each primary field, Φ_{h_n}, we have a sector of Hilbert space which is built on the highest weight state $|h_n\rangle$ by acting with raising operators. These states form a *Verma module*:

L_0 **eigenvalues**	**States**
h	$\|h\rangle$
$h+1$	$L_{-1}\|h\rangle$
$h+2$	$L_{-1}^2\|h\rangle\ ,\ L_{-2}\|h\rangle$
$h+3$	$L_{-1}^3\|h\rangle\ ,\ L_{-1}L_{-2}\|h\rangle\ ,\ L_{-3}\|h\rangle$
$h+4$	$L_{-1}^4\|h\rangle\ ,\ L_{-1}^2L_{-2}\|h\rangle\ ,\ L_{-1}L_{-3}\|h\rangle\ ,\ L_{-2}^2\|h\rangle\ ,\ L_{-4}\|h\rangle$
$\vdots$	$\vdots$

The states which are obtained from the primary states by acting with the L_ns are called descendent states. The fields to which they correspond are descendent fields. Descendent states must be orthogonal to primary states. A primary state $|\chi\rangle$ is defined by the condition $L_m|\chi\rangle = 0$ for all $m > 0$. A descendent state, $|\psi\rangle$ is given by $|\psi\rangle = L_{-m_1}\dots L_{-m_n}|h\rangle$ for some $-m_1, \dots, -m_n < 0$. Then, the inner product $\langle\psi|\chi\rangle = (\langle\psi|L_{-m_1}\dots L_{-m_n})\,|\chi\rangle = \langle\psi|\,(L_{m_1}\dots L_{m_n}|\chi\rangle) = 0$. A primary descendent state, i.e. a state which is simultaneously a primary state and a descendent state, has vanishing inner product with itself and all other states, so it should be set to zero in any unitary representation of the Virasoro algebra.

In (15.74), we saw that $L_{-1}\Phi_h = \partial\Phi_h$ By keeping the less singular terms of (15.74), we can find the other descendents of Φ_h:

$$\Phi_h^{-n}(w) \equiv L_{-n}\Phi_h(w) = \oint \frac{dz}{2\pi i}\,\frac{1}{(z-w)^{n-1}}\,T(z)\Phi_h(w) \tag{15.78}$$

The descendent state can thus be obtained by acting on the vacuum with $L_{-n}|h\rangle = \Phi_h^{-n}(0)$. From our earlier discussion of the identity operator, we see that the energy-momentum tensor is a descendent of the identity. This is one way of understanding why it is not a primary field.

Each Verma module is a single irreducible representation of the Virasoro algebra. There are infinitely many states in each representation. This should not be surprising since the Virasoro algebra is infinite-dimensional. The Hilbert space of any conformal field theory will contain some number (a finite number, if the theory is a rational conformal field theory) of such representations. It is useful to keep in mind the analogy with SU(2). A theory which is invariant under the $SU(2)$ symmetry of spin roations will have a Hilbert space which can be broken into irreducible representations of $SU(2)$: spin-0 representations, spin-1/2 representations, spins-1 representations, etc. A spin-s representation will be $2s+1$-dimensional and there will, in general, be many of them for each s. Thus, $SU(2)$ symmetry introduces some simplification, but only a finite amount of simplification, which is not enough to determine the spectrum of a theory with an infinite-dimensional Hilbert space.

The Virasoro algebra is infinite-dimensional, so it imposes far more structure on Hilbert space. If there are infinitely many primary fields, then this may still not be enough to render the theory tractable. However, for the case of rational conformal field theories, the Virasoro algebra reduces the problem of determining the spectrum of the theory to that of determining a finite set of numbers h_1, h_2, ..., h_n. Note that the Virasoro algebra – and, therefore, the algebra of infinitesimal conformal transformations – is not a symmetry algebra of the theory in the usual sense (i.e. in the sense in which we used it above for $SU(2)$). The operators in this algebra do not commute with the Hamiltonian, L_0. Rather, the Virasoro algebra is a *spectrum-generating algebra* of the theory, analogous to a, $a^\dagger$ for the simple harmonic oscillator.

Our discussion of Verma modules appeared not to depend at all on c or h. The latter was merely an offset from zero by which the entire spectrum was rigidly shifted. This is misleading. We have not imposed any of the commutation relations other than the $[L_0, L_n]$ commutation relations. In fact, it turns out that for some c, h, these commutation relations require that some of the states in a Verma module with a given L_0 eigenvalue are not linearly independent. Hence, some Verma modules are smaller than one would naively expect from the above construction.

The simplest example of this is the Verma module built on the identity operator – or, in other words, the Verma module built on the vacuum state $|0\rangle$. The vacuum state of a conformal field theory should be invariant under $SL(2,C)$ global conformal transformations so, in particular, $L_{-1}|0\rangle = 0$. As a consequence, there is no state with L_0 eigenvalue 1 in the vacuum sector of the theory. Furthermore, there is only one state with L_0 eigenvalue 2 since

$L_{-1}^2|0\rangle = 0$. Indeed, at every level, there are fewer states than one would naively expect.

The condition $L_{-1}|0\rangle = 0$ follows from the $SL(2,C)$ invariance of the vacuum. Another way of deriving it, which can be generalized to other primary fields is by condiering the inner product

$$|L_{-1}|0\rangle|^2 = \langle 0|L_1\, L_{-1}|0\rangle \tag{15.79}$$

Using the commutation relation $[L_1, L_{-1}] = 2L_0$, and the condition $L_1|0\rangle = 0$, we see that

$$|L_{-1}|0\rangle|^2 = \langle 0|2L_0|0\rangle = 2h = 0 \tag{15.80}$$

since the vacuum state has $h = 0$. A state with vanishing norm is called a *null state*. In order to have a unitary representation of the Virasoro algebra, we remove such states from the Hilbert space which we are constructing. In a similar way, we will use the constraints imposed by the Virasoro algebra for a given c on a representation with a given h to find other null states.

Before we do so, let's pause for a moment to see why this is so important. Knowing the spectrum of a quantum field theory is not a full solution of the theory. In order to compute correlation functions, we also need the matrix elements of the operators of interest. In a conformal field theory, we can focus on primary operators because the correlation functions of descendent operators can be obtained from them. In order to compute correlation functions of primary operators, we need to know how to decompose the tensor products of irreducible representations of the Virasoro algebra into a sum of irreducible representations. It is useful to consider the analogy with $SU(2)$. Suppose that we have the correlation function

$$\langle 0|\phi^{i_1}_{\alpha_1}\phi^{i_2}_{\alpha_2}\ldots\phi^{i_n}_{\alpha_n}|0\rangle \tag{15.81}$$

where the ϕ^i_α, $\alpha = \uparrow, \downarrow$ are spin-1/2 fields. This correlation function will only be non-vanishing if these fields are taken in some spin-singlet combination. For example, for $n = 2$, the result is proportional to $\epsilon_{\alpha_1\alpha_2}$; for $n = 3$, there is no way of making an invariant combination, so the correlation function must vanish; for $n = 4$, there will be a contribution proportional to $\epsilon_{\alpha_1\alpha_2}\epsilon_{\alpha_3\alpha_4}$ and a contribution proportional to $(\sigma^y\vec{\sigma})_{\alpha_1\alpha_2}\cdot(\sigma^y\vec{\sigma})_{\alpha_3\alpha_4}$; and so on. In short, we can determine the spin structure of this correlation function if we know how to construct $SU(2)$ invariants out of tensor products of spin-1/2 fields. More generally, we can consider correlation functions of fields of arbitrary spins; again, we simply need to know how to construct invariants. In the case of $SU(2)$, there is a simple decomposition

$$j_1 \otimes j_2 \oplus_{j_3=|j_1-j_2|}^{j_1+j_2} j_3 \tag{15.82}$$

By applying this relation $n-1$ times to a correlation function of n fields and, finally, keeping only the spin $j=0$ piece after the last step, we obtain the spin structure of the correlation function.

In order to calculate correlation functions in a conformal field theory, we need to know how to multiply representations together in this way in order to get invariants. In the case of two-point functions $\langle\Phi_1\Phi_2\rangle$, we know that we need $h_1 = h_2$. In the case of three-point functions, $\langle\Phi_1\Phi_2\Phi_3\rangle$, we need to know whether the OPE of Φ_1 and Φ_2 contains Φ_3. This is is essentially the question of whether the product of the representations (c,h_1) and (c,h_2) contains (c,h_3). If we know this, then we can compute the correlation function because the spatial dependence is essentially determined by conformal invariance. Consider the four-point function $\langle\Phi_1\Phi_2\Phi_3\Phi_4\rangle$. Using the OPE of Φ_1 with Φ_2 and the OPE of Φ_3 with Φ_4, we can write this as

$$\begin{aligned}\langle\Phi_1(z_1)\,\Phi_2(z_2)\,\Phi_3(z_3)\,\Phi_4(z_4)\,\rangle &= \langle\sum_k c_{12k}\,(z_1-z_2)^{h_k-h_1-h_2}\Phi_k(z_2)\,\times \\ &\qquad \sum_n c_{34n}\,(z_3-z_4)^{h_n-h_3-h_4}\Phi_n(z_4)\rangle \\ &= \sum_{k,n}\frac{c_{12k}c_{34n}}{z_{12}^{h_k-h_1-h_2}\,z_{34}^{h_n-h_3-h_4}}\,\langle\Phi_k(z_2)\,\Phi_n(z_4)\rangle\end{aligned} \tag{15.83}$$

In principle, the sums on the right-hand-side run over all fields in the theory, and are, hence, unmanageable. However, we can group each primary field with its descendents, and thus reduce the sum, formally, to a sum over primary fields. This is still unmanageable if there are infinitely many primary fields. Theories with a finite number of primary fields are called *rational conformal field theories*. In these theories, the sum on the right-hand-side can be reduced to a finite sum and some progress can be made.

Even when there is a finite number of primary fields, how can we determine the c_{ijk}s? If we knew more about the theory, for instance if we knew its action and had some sense of how the different primary fields were related physically, then we might be able to deduce which primary fields appear in the operator product of two others. If we had additional symmetries in the theory, then we might be able to derive further restrictions. However, suppose that we wish to proceed purely algebraically, knowing only c and h and no further information about the theory. In such a case, we must take advantage of the existence of null states.

If we take the tensor product of two representations of the Virasoro algebra (c,h_1) and (c,h_2) and one or both of them have null states, then this limits the representations (c,h_k) which can appear in their product.

(c, h_k) must also contain null states, and at certain specific levels which are predicated by those of (c, h_1) and (c, h_2). Thus, the existence of null states is a boon, not a nuisance.

15.9 Null States

In order to solve a conformal field theory with a given central charge, c, we need to determine the primary fields of the theory, i.e. the spectrum of hs, and also how these different irreducible representations are tensored together. In general, there is no way to solve either problem purely algebraically. However, for certain values of c, the Virasoro algebra will not allow unitary representations for most values of h; it will require the existence of negative norm states. Hence, at these values of c, we can figure out the allowed primary fields in the theory since they correspond to those values of h ate which a unitary representation is possible. Note that, in principle, two different conformal field theories with the same c could include different subsets of the allowed hs. In order to solve the second problem – how to decompose the tensor product of representations – we can, as we discussed in the previous section, make progress in those fortunate situations in which the allowed representations have null states. By reducing the size of the Verma module, these constrain the OPE. It may turn out that the OPE requires that all of the allowed hs must actually be in the theory in order for the OPE algebra to close; alternatively, there may be a consistent OPE involving some subset.

From a logical standpoint, the uses of negative norm states and null states are different: the former determines the allowed hs; the latter, the decomposition of the products of these hs. However, they are usually discussed together because the existence of both can be derived with one fell swoop. Here, we will eschew this approach, and discuss them somewhat separately and in the reverse of the normal order.

Let us see how null states can be used in a particular example, which will turn out to be the Ising model. We consider a theory with $c = 1/2$. If $h = 0$, which corresponds to the identity operator, then there is a null state at level one since $L_{-1}|0\rangle = 0$, as we saw in the previous section. Now, let's consider some representation in this theory with $h \neq 0$. Consider the two states at level 2, $L_{-1}^2|h\rangle = 0$ and $L_{-2}|h\rangle = 0$. Suppose that they are not linearly independent, so that there's a null state at this level. Then,

$$L_{-2}|h\rangle + aL_{-1}^2|h\rangle = 0 \tag{15.84}$$

for some a. Acting on this with L_1, and using the commutation relations of the Virasoro algebra together with the fact that $L_1|h\rangle = 0$, we find

$$\begin{aligned} 0 &= L_1 \left(L_{-2}|h\rangle + aL_{-1}^2|h\rangle \right) \\ &= 0 + \left([L_1, L_{-2}] + a \left[L_1, L_{-1}^2 \right] \right) |h\rangle \\ &= \left(3L_{-1} + aL_{-1} \left(2L_0 \right) + a \left(2L_0 \right) L_{-1} \right) |h\rangle \\ &= \left(3 + 2a(2h+1) \right) |h\rangle \end{aligned} \tag{15.85}$$

Hence, the state (15.84) is a null state only if

$$a = -\frac{3}{2(2h+1)} \tag{15.86}$$

If this state is null, then it must also be true that

$$\begin{aligned} 0 &= L_2 \left(L_{-2}|h\rangle + aL_{-1}^2|h\rangle \right) \\ &= 0 + \left([L_2, L_{-2}] + a \left[L_2, L_{-1}^2 \right] \right) |h\rangle \\ &= \left(4L_0 + \frac{c}{2} + aL_{-1} \left(3L_1 \right) + a \left(3L_1 \right) L_{-1} \right) |h\rangle \\ &= \left(4h + \frac{c}{2} + 6ah \right) |h\rangle \end{aligned} \tag{15.87}$$

Substituting the value of a obtained above, this implies that

$$16h^2 - (10 - 2c)h + c = 0 \tag{15.88}$$

For $c = 1/2$, this has the solutions $h = 1/16, 1/2$.

As we will see, these correspond to the spin field, σ and the energy operator, ε in the Ising model. According to this identification, we can obtain the critical exponents η and ν from the two-point functions of the dimension $1/16$ and $1/2$ operators. In the Ising model, the physical fields are left-right symmetric cominations, so these dimensions are effectively doubled by the anti-holomorphic dependence.

$$\langle \sigma(z, \overline{z})\, \sigma(0,0) \rangle \sim \frac{1}{z^{1/8}\, \overline{z}^{1/8}} \sim \frac{1}{r^{\eta}} \tag{15.89}$$

from which we see that $\eta = 1/4$. Similarly,

$$\langle \varepsilon(z, \overline{z})\, \varepsilon(0,0) \rangle \sim \frac{1}{z\, \overline{z}} \sim \frac{1}{r^{2(d-1/\nu}} \tag{15.90}$$

from which we see that $\nu = 1$. These are, indeed, the well-known critical exponents of the $2D$ Ising model.

Suppose that we want to compute a more non-trivial correlation function in the critical Ising model. We can use the existence of null states is the $h = 1/16$ and $h = 1/2$ representations. Consider, for example, the four-point function of the spin field σ. Since

$$\left(L_{-2} - \frac{3}{2\left(2 \cdot \frac{1}{16} + 1\right)} L_{-1}^2 \right) |1/16\rangle = 0 \tag{15.91}$$

it is equivalently true that

$$\left(L_{-2} - \frac{3}{2\left(2 \cdot \frac{1}{16} + 1\right)} L_{-1}^2 \right) \sigma = 0 \tag{15.92}$$

when $\sigma(z)$ is inside of some correlation function. As we will see in a moment, this will give us a differential equation satisfied by the four-point function.

In order to derive this differential equation, we begin by noting that the following correlation function,

$$\langle \phi_1 (w_1) \ \ldots \ \phi_{n-1} (w_{n-1}) \ L_{-k} \phi_n (z) \rangle \tag{15.93}$$

would vanish if we move the L_{-k} to the left, where it could act on the vacuum state and annihilate it. However, in the process of moving it to the left, it must commute with the $\phi_i (w_i)$s. Using equation (15.78), this means that

$$\left\langle \phi_1 (w_1) \ \ldots \ \phi_{n-1} (w_{n-1}) \oint \frac{dz}{2\pi i} \frac{1}{(z' - z)^{n-1}} T(z') \phi_n(z) \right\rangle \tag{15.94}$$

can be simplified by taking breaking the contour into small circles encircling each of the w_is. Thus, we obtain

$$\begin{aligned} &\langle \phi_1 (w_1) \ \ldots \ \phi_{n-1} (w_{n-1}) \ L_{-k} \phi_n (z) \rangle = \\ &- \sum_{j=1}^{n-1} \left[\frac{(1-k) h_j}{(w_j - z)^k} + \frac{1}{(w_j - z)^{k-1}} \frac{\partial}{\partial w_j} \right] \langle \phi_1 (w_1) \ \ldots \ \phi_{n-1} (w_{n-1}) \ \phi_n(z) \rangle \end{aligned} \tag{15.95}$$

Applying this, in conjunction with (15.92) to the σ four-point function, we have

$$\left(\frac{4}{3} \frac{\partial^2}{\partial z_i^2} - \sum_{j=1}^{n-1} \left[\frac{1/16}{(z_4 - z_j)^2} + \frac{1}{z_4 - z_j} \frac{\partial}{\partial z_j} \right] \right) \langle \sigma (z_1) \ \sigma (z_2) \ \sigma (z_3) \ \sigma (z_4) \rangle \tag{15.96}$$

In the first term on the left, we have used $L_{-1}\phi = \partial\phi$ which is simpler than (15.95), but equivalent to it by translational invariance.

Using global conformal invariance, we can write

$$\langle \sigma(z_1)\, \sigma(z_2)\, \sigma(z_3)\, \sigma(z_4)\rangle = \left(\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right)^{1/8} F(x) \qquad (15.97)$$

where $x = z_{12}z_{34}/z_{13}z_{24}$. We have suppressed the dependence on $\overline{z}$ for simplicity.

Substituting this form into the differential equation (15.96), we have the ordinary differential equation

$$\left(x(1-x)\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2} - x\right)\frac{\partial}{\partial x} + \frac{1}{16}\right) = 0 \qquad (15.98)$$

This equation has two independent solutions,

$$f_{1,2}(x) = \left(1 \pm \sqrt{1-x}\right)^{1/2} \qquad (15.99)$$

This is clearly a multiple-valued function. In order to get a single-valued result, we must combine the holomorphic and anti-holomorphic parts of the theory. The only way of doing this is by taking the left-right symmetric combination $f_1(x)f_1(\overline{x}) + f_2(x)f_2(\overline{x})$:

$$\langle \sigma(z_1,\overline{z}_1)\, \sigma(z_2,\overline{z}_2)\, \sigma(z_3,\overline{z}_3)\, \sigma(z_4,\overline{z}_4)\rangle = a\left|\frac{z_{13}z_{24}}{z_{12}z_{23}z_{34}z_{41}}\right|^{1/4} \left(\left|1+\sqrt{1-x}\right| + \left|1-\sqrt{1-x}\right|\right) \qquad (15.100)$$

for some a.

Now, we can determine a_1, a_2 as well as determine the OPE of σ with itself by considering the behavior of this correlation function in various limits. First, let's conside the OPE of σ with itself:

$$\sigma(z_1,\overline{z}_1)\, \sigma(z_2,\overline{z}_2) = \frac{1}{|z_{12}|^{1/4}} + C_{\sigma\sigma\varepsilon}\, |z_{12}|^{3/4}\, \varepsilon(z_2,\overline{z}_2) + \ldots \qquad (15.101)$$

At this stage, we do not yet know whether there are other primary fields in the $c = 1/2$ theory, so the $\ldots$ could, in principle include both primary and descendent fields. In fact, as we will see later, there are none, so the $\ldots$ contains only descendent fields. If we take $z_{12} \to 0$ and $z_{34} \to 0$ in the four-point function and use these OPEs, we have

$$\langle \sigma(z_1,\overline{z}_1)\, \sigma(z_2,\overline{z}_2)\, \sigma(z_3,\overline{z}_3)\, \sigma(z_4,\overline{z}_4)\rangle =$$

$$\frac{1}{|z_{12}|^{1/4}}\frac{1}{|z_{34}|^{1/4}} + C_{\sigma\sigma\varepsilon}^2 |z_{12}|^{3/8} |z_{12}|^{3/8} \langle \varepsilon(z_2,\overline{z}_2)\,\varepsilon(z_4,\overline{z}_4)\rangle + \ldots$$
$$= \frac{1}{|z_{12}|^{1/4}}\frac{1}{|z_{34}|^{1/4}} + C_{\sigma\sigma\varepsilon}^2 \frac{|z_{12}|^{3/4}|z_{12}|^{3/4}}{|z_{24}|^2} + \ldots \quad (15.102)$$

Meanwhile, taking the same limit in (15.100), we have

$$\langle \sigma(z_1,\overline{z}_1)\,\sigma(z_2,\overline{z}_2)\,\sigma(z_3,\overline{z}_3)\,\sigma(z_4,\overline{z}_4)\rangle = a\left|\frac{1}{z_{12}z_{34}}\right|^{1/4}\left(\left|2-\frac{1}{2}\frac{z_{12}z_{34}}{z_{24}^2}\right| + \left|\frac{1}{2}\frac{z_{12}z_{34}}{z_{24}^2}\right|\right) \quad (15.103)$$

Hence, comparing the leading terms, we see that $a = \frac{1}{2}$. Comparing the next terms, we also see that $C_{\sigma\sigma\varepsilon} = \frac{1}{2}$. We also note that there are no operators with $(h,\overline{h}) < (\frac{1}{2},\frac{1}{2})$ appearing in this OPE (which would be natural if there were no other primary fields in the theory).

15.10 Unitary Representations

In fact, the $h = 0, 1/16, 1/2$ representations are the only ones in the $c = 1/2$ theory. If we tried to construct a representation with any other value of h, it would be non-unitary, so it should not arise in most physical theories. (As we will discuss in the next part of the book, systems with quenched random disorder are described by non-unitary field theories, so the requirement of unitarity does not help us there.) Let us see why this is so.

At level 2, the existence of a null state can be determined by taking the determinant of the matrix

$$\det\begin{pmatrix} \langle h|L_2L_{-2}|h\rangle & \langle h|L_1^2L_{-2}|h\rangle \\ \langle h|L_2L_{-1}^2|h\rangle & \langle h|L_1^2L_{-1}^2|h\rangle \end{pmatrix} = \det\begin{pmatrix} 4h+c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix}$$
$$= \left(16h^3 - 10h^2 + 2h^2c + hc\right)$$
$$= 32\,(h - h_{1,1}(c)) \times (h - h_{1,2}(c))\,(h - h_{2,1}(c)) \quad (15.104)$$

where

$$h_{p,q} = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)}$$
$$m = -\frac{1}{2} \pm \frac{1}{2}\sqrt{\frac{25-c}{1-c}} \quad (15.105)$$

$h_{1,1} = 0$ corresponds to the null state at level 1, which must propagate to higher levels. At $c = 1/2$, $h_{1,2} = 1/16$ and $h_{2,1} = 1/2$ are the null states

at level 2. If, in the $c = 1/2$ theory, we had considered a representation at $h = 1/4$, for instance, then we would have found this determinant would be negative. This would imply that there are negatie norm states in the representations, so that it could not be unitary. In order to find the full set of restrictions on the allowed hs, we must also consider the analogous determinant at levels $3, 4, \ldots$.

The determinant of inner products at the N^{th} level is given by the following formula, which generalizes (15.104)

$$\det M_N(c,h) = \alpha_N \prod_{pq \leq N} (h - h_{p,q}(c))^{P(N-pq)} \tag{15.106}$$

where α_N is a constant independent of c, h. We will not prove this formula here, but the basic idea is to write down for each p, q an explicit null state at level pq. Each of these null states leads to $P(N - pq)$ null states $L_{-n_1} \ldots L_{-n_k}|h + pq\rangle$ at level N (where $P(n)$ is the number of partitions of n). A polynomial is determined, up to an overall constant, by its zeroes, which leads to the result above. One can check that the null states constructed are the full set by comparing the highest power of h on both sides.

Let us now consider the question of whether the eigenvalues of $\det M_N(c,h)$ are positive. Keep in mind that the factorization (15.106) gives the product of eigenvalues, but the factors in (15.106) are not the eigenvalues themselves. When the determinant is negative, there are an odd number of eigenvalues. For $1 < c < 25$, $h > 0$, m is not real so the $h_{p,q}$s either have an imaginary part or are negative (the latter only occurs for $p = q$). Hence, the determinant never vanishes. For large h, the eigenvalues are strictly positive since they are just the diagonal elements (which are positive) in this limit. Since the determinant never changes sign, the eigenvalues must remain positive.

For $c = 1$, $m = \infty$, so $h_{p,q} = (p - q)^2/4$. The determinant vanishes at these values, but is nowhere negative. There are null states, which must be set to zero, at these hs but there are no negative norm states, so there are unitary representations at all hs.

For $c < 1$, $h > 0$ not lying on the curves $h_{p,q}(c)$, there exists some level $N = \tilde{p}\tilde{q}$ such that the point (c, h) can be connected to the region $c > 1$ by crossing precisely one of the curves $h_{\tilde{p},\tilde{q}}(c)$. Since an eigenvalue changes sign at these curves and the region $c > 1$ has only positive eigenvalues, this means that there is a single negative norm state at this level for this (c, h). The only exceptions are the the curves $h_{p,q}(c)$ themselves. It can be shown that there is a single negative norm state along these curves except at the "first intersection" points where two of these curves intersect. At these values of

c, h, there are null states but no negative norm states. These crossings occur at a discrete series of c:

$$c = 1 - \frac{6}{m(m+1)} \tag{15.107}$$

At each such c, there is a finite set of allowed hs:

$$h_{p,q}(m) = \frac{[(m+1)p - mq]^2 - 1}{4m(m+1)} \tag{15.108}$$

These theories are called 'minimal models'. The case $m = 3$ is the Ising model, as we mentioned earlier. $m = 4$ is the tricritical Ising model (in $4-\epsilon$ dimensions, this is a scalar field with potential $V(\phi) = \frac{r}{2}\phi^2 + u\phi^4 + v\phi^6$; the tricritical point is at $r = u = 0$), while $m = 5$ is the 3-state Potts model.

As we claimed earlier, there are only three possible primary fields in the $c = 1/2$ theory, with $h = 0, 1/2, 1/16$. Any other h would be a non-unitary representation of the Virasoro algebra for $c = 1/2$.

Using techniques analogous to those which we used in our discussion of the $c = 1/2$ theory, we can compute the OPEs of fields in the minimal models. Focussing on the chiral part of these theories, we can examine the three-point correlation functions to determine which primary fields can appear in the OPE of two primary fields. Such a relation is called a *fusion rule*:

$$\phi_{p_1,q_1} \times \phi_{p_2,q_2} = \sum \phi_{p_3,q_3} \tag{15.109}$$

The rule which specifies which p_3, q_3 can appear on the right-hand-side of this equation can be expressed most neatly by writing $p_i = 2j_i + 1$, $q_i = 2j_i' + 1$. Then the allowed j_3s are $|j_1 - j_2| \leq j_3 \leq \min(j_1 + j_2, m - 2 - j_1 - j_2)$, and the analogous rule holds for j_3'. This is almost the same as the decomposition of the product of the spin j_1 and j_2 representations in $SU(2)$; the only difference is that the upper limit $j_1 + j_2$ is replaced by $\min(j_1 + j_2, m - 2 - j_1 - j_2)$. The origin of this rule will be clearer when we discuss Kac-Moody algebras.

15.11 Free Fermions

In $1+1$ dimensions, the action of a Dirac fermion $\psi = \binom{\psi_R}{\psi_L}$ can be written in the form:

$$\begin{aligned} S &= \int \psi^\dagger \gamma^0 \gamma^\mu \partial_\mu \psi \\ &= \int \psi_R^\dagger \overline{\partial} \psi_R + \psi_L^\dagger \partial \psi_L \end{aligned} \tag{15.110}$$

where $\gamma^0 = \sigma_x$, $\gamma^1 = \sigma_y$. Thus, the fermion splits into two independent fields, one right-moving and the other left-moving. The right-moving field satisfies the equation of motion $\overline{\partial}\psi_R = 0$ while the left-moving field satisfies $\partial\psi_L = 0$.

A right-moving Dirac fermion can be written as the sum of two right-moving Majorana fermions which are its real and imaginary parts $\psi = \psi_1 + i\psi_2$. In radial quantization, a right-moving real fermion is one whose mode expansion

$$i\,\psi(z) = \sum \psi_n\, z^{-n-1/2} \tag{15.111}$$

satisfies $\psi_n^\dagger = \psi_{-n}$. This is what one usually means by a real field, but it is somewhat masked by the fact that we're in imaginary time, $z = e^{\tau - ix}$, and the extra 1/2 in the exponent which results from the passage from the cylinder to the plane in radial quantization. The inverse Fourier transform is:

$$\psi_n = \oint \frac{dz}{2\pi i}\, i\psi(z)\, z^{n-1/2} \tag{15.112}$$

The action for a right-moving Majorana fermion $\psi(z)$ is

$$S = \int \psi\overline{\partial}\psi \tag{15.113}$$

(The fact that we can simultaneously diagonalize chirality and charge conjugation is special to $4k+2$ dimensions. In general, we can either have a right-moving fermion (a Weyl fermion) *or* a Majorana fermion.) From the action (or, essentially, by scaling), we can compute the OPE of a Majorana fermion:

$$\psi(z)\,\psi(w) = -\frac{1}{z-w} + \dots \tag{15.114}$$

The energy-momentum tensor is

$$T = -\frac{1}{2}\,:\psi\partial\psi: \tag{15.115}$$

The central charge is obtained from

$$\begin{aligned} T(z)T(w) &= \frac{1}{4}(-1)\left(-\frac{1}{z-w}\right)\frac{2}{(z-w)^3} \;+\; \frac{1}{4}\,\frac{1}{(z-w)^2}\,\frac{-1}{(z-w)^2} + \\ &\qquad 2\,\frac{1}{(z-w)^2}\,T(w) \\ &= \frac{1/4}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} \end{aligned} \tag{15.116}$$

The (-1) in the first term on the first line results from an anticommutation. From this calculation, we see that $c = 1/2$. Hence, a theory of a free Majorana fermion is the same as the critical theory of the Ising model. This should not come as an enormous surprise if one recalls our rewriting of the partition function of the $2D$ Ising model as a Grassman integral. Thus, a Dirac fermion, composed of two Majorana fermions, has $c = 1$.

From the OPE (15.114), we can compute the anticommutator of the modes of ψ:

$$\begin{aligned}\{\psi_m, \psi_n\} &= i^2 \oint \frac{dw}{2\pi i} w^{m-1/2} \oint \frac{dz}{2\pi i} z^{n-1/2} \psi(z)\,\psi(w) \\ &= i^2 \oint \frac{dw}{2\pi i} w^{m-1/2} \oint \frac{dz}{2\pi i} z^{n-1/2} \frac{-1}{z-w} \\ &= \delta_{n+m,0}\end{aligned} \tag{15.117}$$

Fermionic fields can be either periodic or antiperiodic as one goes around the cylinder, $z \to e^{2\pi i} z$. Physically measurable quantities are always bosonic, so they must be composed of fermion bilinears, which are single-valued. For periodic boundary conditions, we need $n \in Z + \frac{1}{2}$ in the mode-expansion (15.111), while $n \in Z$ gives antiperiodic boundary conditions. Note that for anti-periodic boundary conditions, $n \in Z$, there is a zero mode, ψ_0. According to the canonical anticommutation relations, $\{\psi_0, \psi_0\} = 1$, or $\psi_0^2 = 1/2$. A little bit later, we will discuss the zero mode operator a little further.

Let us compute the fermion propagator in these two cases. For periodic boundary conditions,

$$\begin{aligned}\langle \psi(z)\,\psi(w)\rangle &= i^2 \left\langle \sum \psi_n\, z^{-n-1/2} \sum \psi_m\, w^{-m-1/2} \right\rangle \\ &= -\sum_{n=1/2}^{\infty} z^{-n-1/2}\, w^{n-1/2} \\ &= -\frac{1}{z} \sum_{n=0}^{\infty} \Big(\frac{w}{z}\Big)^n \\ &= -\frac{1}{z-w}\end{aligned} \tag{15.118}$$

For antiperiodic boundary conditions,

$$\begin{aligned}\langle \psi(z)\,\psi(w)\rangle &= i^2 \left\langle \sum \psi_n\, z^{-n-1/2} \sum \psi_m\, w^{-m-1/2} \right\rangle \\ &= -\sum_{n=1}^{\infty} z^{-n-1/2}\, w^{n-1/2} + \frac{1}{\sqrt{zw}} \left\langle \psi_0^2 \right\rangle\end{aligned}$$

$$= -\frac{1}{\sqrt{zw}}\left(\frac{w}{z-w} + \langle\psi_0^2\rangle\right) \tag{15.119}$$

using $\psi_0^2 = \frac{1}{2}$, we have:

$$\langle\psi(z)\,\psi(w)\rangle = \frac{\frac{1}{2}\left(\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}\right)}{z-w} \tag{15.120}$$

As expected, the short-distance behavior of the propagator is the same for both boundary conditions, but the global analytic structure is different.

We can introduce a 'twist' operator, σ, which, when placed at z_1 and z_2 introduces a branch cut for the fermions which extends from z_1 to z_2. The OPE of a Majorana fermion $\psi(z)$ with its twist field $\sigma(z)$ is:

$$\psi(z)\,\sigma(0) = \frac{1}{z^{1/2}}\,\mu(0) \tag{15.121}$$

where $\mu(z)$ is an 'excited' twist field of the same dimension as σ (we will discuss this more later). The $\frac{1}{z^{1/2}}$ on the right-hand-side ensures that a minus sign results when $\psi(z)$ is taken around $\sigma(0)$. If we place twist fields at the origin and at ∞, this will exchange periodic and antiperiodic boundary conditions. Thus,

$$\langle 0;\mathrm{P}\,|\sigma(\infty)\,\psi(z)\,\psi(w)\,\sigma(0)|\,0;\mathrm{P}\rangle = \langle 0;\mathrm{AP}\,|\psi(z)\,\psi(w)|\,0;\mathrm{AP}\rangle \tag{15.122}$$

Differentiating both sides with respect to w, we have

$$\langle 0;\mathrm{P}\,|\sigma(\infty)\,\psi(z)\,\partial_w\psi(w)\,\sigma(0)|\,0;\mathrm{P}\rangle = \partial_w\langle 0;\mathrm{AP}\,|\psi(z)\,\psi(w)|\,0;\mathrm{AP}\rangle \tag{15.123}$$

We know the right-hand-side. The left-hand-side can be re-expressed in terms of the energy-momentum tensor, T by taking $z \to w$ and subtracting an infinite constant $-1/(z-w)^2$.

$$\begin{aligned}\langle 0;\mathrm{AP}\,|\psi(z)\,\partial_w\psi(w)|\,0;\mathrm{AP}\rangle &= -\frac{\sqrt{\frac{z}{w}} + \sqrt{\frac{w}{z}}}{(z-w)^2} + \frac{1}{4}\,\frac{1}{w^{3/2}z^{1/2}} \\ &= -\frac{1}{(z-w)^2} + \frac{1}{8}\,\frac{1}{w^2} + O(z-w)\end{aligned} \tag{15.124}$$

Thus,

$$\langle T(z)\rangle_{AP} = \frac{1}{16}\,\frac{1}{z^2} \tag{15.125}$$

Hence, $h_\sigma = 1/16$. Thus, our use of the notation σ is more than merely suggestive. The twist field, σ, is the spin field of the Ising model.

To summarize, antiperiodic boundary conditions are simply the sector of the full $c = 1/2$ theory given by $h = 1/16$ Verma modules. In this sector, there are actually two states with L_0 eigenvalue $1/16$ because there is a zero mode ψ_0. This zero mode can be either occupied or unoccupied. For a Majorana fermion, ψ is a linear combination of a creation and an annihilation operator (with coefficient $1/\sqrt{2}$), so acting twice with ψ_0 will create and then annihilate (or the reverse if the zero mode were initially occupied) a fermion in this mode, thereby leaving the state unchanged, up to a factor of $1/2$. One of the two states is $\sigma(0)\,|0\rangle$ (zero mode unoccupied) while the other is $\mu(0)\,|0\rangle$ (zero mode occupied). σ is the spin field, or order operator, while μ is the disorder operator to which it is dual.

Thus, the $c = 1/2$ theory of a free Majorana fermion has one copy of the $h = 0$ representation, one copy of the $h = 1/2$ representation, and two copies of the $h = 1/16$ representation. These representations are obtained by acting with the L_{-n}s, on, respectively, $|0\rangle$, $\psi_{-1/2}|0\rangle$, $\sigma(0)|0\rangle$, $\psi_0\,\sigma(0)|0\rangle = \mu(0)\,|0\rangle$.

$h = 0$ Representation

L_0 **eigenvalues**	**States**
0	$\|0\rangle$
1	
2	$\psi_{-3/2}\,\psi_{-1/2}\|0\rangle$
3	$\psi_{-5/2}\,\psi_{-1/2}\|0\rangle$
$\vdots$	$\vdots$

$h = 1/2$ Representation

L_0 **eigenvalues**	**States**
$1/2$	$\psi_{-1/2}\|0\rangle$
$3/2$	$\psi_{-3/2}\|0\rangle$
$5/2$	$\psi_{-5/2}\|0\rangle$
$7/2$	$\psi_{-7/2}\|0\rangle$
$\vdots$	$\vdots$

$h = 1/16$ Representation

L_0 **eigenvalues**	**States**
$\frac{1}{16}$	$\|\frac{1}{16}\rangle$
$1+\frac{1}{16}$	$\psi_{-1}\|\frac{1}{16}\rangle$
$2+\frac{1}{16}$	$\psi_{-2}\|\frac{1}{16}\rangle$
$3+\frac{1}{16}$	$\psi_{-3}\|\frac{1}{16}\rangle$
$\vdots$	$\vdots$

$h = 1/16$ Representation

L_0 **eigenvalues**	**States**
$\frac{1}{16}$	$\psi_0\|\frac{1}{16}\rangle$
$1+\frac{1}{16}$	$\psi_{-1}\psi_0\|\frac{1}{16}\rangle$
$2+\frac{1}{16}$	$\psi_{-2}\psi_0\|\frac{1}{16}\rangle$
$3+\frac{1}{16}$	$\psi_{-3}\psi_0\|\frac{1}{16}\rangle$
$\vdots$	$\vdots$

The Ising model itself has both right- and left-moving Majorana fermions. One might naively think that its Hilbert space could have representations of all possible right-left combinations, $(0,0)$, $(1/16,0)$, $(1/16,1/2)$, etc. However, this is not the case. In fact, there are only the right-left symmetric combinations: $(0,0)$, $(1/16,1/16)$, $(1/2,1/2)$. This can be derived by direct computation on the Ising model, of course. It can also be obtained by putting the theory on a torus. The constraints associated with consistently

putting a theory on a torus ('modular invariance') restrict the allowed right-left combinations of the set of possible $(h, \overline{h})$s at a given $(c, \overline{c})$. In the case of the Ising model, the only allowed combinations are the symmetric ones.

15.12 Free Bosons

Earlier, we considered a theory of a free boson, $\phi = \frac{1}{2}\left(\varphi(z) + \overline{\varphi}(\overline{z})\right)$, which has $c = \overline{c} = 1$. This theory has $(1,0)$ and $(0,1)$ fields $\partial\varphi$ and $\overline{\partial}\overline{\varphi}$. Let us now consider the exponential operators $:e^{i\alpha\varphi}:$. By Wick's theorem,

$$\begin{aligned}\left\langle :e^{i\alpha\varphi(z)}::e^{i\alpha\varphi(0)}:\right\rangle &= e^{\alpha^2\left(\langle\varphi(z)\varphi(0)\rangle - \langle\varphi(0)\rangle^2\right)} \\ &= \frac{1}{z^{\alpha^2}}\end{aligned} \tag{15.126}$$

Hence, $:e^{i\alpha\varphi}:$ is a dimension $\alpha^2/2$ operator. In fact, it is a primary operator, as we now show. These exponential operators have the following OPE with $i\partial\varphi$

$$i\partial\varphi(z) :e^{i\alpha\varphi(0)}: = \frac{\alpha}{z} :e^{i\alpha\varphi(0)}: + \dots \tag{15.127}$$

Hence, the OPE with the energy-momentum tensor, $T(z)$, is:

$$\begin{aligned}-\frac{1}{2} :\partial\varphi(z)\partial\varphi(z)::e^{i\alpha\varphi(0)}: &= -\frac{1}{2}\left(\frac{i\alpha}{z-w}\right)^2 + \dots \\ &= \frac{\alpha^2/2}{(z-w)^2} + \dots\end{aligned} \tag{15.128}$$

Thus, as claimed, the operator $:e^{i\alpha\varphi}:$ is a dimension $\alpha^2/2$ primary field.

Now, suppose that ϕ is an angular variable $\phi \equiv \phi + 2\pi R$. In such a case, $\partial\varphi$ and $\overline{\partial}\overline{\varphi}$ are still fine, but not all exponential operators are single-valued under $\phi \to \phi + 2\pi R$. Consider operators of the form $:e^{im\phi/R}:=: e^{im(\varphi+\overline{\varphi})/2R}:$. They are clearly well-defined, and have dimensions $(h, \overline{h}) = \left(\frac{1}{2}\left(\frac{m}{2R}\right)^2, \frac{1}{2}\left(\frac{m}{2R}\right)^2\right)$. Thus, these operators are among the primary fields of the theory. As we will see shortly, they are not the only primary fields in such a theory.

In order to derive the full set of primary fields, it is useful to consider the mode expansion for a system of free bosons:

$$i\,\partial\varphi(z) = \sum_n \alpha_n z^{-n-1} \tag{15.129}$$

with

$$\alpha_n = \oint \frac{dz}{2\pi i}\, z^n\, i\, \partial\varphi(z) \tag{15.130}$$

The commutators of the modes are:

$$\begin{aligned}
[\alpha_m, \alpha_n] &= i^2 \oint \frac{dw}{2\pi i} w^m \oint \frac{dz}{2\pi i} z^n \partial\varphi(z)\, \partial\varphi(w) \\
&= i^2 \oint \frac{dw}{2\pi i} w^m \oint \frac{dz}{2\pi i} z^n \frac{-1}{(z-w)^2} \\
&= \oint \frac{dw}{2\pi i} w^m\, n w^{n-1} \\
&= n\, \delta_{n+m,0}
\end{aligned} \tag{15.131}$$

As a result of the n on the right-hand-side, the zero mode, α_0, commutes with itself, as it had better. Consider the mode expansions of φ, $\overline{\varphi}$, and ϕ itself:

$$\begin{aligned}
\varphi(z) &= \varphi_0^R + \alpha_0 \ln z + \sum_{n\neq 0} \frac{1}{n}\, \alpha_n z^{-n} \\
\overline{\varphi}(\overline{z}) &= \varphi_0^L + \overline{\alpha}_0 \ln \overline{z} + \sum_{n\neq 0} \frac{1}{n}\, \overline{\alpha}_n \overline{z}^{-n} \\
\phi(z,\overline{z}) &= \phi_0 + \alpha_0 \ln z + \overline{\alpha}_0 \ln \overline{z} + \frac{1}{2} \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n z^{-n} + \overline{\alpha}_n \overline{z}^{-n}\right)
\end{aligned} \tag{15.132}$$

where $\phi_0 = (\varphi_0^R + \varphi_0^L)/2$. The zero modes $\varphi_0^{R,L}$ disappear from $\partial\varphi$ and $\overline{\partial}\overline{\varphi}$, but not from the exponential operators $: e^{im\phi/R} :$. The commutation relations of $\varphi_0^{R,L}$ can be derived from the OPE of φ, $\overline{\varphi}$ with $\partial\varphi$,$\overline{\partial}\overline{\varphi}$:

$$\begin{aligned}
\left[\varphi_0^R, \alpha_0\right] &= \oint \frac{dz}{2\pi i} \varphi(z) \partial\varphi(0) \\
&= \oint \frac{dz}{2\pi i} \frac{i}{2z} \\
&= i
\end{aligned} \tag{15.133}$$

By similar steps, $\left[\varphi_0^L, \overline{\alpha}_0\right] = i$. Hence, $[\phi_0, (\alpha_0 + \overline{\alpha}_0)] = i$ while $[\phi_0, (\alpha_0 - \overline{\alpha}_0)] = 0$. Alternatively, we can go back to the cylinder with coordinates x, τ with $z = e^{\tau - ix}$. Then, the mode expansion is:

$$\phi(x,\tau) = \phi_0 + \frac{1}{2}\,(\alpha_0 + \overline{\alpha}_0)\; t + \frac{1}{2}\,(\alpha_0 - \overline{\alpha}_0)\; x + \sum_{n\neq 0} \frac{1}{n} \left(\alpha_n e^{n\tau - inx} + \overline{\alpha}_n e^{n\tau + inx}\right) \tag{15.134}$$

Meanwhile, according to the canonical commutation relations for ϕ,

$$[\phi(x,\tau), \partial_0\phi(0,\tau)] = i\,\delta(x) \tag{15.135}$$

from which we obtain

$$[\phi_0, (\alpha_0 + \overline{\alpha}_0)] = i \tag{15.136}$$

With this commutation relation in hand, we can derive the spectrum of $\alpha_0 + \overline{\alpha}_0$, as we show below.

Now, let us turn to the energy-momentum tensor and its eigenvalues. Since $T(z) = -\frac{1}{2} : \partial\varphi\partial\varphi :$,

$$\begin{aligned} L_0 &= \sum_{n>0} \alpha_{-n}\alpha_n + \frac{1}{2}\alpha_0^2 \\ L_{m\neq 0} &= \frac{1}{2}\sum_n \alpha_{m-n}\alpha_n \end{aligned} \tag{15.137}$$

For $m \neq 0$, α_{m-n} and α_n commute, so we do not need to worry about normal ordering in the second line. There is a similar expression for the anti-holomorphic modes. From the commutation relations (15.131), we see that α_0 commutes with the Hamiltonian. Hence, we can label states by their α_0, $\overline{\alpha}_0$ eigenvalues. Let's define a set of states $|\alpha, \overline{\alpha}\rangle$ by

$$\begin{aligned} \alpha_0|\alpha,\overline{\alpha}\rangle &= \alpha|\alpha,\overline{\alpha}\rangle \\ \overline{\alpha}_0|\alpha,\overline{\alpha}\rangle &= \overline{\alpha}|\alpha,\overline{\alpha}\rangle \\ \alpha_n|\alpha,\overline{\alpha}\rangle &= 0 \quad \text{for } n>0 \\ \overline{\alpha}_n|\alpha,\overline{\alpha}\rangle &= 0 \quad \text{for } n>0 \end{aligned} \tag{15.138}$$

We can build towers of states on these by acting with the α_ns for $n<0$ or, equivalently, with the L_ns for $n<0$ and also with their anti-holomorphic counterparts. These are, in fact, Verma modules, and the states $|\alpha,\overline{\alpha}\rangle$ are highest weight states. Their L_0, $\overline{L}_0$ eigenvalues are:

$$\begin{aligned} L_0|\alpha,\overline{\alpha}\rangle &= \frac{1}{2}\alpha_0^2|\alpha,\overline{\alpha}\rangle = \frac{1}{2}\alpha^2|\alpha,\overline{\alpha}\rangle \\ \overline{L}_0|\alpha,\overline{\alpha}\rangle &= \frac{1}{2}\overline{\alpha}^2|\alpha,\overline{\alpha}\rangle \end{aligned} \tag{15.139}$$

These highest weight states are created by exponential operators. The simplest is the vacuum state itself, defined by

$$\alpha_0|0\rangle = 0$$

$$\overline{\alpha}_0|0\rangle = 0 \tag{15.140}$$

Now, consider the state $:e^{i\beta\phi(0)}:|0\rangle$,

$$:e^{i\alpha\phi(0)}:|0\rangle = e^{i\alpha\phi_0}|0\rangle \tag{15.141}$$

the right-hand-side follows from the fact that $\alpha_n|0\rangle = 0$ for $n \geq 0$, while the $n < 0$ terms in the expansion vanish when we take $z = 0$. Thus, acting with α_0, $\overline{\alpha}_0$,

$$\begin{aligned} \alpha_0 :e^{i\beta\phi(0)}:|0\rangle &= \alpha_0 e^{i\beta\phi_0}|0\rangle \\ &= \left(e^{i\beta\phi_0}\alpha_0 \;+\; e^{i\beta\phi_0}\frac{1}{2}\beta\right)|0\rangle \\ &= \left(\frac{1}{2}\beta\right) :e^{i\beta\phi(0)}:|0\rangle \end{aligned} \tag{15.142}$$

Similarly,

$$\overline{\alpha}_0 :e^{i\beta\phi(0)}:|0\rangle = \left(\frac{1}{2}\beta\right) :e^{i\beta\phi(0)}:|0\rangle \tag{15.143}$$

To summarize, the different Verma modules of the theory correspond to different values of the zero modes, $|\alpha,\overline{\alpha}\rangle$. Some of these highest weight states are created by the primary operators $:e^{i\beta\phi(0)}:$ acting on the vacuum:

$$:e^{i\beta\phi(0)}:|0\rangle = |\beta/2,\beta/2\rangle \tag{15.144}$$

Let us find the other primary fields in the case in which ϕ is an angular variable, $\phi \equiv \phi + 2\pi R$. Then, $\phi_0 \equiv \phi_0 + 2\pi R$. Since $[\phi_0,(\alpha_0+\overline{\alpha}_0)] = i$, the eigenvalues of $(\alpha_0+\overline{\alpha}_0)$ are quantized, $(\alpha_0+\overline{\alpha}_0) = \frac{m}{R}$.

Periodic boundary conditions around the cylinder require, according to (15.134), that $\alpha_0 - \overline{\alpha}_0 = 2nR$ for some integer n, so that $\phi \to \phi + 2\pi nR$ when $x \to x + 2\pi$. Putting these two together, we have

$$\begin{aligned} \alpha_0 + \overline{\alpha}_0 &= \frac{m}{R} \\ \alpha_0 - \overline{\alpha}_0 &= 2nR \end{aligned} \tag{15.145}$$

or

$$\alpha_0 = \frac{m}{2R} + nR\ , \qquad \overline{\alpha}_0 = \frac{m}{2R} - nR \tag{15.146}$$

Thus, the highest weight states of a free boson with angular identification $\phi \equiv \phi + 2\pi R$ are:

$$\left|\frac{m}{2R}+nR,\ \frac{m}{2R}-nR\right\rangle = :e^{i\left(\frac{m}{2R}+nR\right)\varphi}\, e^{i\left(\frac{m}{2R}-nR\right)\overline{\varphi}}:|0\rangle$$

$$= : e^{im\phi/R}\, e^{2inR\tilde{\phi}} : |0\rangle \tag{15.147}$$

where we have defined the dual field $\tilde{\phi} = \frac{1}{2}(\varphi - \overline{\varphi})$. These states have $(h, \overline{h}) = \left(\frac{1}{2}\left(\frac{m}{2R} + nR\right)^2, \frac{1}{2}\left(\frac{m}{2R} - nR\right)^2\right)$.

The action of the exponential of the dual field, $: e^{2inR\tilde{\phi}} :$ is to increase $(\alpha_0 - \overline{\alpha}_0)/2$ by nR. This causes ϕ to wind n more times around 2π as x winds around the cylinder or z winds around the origin in the complex plane. In other words, $: e^{2inR\tilde{\phi}(0)} :$ creates an n-fold vortex at the origin of the complex plane.

Let us consider the special case $R = 1$. At this radius, the primary states are $\left|\frac{m}{2} + n,\ \frac{m}{2} - n\right\rangle$ Let us divide these into the states in which $m \in 2Z + 1$ and those in which $m \in 2Z$. The states with $m \in 2Z$ are of the form

$$L_0|n_R, n_L; P\rangle = \frac{1}{2}\, n_R^2 |n_R, n_L; P\rangle \tag{15.148}$$

with the obvious counterpart for $\overline{L}_0$. Here, $n_{R,L} = \frac{m}{2} \pm n \in Z$. The states with $m \in 2Z + 1$ are of the form

$$L_0|n_R, n_L; AP\rangle = \frac{1}{2}\left(n_R + \frac{1}{2}\right)^2 |n_R, n_L; AP\rangle \tag{15.149}$$

with the obvious counterpart for $\overline{L}_0$.

These are the primary states of a Dirac fermion, which also has $c = 1$ since it is composed of two Majorana fermions, $\psi_R = \psi_{R,1} + i\psi_{R,2}$, each with $c = 1/2$. The same holds for the anti-holomorphic part of the theory, with $\overline{c} = 1$. If we make the identifications

$$\begin{aligned} e^{i\varphi} &= \psi_R = \psi_{R,1} + i\psi_{R,2} \\ e^{i\overline{\varphi}} &= \psi_L = \psi_{L,1} + i\psi_{L,2} \\ e^{2i\phi} &= \psi_R \psi_L \\ e^{2i\tilde{\phi}} &= \psi_L^\dagger \psi_R \\ e^{i\phi} &= \sigma_{R,1}\sigma_{R,2}\, \sigma_{L,1}\sigma_{L,2} \end{aligned} \tag{15.150}$$

$\sigma_{R,1}, \sigma_{R,2}, , \sigma_{L,1}, \sigma_{L,2}$ are the twist fields for $\psi_{R,L;1,2}$ or, simply the product of the twist fields for the Dirac fermions ψ_R, ψ_L. This product has dimension $(\frac{1}{8}, \frac{1}{8})$.

From these formulae, we see that

$$\begin{aligned} \psi_R^\dagger \psi_R &= e^{-i\varphi(z)} e^{i\varphi(w)} \\ &= \frac{1}{z - w} e^{-i(\varphi(z) - \varphi(w))} \end{aligned}$$

$$= \frac{1}{z-w}\left(1 - i(w-z)\partial\varphi(w) + \ldots\right)$$
$$= \frac{1}{z-w} + i\partial\varphi \tag{15.151}$$

Hence, upon normal ordering, we get the conserved quantity

$$n_R =: \psi_R^\dagger \psi_R := i\partial\varphi \tag{15.152}$$

similarly, $n_L = i\overline{\partial}\overline{\varphi}$.

One slightly peculiar thing about this theory is that we started with two copies of the $c = 1/2$ theory, which has only 3 primary states, corresponding to the (1) even and (2) odd fermion number in the untwisted sector and the (3) twisted sector (where even and odd fermion number are not distinguished because the zero mode moves one between them). When we tensor these two copies of the theory together, we don't get 9 primary states but, rather, infinitely many primary states. This is not really so peculiar when we recall that we are now talking about primary under the $c = 1$ Virasoro algebra of the combined theory, not under the individual $c = 1/2$ Virasoro algebras. The primary states correspond to different n_R and n_L in the twisted and untwisted sectors. There is a big difference between a Dirac fermion and a Majorana fermion because the former has a conserved charge (or, rather, two conserved charges, one right-moving and one left-moving), while the latter does not.

15.13 Kac-Moody Algebras

We now consider conformal field theories which have other symmetries – such as charge conservation – in addition to symmetry under conformal transformations. As we will see, when these other symmetries mesh with conformal invariance in a certain way, they lead to the existence of null states, which render the theory soluble. In such a case, the symmetry generators form a Kac-Moody algebra.

Consider a theory with conserved currents J_μ^a, $\partial^\mu J_\mu^a = 0$, where $a = 1, 2, \ldots, n$ label the currents. The associated conserved charges are $Q^a = \int dx\, J_0^a$. They satisfy some symmetry algebra $\left[Q^a, Q^b\right] = i f_{abc} Q^c$, where the f_{abc}s are the structure constants of some Lie algebra G. (The commutator must equal a linear combination of charges because any state which is invariant under the symmetry transformations generated by the charges must be annihilated by the charges and also by their commutators.) In the case $G = SU(2)$, $f_{abc} = \epsilon_{abc}$.

Now, suppose that the dual current $\epsilon^{\mu\nu} J^a_\nu$ is also conserved, $\partial^\mu \epsilon^{\mu\nu} J^a_\nu = 0$. Then,

$$\partial \overline{J} = \overline{\partial} J = 0 \tag{15.153}$$

where $J = J_0 + iJ_1$, $\overline{J} = J_0 - iJ_1$. In other words, $J(z)$ is holomorphic while $\overline{J}(\overline{z})$ is antiholomorphic.

There two conserved charges, associated with the integrals of the current or the dual current. In more physical terms, the right-moving and left-moving charges are separaely conserved. The charges $Q^R = \oint dz J$ and $Q^R = \oint d\overline{z}\overline{J}$ must be dimensionless since, as conserved quantities, they commute with L_0 and $\overline{L}_0$. This, in turn, implies that J and $\overline{J}$ are dimension $(1,0)$ and $(0,1)$ fields, respectively.

The most general form for the OPE for J^a is

$$J^a(z)\, J^b(0) = \frac{k\delta^{ab}}{z^2} + + \frac{i f^{abc}}{z}\, J^c(0) + \dots \tag{15.154}$$

The second term on the right-hand-side is dictated by the commutators of the Q^as. The first term is allowed by the global symmetry (generated by the Q^as) and by scaling. The normalization of the central extension k is fixed by the normalization of the structure constants in the second term, except in the Abelian case, in which they vanish.

This OPE can be translated into the commutation relations of the modes $J^a(z) = \sum_n J^a_n z^{-n-1}$.

$$\left[J^a_m, J^b_n\right] = i\, f^{abc}\, J^c_{m+n} \;+\; k\, m\, \delta^{ab} \delta_{m+n,0} \tag{15.155}$$

The $m = 0$ modes still satisfy the commutation relations of the Lie algebra G.

The simplest example of a theory with a Kac-Moody algebra is the free boson. The conserved currents are:

$$\begin{aligned} J(z) &= i\partial\varphi \\ \overline{J}(\overline{z}) &= i\overline{\partial}\overline{\varphi} \end{aligned} \tag{15.156}$$

These are the currents associated with the global symmetry $\phi \to \phi + c$, where c is a constant. As a result of the decoupling of the right- and left-handed parts of the theory, there is actually a much large set of symmetries of the classical equation of motion, $\varphi \to \varphi + f(z)$, $\overline{\varphi} \to \overline{\varphi} + \overline{f}(\overline{z})$.

The OPE of the currents is

$$J(z)\, J(0) = -\,\partial\varphi(z)\, \partial\varphi(0)$$

$$= \frac{1}{z^2} + 2T(z) + \dots \tag{15.157}$$

There is no $1/z$ term, as we would expect since the $U(1)$ algebra is Abelian. We see that $k = 1$, irrespective of whether the boson is an angular variable (i.e. is 'compactified'). (In this case, this is somewhat a matter of convention because there is no term on the right-hand-side linear in the currents so we can rescale $J(z)$ to change k to whatever we wish.) The special case of a boson with $R = 1$ is equivalent to a Dirac fermion, which also has a $U(1)$ symmetry, $\psi \to e^{i\theta}\,\psi$, which is promoted to a Kac-Moody symmetry with $J = \psi_R^\dagger \psi_R$ and $\overline{J} = \psi_R^\dagger \psi_L$.

A more interesting example is afforded by a free boson at $R = 1/\sqrt{2}$. Now, there are additional currents, $e^{\pm i\varphi\sqrt{2}} = e^{\pm i(\phi/R + 2\tilde{\phi}R)}$, which have dimension $(1,0)$, and $e^{\pm i\overline{\varphi}\sqrt{2}} = e^{\pm i(\phi/R - 2\tilde{\phi}R)}$ which have dimension $(0,1)$. If we write $J^z = i\partial\varphi$, $J^\pm = e^{\pm i\varphi\sqrt{2}}$ and similarly for the antiholomorphic currents, then these operators have the OPE expected for an $SU(2)$ Kac-Moody algebra:

$$\begin{aligned} J^+(z)\,J^-(0) &= \frac{1}{z^2} + \frac{2i}{z}\,J^z(0) + \dots \\ J^\pm(z)\,J^z(0) &= \pm\frac{i}{z}\,J^z(0) + \dots \end{aligned} \tag{15.158}$$

Again, the level is $k = 1$. Such an algebra will be called $SU(2)_1$. More generally, an $SU(2)$ Kac-Moody algebra at level k wil be called $SU(2)_k$; that associated with an arbitrary Lie algebra G, G_k.

The same algebra arises in a theory of 2 Dirac fermions. We focus on the holomorphic part for simplicity:

$$\int \psi^\dagger_{\alpha;R}\overline{\partial}\psi_{\alpha;R} \tag{15.159}$$

where $\alpha = 1, 2$. This action is invariant under the symmetry $\psi_{\alpha;R} \to U_{\alpha\beta}\psi_{\beta;R}$ where $U^\dagger U = 1$, i.e. $U \in SU(2)$. The associated currents are

$$J^a = \psi^\dagger_{\alpha;R}\,\tau^a_{\alpha\beta}\,\psi_{\alpha;R} \tag{15.160}$$

where $\tau^a_{\alpha\beta}$, $a = 1, 2, 3$ are Pauli matrices. This can be generalized straighforwardly to a theory of N Dirac fermions,

$$\int \psi^\dagger_{A;R}\overline{\partial}\psi_{A;R} \tag{15.161}$$

where $A = 1, 2, \ldots, N$. This theory is invariant under $SU(N)$ transformations, $\psi_{AR} \to U_{AB}\psi_{B;R}$ with associated currents

$$J^a = \psi^\dagger_{A;R}\, T^a_{AB}\, \psi_{B;R} \tag{15.162}$$

and the T^a_{AB}s are the generators of the fundamental representation of $SU(N)$. They satisfy the $SU(N)_1$ Kac-Moody algebra.

The simplest way of obtaining a Kac-Moody algebra with a level other than 1 is to simply take k copies of this theory or, in other words, kN Dirac fermions. Such a theory has an $SU(kN)_1$ symmetry, but if we focus on the $SU(N)$ subgroup of $SU(kN)$, then this algebra realizes an $SU(N)_k$ Kac-Moody algebra since the level is clearly additive when we take k copies of a theory. Note that these fermionic theories all have $U(1)$ Kac-Moody algebras as well.

The bosonic representations of these Kac-Moody algebras are rather interesting and non-trivial, especially in the $k > 1$ case. We will return to them later.

Let us now consider the representation theory of Kac-Moody algebras. We define a primary field $\varphi_{(r)}$ to be a field which transforms in representation r of the group G and is primary under the Kac-Moody algebra,

$$J^a(z)\, \varphi_{(r)}(0) = \frac{T^a_{(r)}\, \varphi_{(r)}(0)}{z} + \ldots \tag{15.163}$$

where the $T^a_{(r)}$s are the matrices representing the generators in representation (r). As usual, primary fields create highest weight states,

$$|(r), \alpha\rangle = \varphi^\alpha_{(r)}|0\rangle \tag{15.164}$$

Here, we have explicitly written the representation vector index α which we suppressed earlier for convenience. The highest weight states form a multiplet under the global symmetry generated by the J^a_0s:

$$J^a_0|(r), \alpha\rangle = \left(T^a_{(r)}\right)_{\alpha\beta}|(r), \beta\rangle \tag{15.165}$$

Not all representations (r) are allowed in the theory at level k. For the sake of concreteness, we consider the case of $SU(2)$. A restriction on the allowed j's in the $SU(2)_k$ theory can be found by considering the mode expansion of J^a, $J^a(z) = \sum_m J^a_m z^{-m-1}$. The J^a_ms satisfy the commutation relations (15.155). From these commutation relations, we see that $I^a \equiv J^a_0$ form an $SU(2)$ Lie algebra. Hence, $2J^3_0$ has integer eigenvalues in any finite-dimensional unitary representation. Similarly, $\tilde{I}^1 \equiv J^1_1$, $\tilde{I}^2 \equiv J^2_{-1}$, $\tilde{I}^3 \equiv$

$\frac{1}{2}k - J_0^3$ also form an $SU(2)$ Lie algebra. Consider a spin j highest weight state $|j, m = j\rangle$, with $I^3|j, m = j\rangle = j|j, m = j\rangle$. Then

$$\begin{aligned} 0 &\leq \langle j, m = j|\tilde{I}^+\tilde{I}^-|j, m = j\rangle \\ &= \langle j, m = j|\left[\tilde{I}^+, \tilde{I}^-\right]|j, m = j\rangle \\ &= \langle j, m = j|k - 2I^3|j, m = j\rangle \\ &= k - 2j \end{aligned} \tag{15.166}$$

Hence, the $SU(2)_k$ theory contains the representations $j = 1/2, 1, \ldots, k/2$ and only these representations.

Since the $\tilde{I}^a$s form an $SU(2)$ algebra, $\tilde{I}^3$ must have integer eigenvalues, from which we conclude that k must be an integer.

One remarkable feature of a Kac-Moody algebra is that it automatically includes the structure of the Virasoro algebra. Consider $T(z)$ defined by

$$T = \frac{1/2}{k + C_A} \, : J^a \, J^a : \tag{15.167}$$

where C_A is the quadratic Casimir in the adjoint representation if the highest root is normalized to length 1. This satisfies the Virasoro algebra,

$$\begin{aligned} T(z)\,T(0) &= \left(\frac{1/2}{k + C_A}\right)^2 \, : J^a(z)\, J^a(z) : \, : J^b(0)\, J^b(0) : \\ &= \left(\frac{1/2}{k + C_A}\right)^2 \times 2 \times (J^a(z) J^a(0)) \left(J^b(z) J^b(0)\right) + \ldots \\ &= \left(\frac{1/2}{k + C_A}\right)^2 \times 2 \times \left(\frac{k\delta^{ab}}{z^2} + \frac{i f^{abc}}{z}\, J^c(0) + \ldots\right) \times \\ &\qquad \left(\frac{k\delta^{ab}}{z^2} + \frac{i f^{abd}}{z}\, J^d(0) + \ldots\right) + \ldots \\ &= \frac{1/2}{(k + C_A)^2} \left(\frac{k^2 \delta^{ab}\delta^{ab}}{z^4} - \frac{f^{abc} f^{abd}}{z^2} J^c J^d\right) \\ &= \frac{1/2}{(k + C_A)^2} \left(\frac{k^2 |G|}{z^4} + \frac{C_A \delta^{cd}}{z^2} J^c J^d\right) \\ &= \frac{1/2}{(k + C_A)^2} \left(\frac{k^2 |G|}{z^4} + \frac{C_A \delta^{cd}}{z^2} \frac{k\delta^{cd}}{z^2}\right) \\ &= \frac{1}{2} \frac{k|G|}{k + C_A} \frac{1}{z^4} \\ &= \frac{c/2}{z^4} \end{aligned} \tag{15.168}$$

with

$$c = \frac{k|G|}{k + C_A} \tag{15.169}$$

where $|G|$ is the dimension of the Lie algebra. One can check that the next term in the OPE is as expected for the energy-momentum tensor, so that

$$T(z)\,T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}\,T(w) + \frac{1}{z-w}\,\partial T(w) + \dots \qquad (15.170)$$

This immediately enables us to compute the dimensions of primary fields. According to (15.167), a primary field transforming under G as representation r has dimension

$$\begin{aligned} T(z)\,\varphi_{(r)}(0) &= \frac{1}{z^2}\,\frac{T^a_{(r)}T^a_{(r)}/2}{k+C_A} \;+\; \dots \\ &= \frac{1}{z^2}\,\frac{C_r}{k+C_A} \;+\; \dots \end{aligned} \qquad (15.171)$$

For the case of $SU(2)_k$, this means that a spin j primary field has dimension $h = j(j+1)/(k+2)$.

The correlation functions of primary fields in a theory with Kac-Moody symmetry can be calculated using the Sugawara construction of $T(z)$. Expanding both sides of (15.167) in modes, the $n = -1$ term is:

$$L_{-1} = \frac{1}{k+C_A}\left(J^a_{-1}J^a_0 + J^a_{-2}J^a_1 + \dots\right) \qquad (15.172)$$

We act with this equation on a primary field, using the fact that J^a_n for $n > 0$ annihilates primary fields, together with (15.163). Hence, there is a null state given by

$$\left(L_{-1} - \frac{1}{k+C_A}J^a_{-1}\,T^a_{(r)}\right)\varphi_{(r)} = 0 \qquad (15.173)$$

This null state condition translates into the following differential equation – the Knizhnik-Zamolodchikov equation – for correlation functions of primary fields:

$$\left((k+C_A)\,\frac{\partial}{\partial z_k} + \sum_{j\neq k}\frac{T^a_{(r_j)}T^a_{(r_k)}}{z_j - z_k}\right)\left\langle\varphi_{(r_1)}(z_1)\dots\varphi_{(r_n)}(z_n)\right\rangle = 0 \qquad (15.174)$$

These first-order differential equations can be solved to determine the correlation functions of primary fields.

The fact that the T given by the Sugawara construction satisfies the Virasoro algebra does not necessarily imply that it is the full energy-momentum tensor of the theory. It may only be the energy-momentum tensor of one

sector of the theory; the full energy-momentum tensor is then the sum of several tensors, each of which obeys the Virasoro algebra. The central charges add.

Consider our theory of two chiral Dirac fermions. This theory has both $U(1)$ and $SU(2)_1$ Kac-Moody symmetries, with

$$T^{SU(2)} = \frac{1/2}{k+2} \, : J^a \, J^a : \tag{15.175}$$

since $C_A = 1(1+1) = 2$; $a = 1, 2, 3$. The central charge associated with the $SU(2)$ part of the theory is $c_{SU(2)} = 1$. Meanwhile,

$$T^{U(1)} = \frac{1}{2} \, : J \, J : \tag{15.176}$$

The normalization is chosen so that the second term in the OPE of $T^{U(1)}$ with itself is $2T/z^2$. As a result, $c_{U(1)} = 1$. The total central charge $c_{SU(2)} + c_{U(1)} = 2$ is the central charge of two Dirac fermions.

Since the level of the $SU(2)$ Kac-Moody algebra is 1, a spin-1/2 primary field has dimension 1/4. Similarly, a charge-1 primary field under $U(1)$ has dimension 1/4. A Dirac fermion carries charge-1 and spin-1/2. Thus, it has dimension 1/2, as expected.

There is not necessarily a unique way of decomposing T into Kac-Moody algebras. For instance, consider a theory of kN Dirac fermions, with $c = kN$. This can be decomposed into $c_{U(1)} = 1$ and, from (15.169) with $|SU(N)| = N^2 - 1$ and $C_A = N$, $c_{SU(kN)_1} = kN - 1$. Alternatively, we can think of our kN fermions as k sets of N fermions. There is an $SU(N)_k$ symmetry among the N fermions (rotating all sets together) and also an $SU(k)_N$ symmetry among the k sets. $c_{SU(N)_k} = \frac{k(N^2-1)}{k+N}$ while $c_{SU(k)_N} = \frac{N(k^2-1)}{N+k}$. Thus, $c_{U(1)} + c_{SU(N)_k} + c_{SU(k)_N} = kN$ is an equally good decomposition of the theory into Kac-Moody algebras. In the multichannel Kondo problem, with $N = 2$ and k channels, this proves to be a particularly useful decomposition.

Earlier, we alluded to the bosonic theories which posess Kac-Moody symmetry at level $k > 1$. We now discuss them. These theories are non-linear σ-models of Wess-Zumino-Witten (WZW) type. In these theories, the basic field, U, takes values in some group G. Since the group G is some curved space, a WZW model is an interacting field theory, similar in spirit but different in detail to, say, the $O(3)$ non-linear σ model.

let us consider such a theory on the cylinder $S^1 \times R$. The WZW action is:

$$S_{WZW} = \frac{k}{4\pi} \int_{S^1 \times R} \mathrm{tr} \left(\partial_\mu \, U^{-1} \partial^\mu U \right) \, +$$

$$\frac{k}{12\pi}\int_{D^2\times R}\epsilon^{\mu\nu\lambda}\mathrm{tr}\left(\partial_\mu U\,U^{-1}\partial_\nu U\,U^{-1}\,\partial_\lambda U\,U^{-1}\right) \qquad (15.177)$$

The second term – often called a Wess-Zumino term – appears to be an integral over the $3D$ manifold given by the solid cylinder, $D^2\times R$, but, in fact, it only depends on the boundary values of U. This is completely analogous to the Berry phase term which we have in the action of a single spin.

The equation of motion which follows from the WZW model is

$$\partial_\mu\left(U^{-1}\partial^\mu U-\epsilon^{\mu\nu}U^{-1}\partial_\nu U\right)=0 \qquad (15.178)$$

The second term in the parentheses results from the Wess-Zumino term in the action. The equation of motion is only for the restriction of U to the boundary, $S^1\times R$, which is one way of seeing that the action is independent of the continuation from $S^1\times R$ to $D^2\times R$.

It is useful to rewrite the equation of motion in terms of $z,\overline{z}$:

$$\overline{\partial}\left(U^{-1}\partial U\right)=0 \qquad (15.179)$$

This automatically implies that

$$\partial\left(\overline{\partial}U\,U^{-1}\right)=0 \qquad (15.180)$$

since

$$\begin{aligned}\overline{\partial}\left(U^{-1}\partial U\right)&=U^{-1}\overline{\partial}\partial U-U^{-1}\overline{\partial}U\,U^{-1}\,\partial U\\&=U^{-1}\partial\left(\overline{\partial}U\,U^{-1}\right)U^{-1}\end{aligned} \qquad (15.181)$$

Thus, we have right- and left-handed currents

$$\begin{aligned}J^a&=\mathrm{tr}\left(T^aU^{-1}\partial U\right)\\\overline{J}^{\,a}&=\mathrm{tr}\left(T^a\overline{\partial}U\,U^{-1}\right)\end{aligned} \qquad (15.182)$$

where the T^as are the generators of the Lie algebra. Note the asymmetry between the definitions of J^a and $\overline{J}^{\,a}$.

We can compute the commutators of the modes of J^a, $\overline{J}^{\,a}$ by first deriving the canonical commutation relations from the action: they are those of a Kac-Moody algebra at level k.

15.14 Coulomb Gas

15.15 Interacting Fermions

15.16 Fusion and Braiding

Part V

Symmetry-Breaking In Fermion Systems

CHAPTER 16

Mean-Field Theory

16.1 The Classical Limit of Fermions

At low-temperatures, there is a canonical guess for the ground state of a system of bosons ψ, the broken-symmetry state $\langle\psi(x)\rangle \neq 0$. In this state, the system becomes rather classical. One can define a classical field $\Psi_c(x) = \langle\psi(x)\rangle$ which describes the properties of the system at the level of the saddle-point approximation to the functional integral. $\Psi_c(x)$ satisfies differential equations (the saddle-point condition) which can be solved in different experimental geometries or with different boundary conditions, just like a classical field.

With fermions we don't have such an option. A fermion field, χ, can't have an expectation value. Consider the functional integral for $\langle\psi\rangle$

$$\langle\psi\rangle = \int d\psi d\overline{\psi} \ldots \psi\, e^{-S} \tag{16.1}$$

Taylor expanding $\exp(-S)$, we obtain only terms with an even number of Grassman fields. Therefore, the integrand contains only terms with an odd number of Grassman fields. Hence, it must vanish. Introducing small bosonic (i.e. physical, at least in principle) symmetry-breaking fields cannot alter this conclusion.

This makes fermion problems intrinsically a little more difficult: there is no stable phase which is a natural, 'classical' ground state of the system.

This fact is somewhat obscured by the remarkable stability of the Fermi liquid critical point which, though it is a critical point, can almost disguise itself as a stable phase. In strongly-interacting fermions systems, however, the Fermi liquid can be unstable to several different phases. In such a regime, perturbation theory about the Fermi liquid critical point is, of course, hopeless. A brute force numerical solution is almost certain to run into difficulties of the same origin: in many problems, algorithms will converge very slowly because they must decide between many possible competing ground states.

To understand one or another of the stable phases, one is better off starting deep within one of these phases. These stable phases fall into several categories. In this part of the book, we focus on phases in which the fermions organize themselves into a bosonic degree of freedom which condenses, breaking a symmetry, and producing a 'classical field' describing the ground state. In later parts of this book, we will consider two others: localized phases in dirty systems and topologically-ordered phases.

16.2 Order Parameters, Symmetries

The simplest bosonic degrees of freedom which can arise in fermionic systems are composed of fermion bilinears. Broken symmetry states occur when one of these bilinears acquires a non-zero expectation value. It is sometimes useful to think of these fermion bilinears as annihilation operators for 'bound states' formed by two electrons or by an electron and a hole. The broken symmetry state is a state in which the 'bound state' condenses. The reason for the quotation marks is that there need not actually be a stable bound state. In this section, we will discuss some of the possible order parameters in a fermionic system.

Let us first consider superfluid/superconducting order parameters. As we discussed in chapter ..., a superconductor is is characterized by the breaking of the global $U(1)$ symmetry associated with charge conservation (if we ignore for the moment the electromagnetic field and issues of gauge invariance in the superconducting case). In other words, the symmetry

$$\psi_\alpha \to e^{i\theta}\,\psi_\alpha \tag{16.2}$$

must be broken. This occurs when one of the following bilinears acquires an expectation value:

$$\langle \psi_\alpha(k,t)\,\psi_\beta(-k,t)\rangle \tag{16.3}$$

Focussing for the moment on the spin structure of the order parameter,

we see that there are two categories, spin-singlet and spin-triplet superfluids/superconductors. The former occur when

$$\langle \psi_\alpha(\mathbf{k},t)\,\psi_\beta(-\mathbf{k},t)\rangle = \Psi(\mathbf{k})\,\epsilon_{\alpha\beta} \tag{16.4}$$

while the latter are characterized by the expectation value

$$\langle \psi_\alpha(k,t)\,\psi_\beta(-k,t)\rangle = \vec{\Psi}(\mathbf{k})\cdot\vec{\sigma}_\alpha{}^\gamma\epsilon_{\gamma\beta} \tag{16.5}$$

Fermi statistics requires that in a singlet superfluids/superconductor $\Psi(\mathbf{k})$ must be even in $\mathbf{k}$, while in a triplet superfluid/superconductor, $\vec{\Psi}(\mathbf{k})$ must be odd in $\vec{k}$.

In order to discuss the $\mathbf{k}$ dependence of the functions $\Psi(\mathbf{k})$, $\vec{\Psi}(\mathbf{k})$, we must make some further assumptions about the system. For the superfluid/superconducting case, we will consider a translationally- and rotationally-invariant system, since this case applies to ^{3}He, as well as the case of a system on a $2D$ square lattice. For the case of density-wave order parameters, to be introduced later in this section, we will focus on the $2D$ square lattice. Other lattices will be considered in the problems at the end of the chapter.

Let us first consider superfluids in the continuum. A singlet superfluid can be thought of as a condensate of spin-singlet fermion pairs which have orbital wavefunction $\Psi(k)$. In the continuum, an energy eigenstate of such a pair will have well-defined L^2, L_z (let us ignore spin-orbit coupling), thanks to rotational invariance. It is possible for two different pairing states with different orbital and/or spin angular momenta to condense. The simplest way for this to happen is if a metal has several bands, and the order parameters in different bands have different symmetries. It is also possible, in principle, for this to occur within a single band. However, two different angular momentum eigenstates would somehow have to become degenerate, which would require fine-tuning, so we will not discuss this further here (see problems, however?). Once we have determined the L^2 eigenvalue, $\ell(\ell+1)$, there is still freedom in the choice of L_z eigenvalue m. Thus, we can focus on angular momentum eigenstates of the form:

$$\langle \psi_\alpha(\mathbf{k},t)\,\psi_\beta(-\mathbf{k},t)\rangle = \varphi(k)\left(\sum_{m=-\ell}^{\ell} d_m Y_m^\ell(\hat{\mathbf{k}})\right)\epsilon_{\alpha\beta} \tag{16.6}$$

where ℓ must be even by Fermi statistics. The cases $\ell = 0, 2, 4$ are called s-wave, d-wave, and g-wave superconductors. Since the d_m's are arbitrary complex numbers, the order parameters for these three cases have 2-, 10-,

and 18-component order parameters, d_m, respectively. $\varphi(k)$ is the radial part of the pair wavefunction.

In order to allow for spatially inhomogeneous situations, such as those resulting from boundary conditions of the presence of a magnetic field, we should allow the fermion fields to be at arbitrary momenta, not necessarily $\mathbf{k}$ and $-\mathbf{k}$.

$$\left\langle \psi_\alpha(\mathbf{k},t)\,\psi_\beta(-\mathbf{k}',t)\right\rangle = \epsilon_{\alpha\beta}\,\varphi(|\mathbf{k}-\mathbf{k}'|)\times \int \frac{d^3\mathbf{R}}{(2\pi)^3}\,e^{i\mathbf{R}\cdot(\mathbf{k}+\mathbf{k}')}\left(\sum_{m=-\ell}^{\ell} d_m(\mathbf{R})Y_m^\ell(\hat{\mathbf{k}})\right) \quad (16.7)$$

Here, we have Fourier transformed the center-of-mass momentum of the pair in order to reveal the possible spatial dependence of $d_m(\mathbf{R})$.

The underlying symmetry of the system is $U(1)\times O(3)\times Z_2^{\rm T}$, where $Z_2^{\rm T}$ is the Z_2 symmetry of time-reversal (there is also an $SU(2)$ spin symmetry which is unaffected by singlet ordering). This is broken down to $Z_2\times U(1)\times Z_\ell\times Z_2^{\rm T}$. The order parameter is composed of a product of two fermion operators, so the $Z_2\in U(1)$ transformation $\psi\to-\psi$ leaves the order parameter unchanged. $U(1)\in O(3)$ is the subgroup of rotations about the direction of the angular momentum vector of the pair. $Z_\ell\in O(3)$ is the discrete set of rotations which leave $Y_m^\ell(\hat{\mathbf{k}})$ invariant. Finally, ... **needs to be finished**

In a triplet superfluid in the continuum, we have

$$\langle\psi_\alpha(k,t)\,\psi_\beta(-k,t)\rangle = \varphi(k)\,\left(\sum_{m=-\ell}^{\ell} Y_m^\ell(\mathbf{k})\vec{d_m}\right)\cdot\vec{\sigma}_\alpha{}^\gamma\epsilon_{\gamma\beta} \quad (16.8)$$

Thus, an $\ell=1$ triplet superfluid in the continuum has an 18-component order parameter $\vec{d_m}$.

The symmetry group of the system is $U(1)\times O(3)\times SU(2)\times Z_2^{\rm T}$. The symmetry-breaking pattern depends on the $\vec{d_m}$s. In the A phase of ^{3}He, $\vec{d_0}=\vec{d_-}=0$, while $\vec{d_+}=\vec{d}$, for some real vector $\vec{d}$:

$$\langle\psi_\alpha(k,t)\,\psi_\beta(-k,t)\rangle = \varphi(k)\,(k_x+ik_y)\,\vec{d}\cdot\vec{\sigma}_\alpha{}^\gamma\epsilon_{\gamma\beta} \quad (16.9)$$

The angular momentum of the pair thus points in the z-direction (which we have arbitrarily chosen, without loss of generality; any other direction is equally good). The pairs have spins $\uparrow\downarrow+\downarrow\uparrow$ in the $\vec{d}$-direction, i.e. the total spin of each pair has vanishing component along the d-direction. (We have ignored spin-orbit coupling, which would break independent orbital

angular momentum and spin conservation down to total angular momentum conservation. The spin-orbit interaction of ^{3}He favors alignment of d with the angular momentum.) In this phase, the remaining symmetry is $U(1) \times U(1)$, where the two $U(1)$ factors correspond to spin rotations about the $\vec{d}$-direction and rotations about the z-axis combined with gauge transformations (the former yield a phase factor which is cancelled by the latter). Time-reversal is broken by the selection of $m = +1$, which transforms into $m = -1$ under T.

In the B phase, $\vec{d_0} = \hat{\mathbf{z}}$, $\vec{d_\pm} = (\hat{\mathbf{x}} \mp i\hat{\mathbf{y}})/\sqrt{2}$ or, simply,

$$\langle \psi_\alpha(k,t)\, \psi_\beta(-k,t) \rangle = \varphi(k)\, \hat{\mathbf{k}} \cdot \vec{\sigma}_\alpha{}^\gamma \epsilon_{\gamma\beta} \tag{16.10}$$

The pairs have orbital angular momentum $\ell = 1$ and spin angular momentum $S = 1$ which add together to give total angular momentum $J = 0$. Thus, the B phase is invariant under $SO(3) \times Z_2^{\mathrm{T}}$, where the $SO(3)$ is the group of simultaneous rotations of both space and spin.

The forms which we have chosen are actually too restrictive. To be completely general, we should allow the fermion operators to be at different times t, t' and allow the right-hand-side to have non-trivial dependence on $t-t'$. If it is an odd function of $t-t'$, so that the correlation function actually vanishes for $t = t'$, then the order is called odd-gap superconductivity.

Now, let us consider a system of electrons in $2D$ on a square lattice of side a. The symmetry group of the square lattice is D_4, with 8 elements: 4 rotations, including the identity – by 0, $\pi/2$, π, $3\pi/2$ – and 4 reflections – through the x-axis, the y-axis, and the lines $x = \pm y$. An s-wave superconductor has an order parameter of the form:

$$\langle \psi_\alpha(\mathbf{k},t)\, \psi_\beta(-\mathbf{k},t) \rangle = \Psi_0\, \epsilon_{\alpha\beta} \tag{16.11}$$

Turning to p-wave superconductors, we see that the analog of the A phase is:

$$\langle \psi_\alpha(k,t)\, \psi_\beta(-k,t) \rangle = \Psi_0\, (\sin k_x a + i \sin k_y a)\, \vec{d} \cdot \vec{\sigma}_\alpha{}^\gamma \epsilon_{\gamma\beta} \tag{16.12}$$

A p_x superconductor with vanishing spin component along the $\vec{d}$-direction is even simpler:

$$\langle \psi_\alpha(k,t)\, \psi_\beta(-k,t) \rangle = \Psi_0\, (\sin k_x a)\, \vec{d} \cdot \vec{\sigma}_\alpha{}^\gamma \epsilon_{\gamma\beta} \tag{16.13}$$

Note that the order parameter vanishes along the direction $k_x = 0$. A p_y superconductors has $\sin k_x a$ replaced with$\sin k_y a$

A d-wave superconductor must be a spin-singlet superconductor. A $d_{x^2-y^2}$ superconductor has

$$\langle \psi_\alpha(k,t)\, \psi_\beta(-k,t)\rangle = \Delta_0\ (\cos k_x a - \cos k_y a)\ \epsilon_{\alpha\beta} \tag{16.14}$$

while a d_{xy} superconductor has $\cos k_x a - \cos k_y a$ replaced by $\sin k_x a\ \sin k_y a$. A $d_{x^2-y^2} + i d_{xy}$ superconductor breaks T with the order parameter:

$$\begin{aligned}&\langle \psi_\alpha(k,t)\, \psi_\beta(-k,t)\rangle = \\ &\quad \Delta_0\,(\cos k_x a - \cos k_y a + i \sin k_x a\ \sin k_y a)\,\epsilon_{\alpha\beta}\end{aligned} \tag{16.15}$$

We can define analogous neutral order parameters which do not break $U(1)$. However, the spin structures will no longer be determined by Fermi statistics. Let us first consider the singlet orderings:

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\, \psi_\beta(k,t)\right\rangle = \Phi_Q(\mathbf{k})\, \delta^\alpha_\beta \tag{16.16}$$

If $\Phi_Q(\mathbf{k}) = 1$ or if its integral over $\mathbf{k}$ is non-zero, the state is a *charge-density-wave* (CDW):

$$\langle \rho(\mathbf{Q})\rangle = \int \frac{d^2k}{(2\pi)^2} \left\langle \psi^{\alpha\dagger}(k+Q,t)\, \psi_\alpha(k,t)\right\rangle = \int \frac{d^2k}{(2\pi)^2} \Phi_Q(\mathbf{k}) \neq 0 \tag{16.17}$$

The triplet orderings are of the form

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\, \psi_\beta(k,t)\right\rangle = \vec{\Phi}_Q(\mathbf{k})\ \cdot \vec{\sigma}^\alpha_\beta \tag{16.18}$$

If $\vec{\Phi}_Q(\mathbf{k}) = 1$ or if its integral over $\mathbf{k}$ is non-zero, the state is a *spin-density-wave* (SDW):

$$\langle \mathbf{S}(\mathbf{Q})\rangle = \int \frac{d^2k}{(2\pi)^2} \left\langle \psi^{\alpha\dagger}(k+Q,t)\, \psi_\beta(k,t)\right\rangle\ \vec{\sigma}^\beta_\alpha = \int \frac{d^2k}{(2\pi)^2} \vec{\Phi}_Q(\mathbf{k})\ \neq 0 \tag{16.19}$$

Let us consider these order parameters in more detail on the square lattice. $f(k)$ is an element of some representation of the space group of the vector $\vec{Q}$ in the square lattice. A singlet s-wave density wave is simply a charge-density-wave:

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\, \psi_\beta(k,t)\right\rangle = \Phi_Q\ \delta^\alpha_\beta \tag{16.20}$$

In the higher angular momentum cases, we must distinguish commmensurate and incommensurate ordering. For commensurate ordering such that $2Q$ is

a reciprocal lattice vector, e.g. $Q = (\pi/a, 0)$ or $Q = (\pi/a, \pi/a)$, we can take the hermitian conjugate of the order parameter:

$$\begin{aligned} \left\langle \psi^{\dagger\beta}(k,t)\,\psi_\alpha(k+Q,t)\right\rangle &= \Phi_Q^*\, f^*(k)\,\delta_\beta^\alpha \\ \left\langle \psi^{\beta\dagger}(k+Q+Q,t)\,\psi_\alpha(k+Q,t)\right\rangle &= \Phi_Q^*\, f^*(k)\,\delta_\beta^\alpha \\ \Phi_Q\, f(k+Q)\,\delta_\beta^\alpha &= \Phi_Q^*\, f^*(k)\,\delta_\beta^\alpha \end{aligned} \tag{16.21}$$

Therefore, for Q commensurate

$$\frac{f(k+Q)}{f^*(k)} = \frac{\Phi_Q^*}{\Phi_Q} \tag{16.22}$$

Hence, if $f(k+Q) = -f^*(k)$, Φ_Q must be imaginary. For singlet p_x ordering, this will be the case if $Q = (\pi/a, 0)$ or $Q = (\pi/a, \pi/a)$. For singlet $d_{x^2-y^2}$ ordering, this will be the case if $Q = (\pi/a, \pi/a)$. If $f(k+Q) = f^*(k)$, Φ_Q must be real. For singlet p_x ordering, this will be the case if $Q = (0, \pi/a)$. For singlet d_{xy} ordering, this will be the case if $Q = (\pi/a, \pi/a)$.

A commensurate singlet p_x density-wave state has ordering

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\,\psi_\beta(k,t)\right\rangle = \Phi_Q \sin k_x a\ \delta_\beta^\alpha \tag{16.23}$$

The commensurate singlet $p_x + ip_y$ density-wave states are defined by:

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\,\psi_\beta(k,t)\right\rangle = \Phi_Q\ (\sin k_x a + i \sin k_y a)\ \ \delta_\beta^\alpha \tag{16.24}$$

Further insight into these states is obtained by considering their real-space forms. From (16.22), a commensurate singlet p_x density-wave state with $Q = (\pi/a, 0)$ must have imaginary Φ_Q:

$$\begin{aligned} \left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}+a\hat{x},t) - \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}-a\hat{x},t)\right\rangle = \\ \ldots + |\Phi_Q|\ e^{i\vec{Q}\cdot\vec{x}}\,\delta_\beta^\alpha \end{aligned} \tag{16.25}$$

The singlet state of this type breaks no other symmetries; it is usually called the *Peierls state* or bond order wave. If $Q = (0, \pi/a)$, Φ_Q must be real.

$$\begin{aligned} \left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}+a\hat{x},t) - \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}-a\hat{x},t)\right\rangle = \\ \ldots - i\,|\Phi_Q|\ e^{i\vec{Q}\cdot\vec{x}}\,\delta_\beta^\alpha \end{aligned} \tag{16.26}$$

As a result of the i, the $Q = (0, \pi/a)$ singlet p_x density-wave states break T. However, the combination of T and translation by an odd number of

lattice spacings remains unbroken. The same is true of the commensurate singlet $p_x + ip_y$ density-wave states. Examples of commensurate and incommensurate singlet p_x and $p_x + ip_y$ density-wave states are depicted in figure **??**.

Similarly, the commensurate singlet $d_{x^2-y^2}$ density-wave states have

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\,\psi_\beta(k,t)\right\rangle = \Phi_Q\,(\cos k_x a - \cos k_y a)\;\;\delta^\alpha_\beta \tag{16.27}$$

while the commensurate singlet $d_{x^2-y^2} + id_{xy}$ density-wave states have

$$\left\langle \psi^{\alpha\dagger}(k+Q,t)\,\psi_\beta(k,t)\right\rangle = \Phi_Q\,(\cos k_x a - \cos k_y a + i\sin k_x a\,\sin k_y a)\;\;\delta^\alpha_\beta \tag{16.28}$$

The commensurate $Q = (\pi/a, \pi/a)$ singlet $d_{x^2-y^2}$ density-wave states must have imaginary Φ_Q, according to (16.22). In real space, it takes the form:

$$\begin{aligned}\left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}+a\hat{x},t)\right\rangle + \left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}-a\hat{x},t)\right\rangle - \\ \left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}+a\hat{y},t)\right\rangle + \left\langle \psi^{\dagger\alpha}(\vec{x},t)\,\psi_\beta(\vec{x}-a\hat{y},t)\right\rangle = \\ \ldots + \frac{i}{2}\,|\Phi_Q|\;e^{i\vec{Q}\cdot\vec{x}}\,\delta^\alpha_\beta\end{aligned} \tag{16.29}$$

As a result of the i, the singlet $d_{x^2-y^2}$ density-wave breaks T as well as translational and rotational invariance. The combination of time-reversal and a translation by one lattice spacing is preserved by this ordering. The commensurate $Q = (\pi/a, \pi/a)$ singlet $d_{x^2-y^2}$ density-wave state is often called the *staggered flux state*. There is also a contribution to this correlation function coming from $\psi^\dagger(k)\psi(k)$ which is uniform in space (the $\ldots$); as a result, the phase of the above bond correlation function – and, therefore, the flux through each plaquette – is alternating. The commensurate $Q = (\pi/a, \pi/a)$ singlet d_{xy} must have real Φ_Q; therefore, it does not break T. On the other hand, the singlet $d_{x^2-y^2} + id_{xy}$ state does break T. Note that the nodeless commensurate singlet $d_{x^2-y^2} + id_{xy}$ density-wave state does not break more symmetries than the commensurate singlet $d_{x^2-y^2}$ density-wave state, in contrast to the superconducting case. Examples of singlet $d_{x^2-y^2}$, d_{xy}, and $d_{x^2-y^2} + id_{xy}$ density-wave states are depicted in figure **??**.

The incommensurate cases will be considered in the problems at the end of the chapter.

It is also possible to break spin-rotational invariance without breaking translational symmetry or $U(1)$ gauge symmetry. This is accomplished in a

ferromagnet:

$$\left\langle \psi^{\alpha\dagger}(k,t)\,\psi_\beta(k,t)\right\rangle = \mathbf{M}\cdot\vec{\sigma}^\alpha_\beta\; g(\mathbf{k}) \tag{16.30}$$

Integrating over $\mathbf{k}$, we have:

$$\langle \mathbf{S}\rangle = \mathbf{M}\,\left(\int \frac{d^dk}{(2\pi)^d}\, g(\mathbf{k})\right) \tag{16.31}$$

It is also possible to break spatial rotational symmetries (either in the continuum or on the lattice) without breaking translational symmetry or spin rotational symmetry:

$$\left\langle \psi^{\alpha\dagger}(k,t)\,\psi_\beta(k,t)\right\rangle = \delta^\alpha_\beta\; f(\mathbf{k}) \tag{16.32}$$

where $f(\mathbf{k})$ transforms non-trivially under rotations.

16.3 The Hubbard-Stratonovich Transformation

There is a formal transformation, the Hubbard-Stratonovich transformation, which allows us to introduce a bosonic degree of freedom to replace one of the possible bilinear combinations of the fermions. When the occurrence of such an order parameter causes a gap in the fermionic spectrum, the fermions can be integrated out and an effective theory for the bosonic variable can be derived. This theory can then be solved by the usual methods applied to bosonic systems, such as the saddle-point approximation. This approach has a good chance of succeeding when one of the possible bosonic degrees of freedom dominates, in particular, when it condenses and the others remain gapped. By studying the instabilities of the RG flows of a Fermi liquid in a periodic potential, it is sometimes possible to detect such a tendency. This approximation can often become exact in some kind of large-N limit.

To illustrate this transformation, let us consider the following integral over $2n$ Grassmann variables χ_a, $\overline{\chi}_a$:

$$I = \int \prod d\chi_a\, d\overline{\chi}_a\; e^{\chi_a\overline{\chi}_a + \frac{1}{2n}(\chi_a\overline{\chi}_a)^2} \tag{16.33}$$

We can rewrite this integral as

$$I = \mathcal{N}\int \prod d\chi_a\, d\overline{\chi}_a\; d\varphi e^{\chi_a\overline{\chi}_a - \varphi\chi_a\overline{\chi}_a - \frac{n}{2}\varphi^2} \tag{16.34}$$

where N is a normalization constant. By performing the Gaussian integral over φ, we recover the integral (16.33). This, in a nutshell, is the Hubbard-Stratonovich transformation; we will be applying it to functional integrals.

The Grassmann integrals are now Gaussian, and they may be performed:

$$\begin{aligned} I &= \mathcal{N} \int d\varphi \, (1-\varphi)^n e^{-\frac{n}{2}\varphi^2} \\ &= \mathcal{N} \int d\varphi \, e^{n \ln(1-\varphi) - \frac{n}{2}\varphi^2} \end{aligned} \tag{16.35}$$

Thus, we have exchanged a non-Gaussian Grassmann integral for a non-Gaussian ordinary integral. This may not seem like such a big success. However, we can now use techniques such as the saddle-point approximation to evaluate the ordinary integral.

Before doing this, however, note that we could have decoupled the quartic term in another way: We can rewrite this integral as

$$\begin{aligned} e^{\frac{1}{2n}(\chi_a \overline{\chi}_a)^2} &= e^{\frac{1}{2n}(\chi_a \overline{\chi}_a)\,(\chi_b \overline{\chi}_b)} \\ &= \mathcal{N} \int d\varphi_{ab} \, e^{\chi_a \varphi_{ab} \overline{\chi}_b - \frac{n}{2}\varphi_{ab}^2} \end{aligned} \tag{16.36}$$

Now the Grassmann integral is transformed into an integral over the matrix φ_{ab}:

$$I = \mathcal{N} \int d\varphi_{ab} \, e^{n \ln(\delta_{ab} - \varphi_{ab}) - \frac{n}{2}\varphi_{ab}^2} \tag{16.37}$$

The two integrals (16.35) and (16.37) are equal so long as they are performed exactly. However, different approximations are suggested by the forms of these integrals. In the analogous field-theoretic context, the underlying physics will dictate which one is a better starting point.

16.4 The Hartree and Fock Approximations

Let us now consider the saddle-point evaluation of these integrals. The saddle-point condition for (16.35) is:

$$-\varphi - \frac{1}{1-\varphi} \tag{16.38}$$

The two saddle-point values of φ are the golden number and the negative of its inverse, $\varphi_{\text{s.p.}} = (1 \pm \sqrt{5})/2$. Adding the contributions from both saddle-points, we have

$$I = \mathcal{N}\{ \frac{(-1)^n}{\left[(\sqrt{5}+1)/2\right]^{n/2}} \, e^{-n\left(1+\frac{1+\sqrt{5}}{2}\right) + n \, \ln\left(\frac{\sqrt{5}-1}{2}\right)}$$

$$+\frac{(-i)^n}{\left[(\sqrt{5}-1)/2\right]^{n/2}}\, e^{-n\left(1+\frac{1-\sqrt{5}}{2}\right)+n\,\ln\left(\frac{\sqrt{5}+1}{2}\right)}\}\qquad(16.39)$$

In the physical context in which we will be using these ideas, we will be considering Grassmann functional integrals which can be rewritten as bosonic functional integrals – or 'decoupled', since the remaining Grassman integral is Gaussian – in a variety of ways. The different bosonic fields which we introduce will be the different possible *order parameters* of the system. Mean-field-theory – or the Hartree approximation – for any of these order parameters is simply the saddle-point approximation for the bosonic functional integral. The saddle-point condition is then a 'gap equation' (a name whose aptness will become clear when we look at examples).

This approach can also be used to decouple an interaction which does not lead to the development of a non-trivial order parameter. For instance, a density-density interaction can be decoupled so that the electrons interact with a bosonic field. At the saddle-point level, the bosonic field is equal to the electron density. This is the Hartree approximation. Alternatively, the interaction can be decoupled so that the bosonic field is equal to the product of a creation operator from one density factor and an annihilation operator from the other density factor. This is the Fock approximation.

16.5 The Variational Approach

There is an equivalent approach within the framework of canonical quantization. One introduces a trial ground state, $|0\rangle$, which is based on the anticipated order parameter. The size of the order parameter is the variational parameter which is tuned to minimize

$$\langle H\rangle = \frac{\langle 0|H|0\rangle}{\langle 0|0\rangle}\qquad(16.40)$$

The condition for minimizing $\langle H\rangle$ is the same as the saddle-point condition in the path integral approach.

The basic form of the trial wavefunction is, in the superconducting case:

$$|\Psi_0\rangle = \prod_{\mathbf{k}}\left(u_{\mathbf{k}} + v_{\mathbf{k}}\psi^\dagger_{k\uparrow}\psi^\dagger_{-k\downarrow}\right)|0\rangle\qquad(16.41)$$

with where the $u_{\mathbf{k}}$'s and $v_{\mathbf{k}}$'s are variational parameters with respect to which $\langle\Psi_0|\,H\,|\Psi_0\rangle$ is minimized. The wavefunction is normalized by taking

$$u_k^2 + v_k^2 = 1\qquad(16.42)$$

Thus, there is only one free variational parameter for each $\mathbf{k}$. The combination $u_{\mathbf{k}}v_{\mathbf{k}}$ is a convenient way of parametrizing it. Calculating the order parameter in (16.41), we see that it is given by $u_{\mathbf{k}}v_{\mathbf{k}}$. Thus, the variational method selects a ground state by minimizing the *energy* with respect to the order parameter, while the saddle-point approximation of the previous section finds a ground state by minimizing the *action* with respect to the order parameter.

In the case of a density-wave at wavevector $\mathbf{Q}$, the trial wavefunction takes the form:

$$|\Psi_0\rangle = \prod_{\mathbf{k}} \left(u_{\mathbf{k}} \psi^\dagger_{\mathbf{k},\alpha} + v_{\mathbf{k}} \psi^\dagger_{\mathbf{k}+\mathbf{Q},\alpha} \right) |0\rangle \tag{16.43}$$

CHAPTER 17

Superconductivity

17.1 Instabilities of the Fermi Liquid

When a fixed point has a relevant perturbation, this perturbation generally leads to a fundamental reorganization of the ground state. We saw a trivial example of this with a shift of the chemical potential of a Fermi liquid. When the instability is due to interaction terms, the general strategy is to use the RG to go to low energies so that the irrelevant variables have all become small and the relevant variable is dominant. The problem with a single relevant interaction must then be solved by finding a new saddle-point (i.e. mean field theory), the variational method, or some other non-perturbative method. This approach has proven very successful in the study of ordering in condensed matter physics. (Sometimes, there are competing instabilities in which case it is very difficult to find a new saddle-point or an appropriate variational ansatz. This occurs in the case of a $1D$ system of fermions.) In the case of electrons in a solid, the Fermi surface need not be rotationally symmetric, and spin- and charge-density wave instabilities are possible when the Fermi surface satisfies certain special conditions ('nesting'). If the Fermi surface is rotationally symmetric, there is only one instability, as we found earlier: the Cooper pairing instability.

Consider the action of electrons in $D = 2$ with $F = 0$ but non-zero V,

$$S = \int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \psi^\dagger_\sigma(\epsilon, k) \left(i\epsilon - v_F k\right) \psi_\sigma(\epsilon, k)$$

$$- \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi} \frac{d\epsilon_3}{2\pi} \psi^\dagger_\uparrow(\epsilon_4, k')\psi^\dagger_\downarrow(\epsilon_3, -k')\, V(k,k')\, \psi_\uparrow(\epsilon_2, -k)\psi_\downarrow(\epsilon_1, k) \tag{17.1}$$

where $V(k,k') \equiv V(\theta_1 - \theta_2)$ is a function of the angles only. Unlike in previous chapters, where we dealt with spinless fermions, we now consider spin 1/2 electrons.

In chapter 14, we showed that the Fourier modes of $V(\theta_1 - \theta_2)$ satisfy the RG equation:

$$\frac{dV_m}{d\ell} = -\frac{1}{2\pi v_F} V_m^2 \tag{17.2}$$

When negative, these are relevant. In the next section, we will find the new saddle point which is appropriate for the case in which V is relevant. We will also mention briefly the equivalent variational ansatz (which was the historical method of solution).

17.2 Saddle-Point Approximation

We introduce a Hubbard-Stratonovich field $\Psi(k,\omega)$ to decouple the BCS interaction:

$$\begin{aligned} S = &\int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \psi^\dagger_\sigma(\epsilon,k)\,(i\epsilon - v_F k)\,\psi_\sigma(\epsilon,k) \\ &- \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi} V(k,k')\,[\psi^\dagger_\uparrow(\epsilon_1,k')\psi^\dagger_\downarrow(\epsilon_2,-k')\,\Psi(\epsilon_1+\epsilon_2,k) \\ &+\psi_\uparrow(\epsilon_1,k')\psi_\downarrow(\epsilon_2,-k')\,\Psi^\dagger(\epsilon_1,\epsilon_2,k) + \Psi^\dagger(\epsilon_1+\epsilon_2,k)\Psi(\epsilon_1+\epsilon_2,k')] \end{aligned} \tag{17.3}$$

We now make the change of variables:

$$\Delta(\epsilon_1+\epsilon_2,k) = \int \frac{d^2k'}{(2\pi)^2} V(k,k')\,\Psi^\dagger(\epsilon_1+\epsilon_2,k') \tag{17.4}$$

Then, the action can be rewritten:

$$\begin{aligned} S = &\int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi}\, \psi^\dagger_\sigma(\epsilon,k)\,(i\epsilon - v_F k)\,\psi_\sigma(\epsilon,k) \\ &- \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} \frac{d\epsilon_1}{2\pi} \frac{d\epsilon_2}{2\pi}\,[\psi^\dagger_\uparrow(\epsilon_1,k')\psi^\dagger_\downarrow(\epsilon_2,-k')\,\Delta(\epsilon_1+\epsilon_2,k) \\ &+\psi_\uparrow(\epsilon_1,k')\psi_\downarrow(\epsilon_2,-k')\,\Delta^\dagger(\epsilon_1+\epsilon_2,k) + \Delta^\dagger(\epsilon_1+\epsilon_2,k)V^{-1}(k,k')\Delta(\epsilon_1+\epsilon_2,k')] \end{aligned} \tag{17.5}$$

where $V^{-1}(k,k')$ is the inverse of $V(k,k')$:

$$\int \frac{d^2k}{(2\pi)^2}\, V^{-1}(k,k')\, V(k',k'') = \delta(k-k'') \tag{17.6}$$

Since the action is quadratic in the fermion fields ψ_σ, we can integrate out the fermions to get an effective action $S[\Delta]$:

$$\begin{aligned} S[\Delta] = & -Tr \ln \left((i\epsilon)^2 - (v_F k)^2 - |\Delta(k)|^2 \right) + \\ & \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} \frac{d\epsilon}{2\pi} \Delta^\dagger(\epsilon, k) V^{-1}(k, k') \Delta(\epsilon, k') \\ = & -\int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \ln \left((i\epsilon)^2 - (v_F k)^2 - |\Delta(k)|^2 \right) + \\ & \int \frac{d^2k}{(2\pi)^2} \frac{d^2k'}{(2\pi)^2} \frac{d\epsilon_1}{2\pi} \Delta^\dagger(\epsilon, k) V^{-1}(k, k') \Delta(\epsilon, k') \end{aligned} \tag{17.7}$$

We look for a frequency-independent solution, $\Delta(\epsilon, k) = \Delta(k)$ of the saddle point equations,

$$\frac{\delta S}{\delta \Delta} = 0 \tag{17.8}$$

From (17.7), we have the saddle-point equations:

$$\int \frac{d^2k}{(2\pi)^2} \frac{d\epsilon}{2\pi} \frac{1}{(i\epsilon)^2 - (v_F k)^2 - |\Delta(k)|^2} = \int \frac{d^2k'}{(2\pi)^2} V^{-1}(k, k') \Delta(k') \tag{17.9}$$

At zero-temperature, the ϵ integral in the first term can be done (at finite-temperature, we must do a Matsubara sum instead), giving:

$$\int \frac{d^2k}{(2\pi)^2} \frac{1}{\sqrt{(v_F k)^2 + |\Delta(k)|^2}} = \int \frac{d^2k'}{(2\pi)^2} V^{-1}(k, k') \Delta(k') \tag{17.10}$$

or

$$\int \frac{d^2k'}{(2\pi)^2} \frac{V(k, k') \Delta(k')}{\sqrt{(v_F k')^2 + |\Delta(k')|^2}} = \Delta(k) \tag{17.11}$$

The is the BCS gap equation. It determines the order parameter Δ which breaks the $U(1)$ symmetry $\Delta \to e^{i\theta}\Delta$ of the action (17.7).

For V attractive, i.e. $V > 0$, this equation always has a solution. Consider the simplest case, of an s-wave attraction, $V(k, k') = V$. Then the gap equation reads:

$$\int \frac{d^2k'}{(2\pi)^2} \frac{V\Delta}{\sqrt{(v_F k')^2 + |\Delta|^2}} = \Delta \tag{17.12}$$

or,

$$\int \frac{d^2k'}{(2\pi)^2} \frac{1}{\sqrt{(v_F k')^2 + |\Delta|^2}} = \frac{1}{V} \tag{17.13}$$

Since the left-hand-side is logarithmically divergent at the Fermi surface if $\Delta = 0$, there is always a non-trivial saddle-point solution when $V > 0$.

$$\frac{m^*}{2\pi} \int d\xi \, \frac{1}{\sqrt{\xi^2 + \Delta^2}} = \frac{1}{V} \tag{17.14}$$

or

$$\Delta = \frac{\Lambda}{\sinh \frac{2\pi}{m^* V}} \tag{17.15}$$

If the attraction is weak, $m^* V / 2\pi \ll 1$, then

$$\Delta = 2\Lambda e^{-\frac{2\pi}{m^* V}} \tag{17.16}$$

Note that the gap is not analytic in V; it could never be discovered in perturbation theory.

As you will show in the problem set, the finite-temperature gap equation is:

$$\int \frac{d^2 k'}{(2\pi)^2} \, \frac{V(k,k')\Delta(k')}{\sqrt{(v_F k')^2 + |\Delta(k')|^2}} \tanh \frac{\beta E_{k'}}{2} = \Delta(k) \tag{17.17}$$

with

$$E_k = \sqrt{(v_F k)^2 + |\Delta(k)|^2} \tag{17.18}$$

For an s-wave attraction, this gap equation has solution $\Delta =$ when:

$$\frac{m^*}{2\pi} \int d\xi \, \frac{1}{\sqrt{\xi^2}} \tanh \frac{\beta \xi}{2} = \frac{1}{V} \tag{17.19}$$

So the critical temperature for the onset of superconductivity is:

$$T_c = 1.14 \, \Lambda e^{-\frac{2\pi}{m^* V}} \tag{17.20}$$

17.3 BCS Variational Wavefunction

For purposes of comparison, consider the route taken by Bardeen, Cooper, and Schrieffer. They wrote down the wavefunction

$$|\Psi_0\rangle = \prod_k \left(u_k + v_k \psi^\dagger_{k\uparrow} \psi^\dagger_{-k\downarrow} \right) |0\rangle \tag{17.21}$$

with where the u_k's and v_k's are variational parameters with respect to which $\langle \Psi_0 | \, H \, | \Psi_0 \rangle$ is minimized. The wavefunction is normalized by taking

$$u_k^2 + v_k^2 = 1 \tag{17.22}$$

For notational simplicity, we assume that the k's are discrete (as they are in a finite-size system). The Hamiltonian which follows from (17.1) is:

$$H = \sum_k \xi_k \psi^\dagger_{k\sigma} \psi_{k\sigma} - \sum_{k,k'} V(k,k') \psi^\dagger_{k\uparrow} \psi^\dagger_{-k\downarrow} \psi_{k'\uparrow} \psi_{-k'\downarrow} \tag{17.23}$$

This Hamiltonian is called the BCS reduced Hamiltonian. It is the Hamiltonian which only contains the relevant interaction. The irrelevant and marginal interactions have been dropped. The expectation value of the Hamiltonian is:

$$\langle \Psi_0 | H | \Psi_0 \rangle = \sum_k 2 v_k^2 \xi_k - \sum_{k,k'} V(k,k') u_k v_k u_{k'} v_{k'} \tag{17.24}$$

Hence,

$$\begin{aligned} \frac{\partial}{\partial v_k} \langle \Psi_0 | H | \Psi_0 \rangle &= 4 v_k \xi_k - \sum_{k'} V(k,k') \left[u_k u_{k'} v_{k'} + 2 \frac{\partial u_k}{\partial v_k} v_k u_{k'} v_{k'} \right] \\ &= 4 v_k \xi_k - 2 \sum_{k'} V(k,k') \left(\frac{u_k^2 - v_k^2}{u_k} \right) u_{k'} v_{k'} \end{aligned} \tag{17.25}$$

The minimum of $\langle \Psi_0 | H | \Psi_0 \rangle$ occurs when

$$2 \xi_k u_k v_k = \sum_{k'} V(k,k') \left(u_k^2 - v_k^2 \right) u_{k'} v_{k'} \tag{17.26}$$

If we define $\Delta(k)$ by

$$u_k v_k = \frac{\Delta(k)}{2 \sqrt{\xi_k^2 + |\Delta(k)|^2}} \tag{17.27}$$

or, equivalently,

$$\begin{aligned} u_k &= \frac{1}{\sqrt{2}} \left(1 + \frac{\xi_k}{E_k} \right)^{\frac{1}{2}} \\ v_k &= \frac{1}{\sqrt{2}} \left(1 - \frac{\xi_k}{E_k} \right)^{\frac{1}{2}} \end{aligned} \tag{17.28}$$

with

$$E_k = \sqrt{\xi_k^2 + |\Delta(k)|^2} \tag{17.29}$$

Then we can rewrite the minimization condition as the BCS gap equation:

$$\sum_k V(k,k') \frac{\Delta(k')}{\sqrt{\xi_k^2 + |\Delta(k')|^2}} = \Delta(k) \tag{17.30}$$

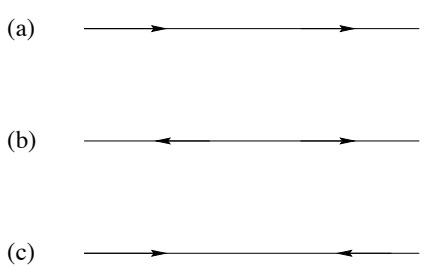


Figure 17.1: The graphical representation of (a) $\mathcal{G}$ (b) $\mathcal{F}$ and (c) $\mathcal{F}^\dagger$.

17.4 Condensate fraction and superfluid density**

17.5 Single-Particle Properties of a Superconductor

17.5.1 Green Functions

When Δ takes a non-zero, frequency-independent value, the action for the fermions is:

$$S = \int \frac{d^2k}{(2\pi)^2}\frac{d\epsilon}{2\pi}\left[\psi^\dagger_\sigma(i\epsilon,k)\left(i\epsilon - v_F k\right)\psi_\sigma(i\epsilon,k)\right. \\ \left. - \;\psi^\dagger_\uparrow(i\epsilon,k')\psi^\dagger_\downarrow(-i\epsilon,-k')\,\Delta(k) \;-\; \psi_\uparrow(i\epsilon,k')\psi_\downarrow(-i\epsilon,-k')\,\Delta^\dagger(k)\right] \quad (17.31)$$

As usual, the propagator is obtained by inverting the quadratic part of the action. This is now a matrix, with an inverse which gives

$$\mathcal{G}_{\sigma\sigma'}(i\epsilon,k) = \left\langle \psi^\dagger_\sigma(i\epsilon,k)\psi_{\sigma'}(i\epsilon,k)\right\rangle = \delta_{\sigma\sigma'}\,\frac{i\epsilon+\xi_k}{(i\epsilon)^2-\xi_k^2-|\Delta(k)|^2} \\ \mathcal{F}_{\sigma\sigma'}(i\epsilon,k) = \langle\psi_\sigma(i\epsilon,k)\psi_{\sigma'}(-i\epsilon,-k)\rangle = \epsilon_{\sigma\sigma'}\,\frac{\Delta(k)}{(i\epsilon)^2-\xi_k^2-|\Delta(k)|^2} \quad (17.32)$$

We denote $\mathcal{G}(i\epsilon,k)$ by a line with two arrows pointing in the same direction. We denote $\mathcal{F}(i\epsilon,k)$ by a line with two arrows pointing away from each other and $\mathcal{F}^\dagger(i\epsilon,k) = \left\langle \psi^\dagger_\sigma(i\epsilon,k)\psi^\dagger_{\sigma'}(-i\epsilon,-k)\right\rangle$ by a line with two arrows pointing towards each other. The electron spectral function is given by

$$A(k,\epsilon) = \mathrm{Im}\left(\frac{\epsilon+\xi_k}{(\epsilon+i\delta)^2-\xi_k^2-|\Delta(k)|^2}\right) \\ = u_k^2\,\delta(\epsilon-E_k) + v_k^2\,\delta(\epsilon+E_k) \quad (17.33)$$

which shows that the electron has spectral weight u_k^2 at E_k and spectral weight v_k^2 at $-E_k$.

Another way of understanding the single-particle properties of a superconductor is to diagonalize the action. The action is diagonalized by the $\gamma(k)$'s

$$\begin{aligned}\gamma_\uparrow(k,\epsilon) &= u_k\psi_\uparrow(k,\epsilon) - v_k\psi_\downarrow^\dagger(-k,\epsilon)\\ \gamma_\downarrow(k,\epsilon) &= u_k\psi_\downarrow(k,\epsilon) + v_k\psi_\uparrow^\dagger(-k,\epsilon)\end{aligned} \tag{17.34}$$

$$S = \int \frac{d^2k}{(2\pi)^2}\,\frac{d\epsilon}{2\pi}\,\gamma_\sigma^\dagger(k,\epsilon)\,(i\epsilon - E_k)\,\gamma_\sigma(k,\epsilon) \tag{17.35}$$

The $\gamma(k)$'s have propagator:

$$\left\langle \gamma_\sigma^\dagger(i\epsilon,k)\gamma_{\sigma'}(i\epsilon,k)\right\rangle = \frac{\delta_{\sigma\sigma'}}{i\epsilon - E_k} \tag{17.36}$$

The $\gamma(k)$'s are the basic single-particle excitations – 'Bogoliubov-DeGennes quasiparticles' – of a superconductor; they are superpositions of fermions and holes. In the case of electrons, the basic excitations have indefinite charge, since they are a superposition of an electron and a hole. Although they are not charge eigenstates, they are spin eigenstates.

Note that $E_k > 0$. When $\xi_k \gg \Delta$, $u_k \to 1$, $v_k \to 0$, so $\gamma_\sigma^\dagger(k)$ creates a fermion above the Fermi surface, costing positive energy. When $\xi_k \ll -\Delta$, $u_k \to 0$, $v_k \to 1$, so $\gamma_\sigma^\dagger(k)$ creates a hole below the Fermi surface, also costing positive energy.

For some purposes – such as the Hebel-Slichter peak in NMR – we can ignore the fact that they are a superposition of an electron and a hole and treat the superconductor as a semiconductor with energy bands $\pm E_k$. Since the density of single quasiparticle states,

$$\frac{dk}{dE} = \frac{m^*}{2\pi}\,\frac{|E|}{\sqrt{E^2-\Delta^2}}\,\theta\left(|E|-\Delta\right) \tag{17.37}$$

is divergent for $|E| \to \Delta$ and vanishing for $|E| < \Delta$, the semiconductor model predicts sharp increases in these quantities for $T \sim \Delta$ and exponential decay for $T \ll \Delta$. However, for other properties – such as the acoustic attenuation – the mixing between between electron and hole state ('coherence factors') is important. The coherence factors can cancel the density of states divergence at $|E| \to \Delta$, and there is no enhancement for $T \sim \Delta$.

Figure 17.2: The two diagrams which contribute to the spin-spin correlation function of a superconductor.

17.5.2 NMR Relaxation Rate

According to (6.93), the NMR relaxation rate is given by:

$$\frac{1}{T_1 T} = \int \frac{d^2q}{(2\pi)^2} A(q) \lim_{\omega\to 0} \frac{1}{\omega} \chi''_{+-}(q,\omega) \tag{17.38}$$

The spin-spin correlation function $\langle S_+(q,i\omega_m)\, S_-(-q,-i\omega_m)\rangle$ is given by the sum of the two diagrams of figure 17.2. Assuming that $\Delta(k) = \Delta$, this is:

$$\begin{aligned}\langle S_+(q,i\omega_m)\, S_-(-q,-i\omega_m)\rangle = &\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, \mathcal{G}_{\downarrow\downarrow}(i\epsilon_n,k)\, \mathcal{G}_{\uparrow\uparrow}(i\epsilon_n+i\omega_m,k+q) \\ &+ \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, \mathcal{F}_{\uparrow\downarrow}(i\epsilon_n,k)\, \mathcal{F}^\dagger_{\downarrow\uparrow}(i\epsilon_n+i\omega_m,k+q)\end{aligned} \tag{17.39}$$

or,

$$\begin{aligned}\langle S_+(q,i\omega_m)\, S_-(-q,-i\omega_m)\rangle = &\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, \frac{i\epsilon_n+\xi_k}{(i\epsilon_n)^2-\xi_k^2-|\Delta|^2}\, \frac{i\epsilon_n+i\omega_m+\xi_{k+q}}{(i\epsilon_n+i\omega_m)^2-\xi_{k+q}^2-|\Delta|^2} \\ &+ \frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\, \frac{\Delta}{(i\epsilon_n)^2-\xi_k^2-|\Delta|^2}\, \frac{\Delta}{(i\epsilon_n+i\omega_m)^2-\xi_{k+q}^2-|\Delta|^2}\end{aligned} \tag{17.40}$$

If we replace the sums over Matsubara frequencies by contour integrals which avoid $z = (2n+1)\pi i/\beta$,

$$\begin{aligned}&\oint_{\mathcal{C}} \frac{dz}{2\pi i} n_F(z) \int \frac{d^3k}{(2\pi)^3}\, \frac{z+\xi_k}{(z)^2-\xi_k^2-|\Delta|^2}\, \frac{z+i\omega_m+\xi_{k+q}}{(z+i\omega_m)^2-\xi_{k+q}^2-|\Delta|^2} \\ &+ \oint_{\mathcal{C}} \frac{dz}{2\pi i} n_F(z) \int \frac{d^3k}{(2\pi)^3}\, \frac{\Delta}{(z)^2-\xi_k^2-|\Delta|^2}\, \frac{\Delta}{(z+i\omega_m)^2-\xi_{k+q}^2-|\Delta|^2}\end{aligned}$$

(17.41)

these integrals receive contributions only from the poles at

$$\begin{aligned} z &= \pm\sqrt{\xi_k^2 + |\Delta|^2} \\ z &= -i\omega_n \pm \sqrt{\xi_{k+q}^2 + |\Delta|^2} \end{aligned} \tag{17.42}$$

Hence,

$$\begin{aligned} \langle S_+(q,i\omega_m)\, S_-(-q,-i\omega_m)\rangle = &\int \frac{d^3k}{(2\pi)^3}\, n_F(E_k)\, \frac{\xi_k + E_k}{2E_k}\, \frac{E_k + i\omega_m + \xi_{k+q}}{(E_k + i\omega_m)^2 - E_{k+q}^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_k)\, \frac{\xi_k - E_k}{-2E_k}\, \frac{-E_k + i\omega_m + \xi_{k+q}}{(-E_k + i\omega_m)^2 - E_{k+q}^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_{k+q})\, \frac{\xi_{k+q} + E_{k+q}}{2E_{k+q}}\, \frac{E_{k+q} - i\omega_m + \xi_k}{(E_{k+q} - i\omega_m)^2 - E_k^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_{k+q})\, \frac{\xi_{k+q} - E_{k+q}}{-2E_{k+q}}\, \frac{-E_{k+q} - i\omega_m + \xi_k}{(E_{k+q} + i\omega_m)^2 - E_k^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_k)\, \frac{\Delta}{2E_k}\, \frac{\Delta}{(E_k + i\omega_m)^2 - E_{k+q}^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_k)\, \frac{\Delta}{-2E_k}\, \frac{\Delta}{(-E_k + i\omega_m)^2 - E_{k+q}^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_{k+q})\, \frac{\Delta}{2E_{k+q}}\, \frac{\Delta}{(E_{k+q} - i\omega_m)^2 - E_k^2} \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_{k+q})\, \frac{\Delta}{-2E_{k+q}}\, \frac{\Delta}{(E_{k+q} + i\omega_m)^2 - E_k^2} \end{aligned} \tag{17.43}$$

If we now take $i\omega_m \to \omega + i\delta$, and $\omega < 2\Delta$ (and, thereby, dropping terms such as $\delta(\omega - E_k - E_{k+q})$ which vanish for $\omega < 2\Delta$), we obtain:

$$\begin{aligned} \chi''_{+-}(q,\omega) = &\int \frac{d^3k}{(2\pi)^3}\, n_F(E_k)\, \frac{(\xi_k + E_k)\,(E_k + \xi_{k+q})}{2E_k E_{k+q}}\, \delta(\omega + E_k - E_{k+q}) \\ &- \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_k)\, \frac{(\xi_k - E_k)\,(-E_k + \xi_{k+q})}{2E_k E_{k+q}}\, \delta(\omega + E_k - E_{k+q}) \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_{k+q})\, \frac{(\xi_{k+q} + E_{k+q})\,(E_{k+q} + \xi_k)}{2E_{k+q} E_k}\, \delta(\omega + E_k - E_{k+q}) \\ &- \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_{k+q})\, \frac{(\xi_{k+q} - E_{k+q})\,(-E_{k+q} + \xi_k)}{2E_{k+q} E_k}\, \delta(\omega + E_{k+q} - E_k) \\ &+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_k)\, \frac{\Delta^2}{2E_k E_{k+q}}\, \delta(\omega + E_{k+q} - E_k) \end{aligned}$$

$$
\begin{aligned}
&- \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_k)\, \frac{\Delta^2}{2E_kE_{k+q}}\, \delta(\omega + E_k - E_{k+q}) \\
&+ \int \frac{d^3k}{(2\pi)^3}\, n_F(E_{k+q})\, \frac{\Delta^2}{2E_{k+q}E_k}\, \delta(\omega + E_k - E_{k+q}) \\
&- \int \frac{d^3k}{(2\pi)^3}\, n_F(-E_{k+q})\, \frac{\Delta^2}{2E_{k+q}E_k}\, \delta(\omega + E_{k+q} - E_k)
\end{aligned}
\tag{17.44}
$$

dropping terms which are odd in ξ_k or ξ_{k+q}, and using $n_F(-E_k) = 1 - n_F(E_k)$, we have:

$$
\chi''_{+-}(q,\omega) = \int \frac{d^3k}{(2\pi)^3}\, (n_F(E_k) - n_F(E_{k+q}))\, \left(1 + \frac{\xi_k\xi_{k+q} + \Delta^2}{2E_kE_{k+q}}\right) \delta(\omega + E_k - E_{k+q}) \tag{17.45}
$$

Let us assume that $A(q) = A$. Then, dropping the term linear in ξ_k and $\xi_{k'}$

$$
\frac{1}{T_1T} = \frac{A}{\omega} \int \frac{d^3k'}{(2\pi)^3} \frac{d^3k}{(2\pi)^3}\, (n_F(E_k) - n_F(E_{k'}))\, \left(1 + \frac{\Delta^2}{2E_kE_{k'}}\right) \delta(\omega + E_k - E_{k'}) \tag{17.46}
$$

or, using the single-particle density of states to re-write the momentum integrals as energy integrals,

$$
\begin{aligned}
\frac{1}{T_1T} = \frac{A}{\omega} \left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \int_\Delta^\Lambda dE' \frac{E}{\sqrt{E^2-\Delta^2}}\, \frac{E'}{\sqrt{E'^2-\Delta^2}} \left(1 + \frac{\Delta^2}{2EE'}\right) \\
\times\, \big(n_F(E) - n_F(E')\big)\, \delta(\omega + E - E')
\end{aligned}
\tag{17.47}
$$

or

$$
\begin{aligned}
\frac{1}{T_1T} = \frac{A}{\omega} \left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \frac{E}{\sqrt{E^2-\Delta^2}}\, \frac{E+\omega}{\sqrt{(E+\omega)^2-\Delta^2}} \left(1 + \frac{\Delta^2}{2E(E+\omega)}\right) \\
\times\, (n_F(E) - n_F(E+\omega))
\end{aligned}
\tag{17.48}
$$

For $\omega \to 0$, we can write this as:

$$
\frac{1}{T_1T} = A \left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \frac{E}{\sqrt{E^2-\Delta^2}}\, \frac{E+\omega}{\sqrt{(E+\omega)^2-\Delta^2}} \left(1 + \frac{\Delta^2}{2E(E+\omega)}\right) \frac{\partial}{\partial E} n_F(E) \tag{17.49}
$$

For $T \to 0$, the right-hand-side is exponentially suppressed as a result of the $\partial n_F(E)/\partial E$, and

$$
\frac{1}{T_1T} \sim e^{\frac{-\Delta}{T}} \tag{17.50}
$$

For $T \sim \Delta$, the exponential suppression is not very strong so the density of states divergence is important. In fact, for $\omega = 0$

$$\lim_{\omega\to 0}\frac{1}{T_1 T} = A\left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \frac{E^2}{E^2-\Delta^2}\left(1+\frac{\Delta^2}{2E^2}\right)\frac{\partial}{\partial E}n_F(E) \quad (17.51)$$

which is a divergent integral at $E = \Delta$. For realistic values of ω, there is a moderate, but clearly observable increase of $1/T_1$ for $T < T_c$ with a maximum which is called the Hebel-Slichter peak.

17.5.3 Acoustic Attenuation Rate

Suppose we compute the acoustic attenuation rate, which is essentially the phonon lifetime. Phonons are coupled to the electron density, so the phonon lifetime is determined by a density-density correlation function. This, too, is given by the diagrams of figure 17.2. However, since there are density operators rather than spin operators at the vertices of these diagrams, there is a crucial minus sign arising from the ordering of the electron operators:

$$\begin{aligned}\langle\rho(q,i\omega_m)\,\rho(-q,-i\omega_m)\rangle = &\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,\mathcal{G}_{\downarrow\downarrow}(i\epsilon_n,k)\,\mathcal{G}_{\uparrow\uparrow}(i\epsilon_n+i\omega_m,k+q)\\ &-\frac{1}{\beta}\sum_n \int \frac{d^3k}{(2\pi)^3}\,\mathcal{F}_{\uparrow\downarrow}(i\epsilon_n,k)\,\mathcal{F}^\dagger_{\downarrow\uparrow}(i\epsilon_n+i\omega_m,k+q)\end{aligned} \quad (17.52)$$

The acoustic attenuation rate, α, of a phonon of frequency ω is essentially given by

$$\alpha = \int \frac{d^3q}{(2\pi)^3}\,g(q)\,\chi''_{\rho\rho}(q,\omega) \quad (17.53)$$

where $g(q)$ is the electron-phonon coupling. From our calculation of $1/T_1$, we see that this is (assuming constant g):

$$\alpha = g\left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \frac{E}{\sqrt{E^2-\Delta^2}}\,\frac{E+\omega}{\sqrt{(E+\omega)^2-\Delta^2}}\left(1-\frac{\Delta^2}{E(E+\omega)}\right)\frac{\partial}{\partial E}n_F(E) \quad (17.54)$$

As a result of the $-$sign, we can take the $\omega \to 0$ limit:

$$\alpha = A\left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \frac{E^2}{E^2-\Delta^2}\left(1-\frac{\Delta^2}{E^2}\right)\frac{\partial}{\partial E}n_F(E)$$

$$= A \left(\frac{m^*}{2\pi}\right)^2 \int_\Delta^\Lambda dE \, \frac{\partial}{\partial E} n_F(E) \tag{17.55}$$

As in the case of $1/T_1$, this is exponentially decreasing at low, T,

$$\alpha \sim e^{\frac{-\Delta}{T}} \tag{17.56}$$

However, the density of states divergence has been cancelled by the quantum interference between particles and holes, so there is no enhancement for $T \sim \Delta$. Since the underlying quasiparticles are a superposition of electrons and holes such that their charge vanishes as the Fermi surface is approached, their contribution to the density-density correlation function is suppressed. This suppression cancels the divergence in the density of states. On the other hand, the quasiparticles carry spin 1/2 (since they are a mixture of an up-spin electron and a down-spin hole) so their contribution to the spin-spin correlation function is unsuppressed; hence the density of states divergence has dramatic consequences leading to the Hebel-Slichter peak.

17.5.4 Tunneling

Tunneling is a classic probe of the single-particle properties of an electron system. Let us suppose we connect a superconductor on the left with another system – which may or may not be a superconductor – on the right. An approximate description of the coupling between the superconductor and the other system is given by the tunneling Hamiltonian:

$$\begin{aligned} H_T &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \left[t(k,k') \, \psi_\sigma^\dagger(k) \, \chi_\sigma(k) + t^*(k,k') \, \chi_\sigma(k) \, \psi_\sigma^\dagger(k) \right] \\ &\equiv B + B^\dagger \end{aligned} \tag{17.57}$$

where $\psi_\sigma^\dagger(k)$ is the creation operator for an electron in the superconductor and $\chi_\sigma^\dagger(k)$ is the creation operator for an electron in the other system. $t(k,k')$ is the tunneling matrix element for an electron of momentum k in the superconductor to tunnelin into a momentum k' state in the other system. Tunneling occurs when there is a voltage difference, V, between the superconductor and the other system,

$$\begin{aligned} H_V &= V \int \frac{d^3k}{(2\pi)^3} \, \psi_\sigma^\dagger(k) \, \psi_\sigma(k) \\ &= V \, N_L \end{aligned} \tag{17.58}$$

The current flowing between the superconductor and the other system is

$$\begin{aligned} I &= i \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \left[t(k,k')\, \psi_\sigma^\dagger(k)\, \chi_{k\sigma} - t^*(k,k')\, \chi_\sigma(k)\, \psi_\sigma^\dagger(k) \right] \\ &\equiv i \left(B - B^\dagger \right) \end{aligned} \tag{17.59}$$

Following the steps by which we derived the conductivity and other response functions in chapter 7, we see that the current, $I(t)$ computed to linear order in H_T is given by:

$$\begin{aligned} \langle I(t) \rangle &= \left\langle T \left\{ e^{iVtN_L + i\int_{-\infty}^t H_T} \right\} I(t)\, T \left\{ e^{-iVtN_L - i\int_{-\infty}^t H_T} \right\} \right\rangle \\ &= i \left\langle \left[I(t), \int_{-\infty}^t H_T e^{iVtN_L} \right] \right\rangle \end{aligned} \tag{17.60}$$

Substituting the above expressions for I and H_T, we have:

$$\begin{aligned} \langle I(t) \rangle = &\int_{-\infty}^{\infty} dt'\, \theta(t-t') \left\{ e^{ieV(t'-t)}\, i \left\langle \left[B(t), B^\dagger(t') \right] \right\rangle - e^{ieV(t-t')}\, i \left\langle \left[B^\dagger(t), B(t') \right] \right\rangle \right\} \\ &+ \int_{-\infty}^{\infty} dt'\, \theta(t-t') \left\{ e^{-ieV(t+t')}\, i \left\langle \left[B(t), B(t') \right] \right\rangle - e^{ieV(t+t')}\, i \left\langle \left[B^\dagger(t), B^\dagger(t') \right] \right\rangle \right\} \end{aligned} \tag{17.61}$$

Suppose that $t(k,k') = t$. Then the real part of the current is

$$\begin{aligned} I = &t^2\, \mathrm{Im} \left\{ \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \sum_n \mathcal{G}_L(k, i\epsilon_n) \mathcal{G}_R(k, i\epsilon_n - i\omega) \right\}_{i\omega \to eV + i\delta} \\ &+ t^2\, \mathrm{Im} \left\{ e^{2ieVt} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \sum_n \mathcal{F}_L(k, i\epsilon_n) \mathcal{F}_R^\dagger(k, i\epsilon_n - i\omega) \right\}_{i\omega \to i\delta} \end{aligned} \tag{17.62}$$

Converting the Matsubara sum in the first term to an integral, analytically continuing, and taking the imaginary part (as we have done so often before), we have:

$$\begin{aligned} I = &t^2 \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\, A_L(k, \epsilon + eV)\, A_R(k', \epsilon)\, \left[n_F(\epsilon) - n_F(\epsilon + eV) \right] \\ &+ t^2\, \mathrm{Im} \left\{ e^{2ieVt} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \sum_n \mathcal{F}_L(k, i\epsilon_n) \mathcal{F}_R^\dagger(k, i\epsilon_n - i\omega) \right\}_{i\omega \to i\delta} \end{aligned} \tag{17.63}$$

Let us first focus on the first term. We will call this current I_E since it results from the tunneling of electrons. It can be rewritten as:

$$I_E = t^2 \frac{m_L^* k_F}{2\pi^2} \frac{m_R^* k_F}{2\pi^2} \int d\xi_k \int d\xi_{k'} \int \frac{d\epsilon}{2\pi} \left[u_k^2 \, \delta(\epsilon + eV - E_k) + v_k^2 \, \delta(\epsilon + eV + E_k) \right] \times A_R(k', \epsilon) \left[n_F(\epsilon) - n_F(\epsilon + eV) \right] \tag{17.64}$$

Suppose the system on the left is a Fermi liquid, with

$$A_R(k', \epsilon) = \delta(\epsilon - \xi_{k'}) \tag{17.65}$$

Then,

$$I_E = t^2 \frac{m_L^* k_F}{2\pi^2} \frac{m_R^* k_F}{2\pi^2} \int_\Delta^\infty dE \, \frac{E}{\sqrt{E^2 - \Delta^2}} \left[n_F(E) - n_F(E - eV) \right] \tag{17.66}$$

For $T = 0$, this vanishes for $eV < \Delta$ and asymptotes $I \propto V$ for V large. For T finite, I is exponentially small for $V < \Delta$. If the system on the right is also a superocnductor, we find:

$$I_E = t^2 \frac{m_L^* k_F}{2\pi^2} \frac{m_R^* k_F}{2\pi^2} \int dE \int_\Delta^\infty \frac{E}{\sqrt{E^2 - \Delta_L^2}} \frac{V - E}{\sqrt{(V - E)^2 - \Delta_R^2}} \left[n_F(E) - n_F(E - eV) \right] \tag{17.67}$$

This is exponentially small (vanishing at $T = 0$) for $eV < \Delta_L + \Delta_R$.

The current I_E resulting from the tunneling of electrons can be understood in terms of the semiconductor. However, the current described by the second term in (17.63) cannot. It vanishes unless the system on the right is a superconductor. We call this current I_J, since it was first discovered by Josephson.

$$I_J = t^2 \, \mathrm{Im} \left\{ e^{2ieVt} \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \sum_n \mathcal{F}_L(k, i\epsilon_n) \mathcal{F}_R^\dagger(k, i\epsilon_n - i\omega) \right\}_{i\omega \to i\delta} \tag{17.68}$$

This is one of the few cases in which it is advantageous to do the momentum integrals first. Let us assume that $|\Delta_L| = |\Delta_R| = \Delta$, $\Delta_L = \Delta_R e^{i\phi}$ and $m_R^* = m_L^*$.

$$\begin{aligned} I_J &= t^2 \left(\frac{m^* k_F}{2\pi^2} \right)^2 \mathrm{Im} \left\{ e^{2ieVt} \sum_n \int d\xi_k \int d\xi_{k'} \, \frac{\Delta_L}{(i\epsilon_n)^2 - \xi_k^2 - |\Delta|^2} \frac{\Delta_R^*}{(i\epsilon_n - i\omega)^2 - \xi_k^2 - |\Delta|^2} \right\}_{i\omega \to i\delta} \\ &= t^2 \left(\frac{m^* k_F}{2\pi^2} \right)^2 \mathrm{Im} \left\{ e^{2ieVt} \sum_n \frac{\pi \Delta_L}{\sqrt{\epsilon_n^2 + \Delta^2}} \frac{\pi \Delta_R^*}{\sqrt{(\epsilon_n - \omega)^2 + \Delta^2}} \right\}_{i\omega \to i\delta} \end{aligned}$$

$$
\begin{aligned}
&= t^2 \left(\frac{m^* k_F}{2\pi^2}\right)^2 \mathrm{Im}\left\{ e^{2ieVt+\phi} \sum_n \frac{(\pi\Delta)^2}{\epsilon_n^2 + \Delta^2} \right\} \\
&= t^2 \left(\frac{m^* k_F}{2\pi^2}\right)^2 \mathrm{Im}\left\{ e^{2ieVt+\phi} (\pi\Delta)^2 \frac{1}{|\Delta|} \tanh\frac{\beta\Delta}{2} \right\} \\
&= t^2 \left(\frac{m^* k_F}{2\pi}\right)^2 \left\{ |\Delta| \tanh\frac{\beta\Delta}{2} \right\} \sin(2eVt+\phi)
\end{aligned}
\tag{17.69}
$$

The Josephson current results from the tunneling of pairs between two superconductors. A DC voltage V leads to an AC Josephson current at frequency $2eV$. Even if the voltage difference is zero, there will be a DC Josephson current if the superconducting order parameters onthe left and right have different phases. (The flow of this DC current will feed back into the electrostatics of the problem and, eventually, turn off this current.)

The Josephson current cannot be understood with the semiconductor model since it is due to the tunneling of pairs. It can be understood as an oscillation of the phase difference between two superconductors. As such, it is an example of a collective mode in a superocnductor.

17.6 Collective Modes of a Superconductor

If we expand the effective action (17.7) in powers of Ψ and its gradients, and include the action of the electromagnetic field, then we have:

$$
S = \int dt\, d^3x \left(\Psi^\dagger (i\partial_t - A_0)\Psi + \frac{1}{2m^*}|(i\partial_i - A_i)\Psi|^2 + V(|\Psi|) + \frac{1}{8\pi}\left(E^2 - B^2\right)\right) \tag{17.70}
$$

$V(|\Psi|)$ is actually a complicated function, but let us, for the sake of simplicity, approximate it by:

$$
V(|\Psi|) = a\left(|\Psi|^2 - \rho_s\right)^2 \tag{17.71}
$$

for some constants a and ρ_s. This action is very similar to our effective action for ^{4}He: the $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$ is broken when Ψ has an expectation value. The principal difference is the electromagnetic field.

Following our analysis of ^{4}He, we write:

$$
\psi = \sqrt{(\rho_s + \delta\rho)}\, e^{i\theta} \tag{17.72}
$$

We can rewrite the action as:

$$
S = \int d\tau d^d x \left(\frac{1}{2}\partial_\tau \delta\rho + \rho_s\left(\partial_t\theta + A_0\right) + \delta\rho\left(\partial_t\theta + A_0\right)\right.
$$

$$+\frac{1}{2\left(\delta\rho+\rho_s\right)}(\nabla\delta\rho)^2+\rho_s\Big(\vec{\nabla}\theta-\vec{A}\Big)^2+\delta\rho\Big(\vec{\nabla}\theta-\vec{A}\Big)^2+a\delta\rho^2$$
$$+\frac{1}{8\pi}\left(E^2-B^2\right)) \tag{17.73}$$

The first two terms can (naively) be neglected since they are total derivatives, so the free part of this action is

$$S=\int d\tau d^dx\left(\delta\rho\ \left(\partial_t\theta+A_0\right)+\ \frac{1}{2\rho_s}\left(\nabla\delta\rho\right)^2+\ \rho_s\left(\vec{\nabla}\theta-\vec{A}\right)^2+\ a\delta\rho^2\right.$$
$$\left.+\frac{1}{8\pi}\left(E^2-B^2\right)\right) \tag{17.74}$$

Let us take the gauge $\theta=0$. Then we have:

$$S=\int d\tau d^dx\,(+\frac{1}{2\rho_s}\left(\nabla\delta\rho\right)^2+\ a\delta\rho^2$$
$$+\left(\vec{\nabla}\cdot\vec{E}-\delta\rho\right)\ A_0$$
$$+\rho_s\,\vec{A}^2-\frac{1}{8\pi}\Big(\vec{\nabla}\times\vec{A}\Big)^2) \tag{17.75}$$

From the third line, we see that the transverse electromagnetic field now aquires a gap. Its equation of motion is:

$$\nabla^2\vec{A}=\rho_s\vec{A} \tag{17.76}$$

which has solutions:

$$\vec{A}(x)=\vec{A}(0)\,e^{-\lambda x} \tag{17.77}$$

where $\lambda^2=1/\rho_s$. This is the Meissner effect: the magnetic field vanishes in the interior of a superconductor. The action (17.75) also implies the London equation:

$$\vec{j}\equiv\frac{\delta S}{\delta\vec{A}}=\rho_s\vec{A} \tag{17.78}$$

from which the infinite conductivity of a superconductor follows.

Although the $U(1)$ symmetry has been broken, there is no Goldstone boson. The would-be Goldstone boson, θ, has been gauged away. To put this more physically, the Goldstone mode would be an oscillation of the density. However, as we saw in chapter 15, the Coulomb interaction pushes the density oscllation up to a high frequency, the plasma frequency. Hence, the would-be Goldstone boson is at the plasma frequency.

From the first term in (17.74), we see that $\delta\rho$ and θ are canonical conjugates and the Hamiltonian is:

$$H = \int d^d k \left(\left(\frac{1}{2\rho_s} k^2 + a \right) |\delta\rho_k|^2 + (A_0)_{-k} \delta\rho_k + \rho_s k^2 |\theta_k|^2 + \frac{1}{8\pi} \left(E^2 + B^2 \right) \right) \tag{17.79}$$

From the constraint (the A_0 equation of motion),

$$k^2 (A_0)_{-k} = \delta\rho_k \tag{17.80}$$

we have:

$$(A_0)_{-k} = \frac{1}{k^2} \delta\rho_k \tag{17.81}$$

Neglecting the magnetic field, since all velocities are much smaller than the speed of light, we have:

$$H = \int d^d k \left(\left(\frac{1}{2\rho_s} k^2 + a + \frac{1}{k^2} \right) |\delta\rho_k|^2 + \rho_s k^2 |\theta_k|^2 \right) \tag{17.82}$$

Since $\delta\rho$ and θ are canonical conjugates, this is of the harmonic osciallator form

$$H = \int d^d k \left(\frac{1}{2m} |P_k|^2 + \frac{1}{2} m \omega_k^2 |X_k|^2 \right) \tag{17.83}$$

with

$$\omega_k = \sqrt{4 (\rho_s k^2) \left(\frac{1}{k^2} + a + \frac{1}{2\rho_s} k^2 \right)} \tag{17.84}$$

In the long-wavelength limit, $k \to 0$,

$$\omega_k = \sqrt{\rho_s} \tag{17.85}$$

i.e. the mode is gapped.

17.7 The Higgs Boson

17.8 Broken gauge symmetry**

17.9 The Josephson Effect-xxx

17.10 Response Functions of a Superconductor-xxx

17.11 Repulsive Interactions

In any real metal, there is a large repulsive force due to Coulomb intractions. This repulsion is much stronger than the weak attraction due to the exchange of phonons, so one might wonder how superconductivity can occur at all. The answer is that the repulsive interaction occurs at short time scales and high-energies. At the low energies at which superconductivity occurs, the repulsion is much weaker. Since a repulsive interaction in the BCS channel is marginally irrelevant, as we saw earlier, it will be logarithmically suppressed.

Consider the following illustrative model:

$$V(k,k') = \begin{cases} V & \text{if } |\xi_k| > \omega_D \text{ or } |\xi_{k'}| > \omega_D \text{—} \\ (V - V_a) & \text{if } |\xi_k|\,, |\xi_{k'}| < \omega_D \end{cases}$$

with $V > 0$ and $V - V_a > 0$ so that the interaction is *repulsive* everywhere, but less repulsive near the Fermi surface – i.e. $-V_a$ is the weak attraction on top of the repulsion V. Let

$$\Delta(k) = \begin{cases} \Delta_1 & \text{if } \omega_D < |\xi_k| < \Lambda \text{ or } |\xi_{k'}| > \omega_D \text{—} \\ \Delta_2 & \text{if } |\xi_k| < \omega_D \end{cases}$$

The gap equation is:

$$\begin{aligned} \Delta_1 &= -V\Delta_1 \frac{m^*}{2\pi}\int_{\omega_D}^{\Lambda} d\xi \, \frac{1}{\sqrt{\xi^2+\Delta_1^2}} - V\Delta_2 \frac{m^*}{2\pi}\int_0^{\omega_D} d\xi \, \frac{1}{\sqrt{\xi^2+\Delta_2^2}} \\ \Delta_1 &= -V\Delta_1 \frac{m^*}{2\pi}\int_{\omega_D}^{\Lambda} d\xi \, \frac{1}{\sqrt{\xi^2+\Delta_1^2}} - (V-V_a)\,\Delta_2 \frac{m^*}{2\pi}\int_0^{\omega_D} d\xi \, \frac{1}{\sqrt{\xi^2+\Delta_2^2}} \end{aligned} \tag{17.86}$$

If we assume that $\Lambda \gg \omega_D$ and $\omega_D \gg \Delta_2$ then we have:

$$\Delta_1 = -V\Delta_1 \frac{m^*}{2\pi} \ln\left(\frac{\Lambda}{\omega_D}\right) - V\Delta_2 \frac{m^*}{2\pi} \ln\left(\frac{\omega_D}{\Delta_2}\right)$$

$$\Delta_1 = -V\Delta_1 \frac{m^*}{2\pi} \ln\left(\frac{\Lambda}{\omega_D}\right) - (V - V_a)\,\Delta_2 \frac{m^*}{2\pi} \ln\left(\frac{\omega_D}{\Delta_2}\right) \quad (17.87)$$

From the first equation, we have:

$$\Delta_1 = -\frac{V}{1 + \frac{m^*}{2\pi} V \ln\left(\frac{\Lambda}{\omega_D}\right)} \Delta_2 \frac{m^*}{2\pi} \ln\left(\frac{\omega_D}{\Delta_2}\right) \quad (17.88)$$

Hence, Δ_1 and Δ_2 must have opposite signs. Substituting into the second equation, we find:

$$\left[V_a - \frac{V}{1 + \frac{m^*}{2\pi} V \ln\left(\frac{\Lambda}{\omega_D}\right)}\right] \frac{m^*}{2\pi} \ln\left(\frac{\omega_D}{\Delta_2}\right) = 1 \quad (17.89)$$

This equation will have a solution if

$$V_a - \frac{V}{1 + \frac{m^*}{2\pi} V \ln\left(\frac{\Lambda}{\omega_D}\right)} > 0 \quad (17.90)$$

even if $V_a - V < 0$. In other words, the bare interaction may be repulsive, but the effective pairing interaction can be attractive because the repulsive part will be logarithmically suppressed.

17.12 Phonon-Mediated Superconductivity-xxx

17.13 The Vortex State***

17.14 Fluctuation effects***

17.15 Condensation in a non-zero angular momentum state***

17.15.1 Liquid ^{3}He***

17.15.2 Cuprate superconductors***

17.16 Experimental techniques***

CHAPTER 18

Density waves in solids

18.1 Spin density wave

Much of the formalism which we used in the previous chapter can be adapted to the case of density-waves in fermion systems with nested or nearly nested Fermi surfaces.

18.2 Charge density wave***

18.3 Density waves with non-trivial angular momentum-xxx

18.4 Incommensurate density waves***

Part VI

Gauge Fields and Fractionalization

CHAPTER 19

Topology, Braiding Statistics, and Gauge Fields

19.1 The Aharonov-Bohm effect

As we have discussed, systems of many particles tend to form energy gaps as a way of lowering their energy. One might be tempted to conclude that their low-energy properties are, as a result, trivial, and that interesting physics occurs only when they are gapless, either because they are tuned to a critical point or because their ground state spontaneously breaks a symmetry. However, non-trivial low-energy physics can occur even when a system is fully gapped. A fully gapped system can have non-trivial *topological* properties, which do not require low-energy local degrees of freedom. As we will see, such properties can be described by *gauge fields*. These topological properties are concomitant with the phenomenon of *fractionalization*, whereby the quantum numbers of the low-energy excitations of a system can be fractions of the quantum numbers of its basic microscopic constituents, presumably electrons. Phases which are characterized by fractionalization are stable against small perturbations: if the electron breaks into n pieces, a small perturbation cannot change this continuously; an electron, unlike the average American family, cannot have 2.4 children. It is the fact that fractionalization is necessarily characterized by integers which guarantees that it is stable if it occurs.[7]

The basic idea can be understood by considering the Aharonov-Bohm effect. Suppose an infinitely-long, thin solenoid at the origin which is threaded

by flux Φ (in units in which $\hbar = e = c = 1$, one flux quantum is $\Phi = 2\pi$) is surrounded by an infinitely-high potential barrier. As result, electrons are prevented from entering the solenoid, and they move in a region in which the magnetic field is zero. As Aharonov and Bohm showed, the cross-section for an electron of momentum p to scatter off the flux tube is:

$$\frac{d\sigma}{d\theta} = \frac{1}{2\pi p \sin^2(\theta/2)} \sin^2 \frac{\Phi}{2} \tag{19.1}$$

In other words, the scattering cross-section is non-trivial and depends on Φ even though the electron never enters the region in which $\mathbf{B} \neq 0$ (the interior of the solenoid).

Any description of the physics of this system in terms of the electric and magnetic fields $\mathbf{E}$, $\mathbf{B}$ alone cannot be local. It must involve some kind of action-at-a-distance so that the magnetic field inside the solenoid can affect the electron outside the solenoid. However, a local description can be given in terms of the vector potential,

$$\mathbf{A}(\mathbf{x}) = \frac{\Phi}{2\pi} \frac{\hat{\mathbf{z}} \times \mathbf{x}}{|\mathbf{x}|^2} \tag{19.2}$$

by simply including this vector potential in the Hamiltonian,

$$H\psi = \frac{1}{2m}(\mathbf{p} - \mathbf{A})^2 \psi \tag{19.3}$$

The electromagnetic potential is an example of a *gauge field*. By this, we mean that the vector potential, A_μ, is not itself a measurable quantity because all physically measurable quantities are invariant under the *gauge transformation*:

$$\begin{aligned} A_\mu(x) &\to A_\mu(x) - \partial_\mu \chi(x) \\ \psi(x) &\to e^{i\chi(x)} \psi(x) \end{aligned} \tag{19.4}$$

The gauge field $A_\mu(x)$ is a redundant way of parametrizing $\mathbf{B}$, $\mathbf{E}$ which satisfy $\nabla \cdot \mathbf{B} = 0$, $\nabla \cdot \mathbf{E} = 4\pi\rho$. This redundancy is the price which must be paid in order to retain a local description.

In particular, when $A_\mu(x) = \partial_\mu f$ for some f, the electromagnetic potential is equivalent under a gauge transformation to zero. However, $\nabla \times \mathbf{A} = 0$ does not always mean that an $f(x)$ exists such that $\mathbf{A} = \nabla f$. The potential (19.2) is an example of such a topologically non-trivial vector field. It is locally equivalent to zero, but not globally, as a result of the singularity at the origin.

If we were to try to gauge away the vector potential (19.2) by taking the singular function

$$f = \frac{\Phi}{2\pi} \tan^{-1} \frac{y}{x} = \frac{\Phi}{2\pi} \theta \tag{19.5}$$

the wavefunction would no longer be single-valued:

$$\psi(r, \theta) \to e^{i\Phi\theta/2\pi} \psi(r, \theta) \tag{19.6}$$

This is because, as the electron encircles the origin, it aquires a gauge-invariant 'Aharonov-Bohm phase'

$$e^{i \oint \mathbf{A} \cdot d\mathbf{l}} = e^{i\Phi} \tag{19.7}$$

which distinguishes the vector potential from a trivial one. However, as the above example shows, we can work with a vanishing vector potential at the cost of having a multi-valued wavefunction.

The phase aquired by the electron is independent of how close the electron comes to the solenoid or how fast it moves and depends only on the topology of the electron's path, namely how many times it winds about the origin. Hence, the gauge field (19.2) gives rise to a 'topological interaction', which is felt by the electron even if it is infinitely far away from the solenoid. As we discuss below, it is customary in certain circumstances to separate such topological interactions from ordinary ones which do depend on distance and lump them into particle 'statistics'.

As we will see, the low-energy excitations of a strongly-interacting electron system can aquire similar phases – i.e. have non-trivial braiding properties – when they encircle each other. These phases result from the electron-electron correlations which are encoded in the ground-state wavefunction. A local description of the physics of these excitations must incorporate gauge fields for the reason which we saw above. Unlike the electromagnetic field, these gauge fields will be a dynamically generated feature of the low-energy properties of the system. Such a system can be fully gapped, in which case the non-trivial braiding properties of the excitations come into play at the finite energies at which these excitations are created. However, even at low-energies, these braiding properties are manifested in the ground state on manifolds of non-trivial topology. The ground state is degenerate, a reflection of the braiding properties of the quasiparticles. The effective field theories of these ground states and of the ground states with a fixed number of quasiparticles are called *topological quantum field theories*.

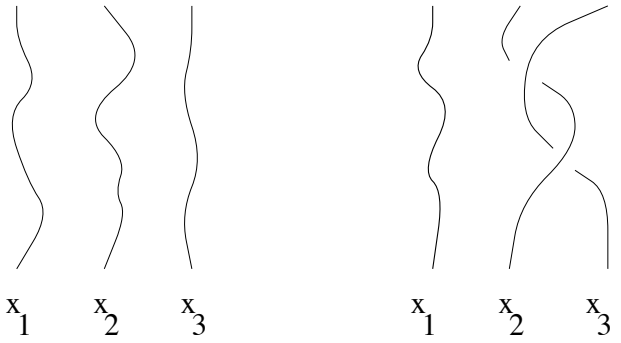


Figure 19.1: Different trajectories of hard-core particles in $2+1$ dimensions which are not adiabatically deformable into each other.

19.2 Exotic Braiding Statistics

Let us consider the braiding properties of particle trajectories in $2+1$-dimensions (2 spatial and 1 time dimension). According to Feynman, the quantum-mechanical amplitude for hard-core particles which are at $x_1, x_2, \ldots, x_n$ at an initial time t_0 to return to these coordinates at a later time t is given by a sum over all trajectories. Each trajectory is summed with weight e^{iS}. This particular assignment of weights gives consistency with the classical limit. However, a peculiarity of two spatial dimensions is that the space of trajectories of hard-core particles is disconnected, as may be seen in figure (19.1).

Consequently, at the quantum mechanical level, we have the freedom, as Leinaas and Myrrheim, Wilczek, ... observed, to weight each of these different components by a different phase factor. Since these components are not continuously deformable into each other, the stationary phase condition associated with the classical limit does not constrain these phases.

These phase factors realize an Abelian representation of the *braid group*, whose elements are the different components of trajectory space with a composition operation obtained by simply following one trajectory by another. Let us consider the case of two identical particles. The braid group is simply the group of integers, with integer n corresponding to the number of times that one particle winds counter-clockwise about the other (negative integers are clockwise windings). If the particles are identical, then we must allow exchanges as well, which we can label by half-integer windings. The different representations of the braid group of two identical particles are la-

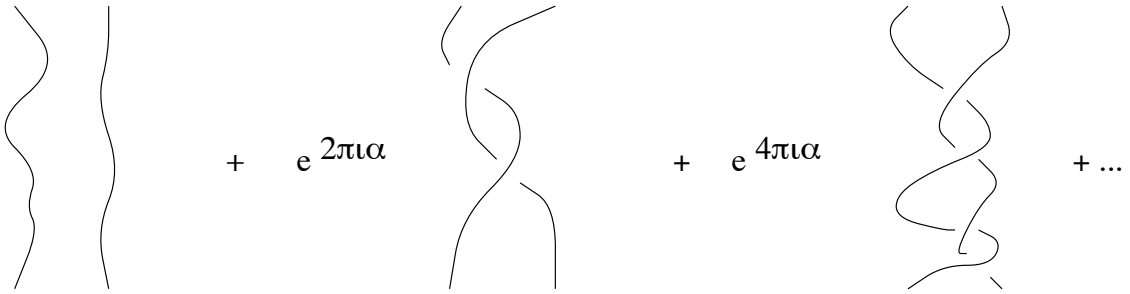


Figure 19.2: An assignment of phases to different disonnected components of the space of trajectories of two particles.

belled by a phase α, so that a trajectory in which one particle is exchanged counter-clockwise with the other n times receives the phase $e^{in\alpha}$.

If $\alpha = 0$, the particles are bosons; if $\alpha = \pi$, the particles are fermions. For intermediate values of α, the particles are called *anyons*. The braid group of N particles has more complicated representations which can be non-abelian, but a class of its representations is just an extension of the two-particle case: whenever any of N identical particles is exchanged counter-clockwise n times with another, the phase associated with this is $e^{in\alpha}$.

In a slight abuse of terminology, we use the term 'statistics' to describe these representations of the braid group. In reality, it is more like a topological interaction since it is not limited to identical particles. Different particle species can have 'mutual statistics' when they wind about each other (since they are not identical, they cannot be exchanged). This is quite different from the case in higher dimensions, where there is no braid group, and we only have the permutation group – which acts only on identical particles – whose only abelian representations are bosonic and fermionic. To emphasize the distinction between this notion of statistics and the usual one, we will use the term 'braiding statistics'.

As we will see in the next chapter, this expanded notion of statistics is more than a mathematical curiosity; it is realized in all of its glory in the quantum Hall effect. First, however, we will discuss its field-theoretical implementation.

19.3 Chern-Simons Theory

Non-trivial braiding statistics can be implemented by taking wavefunctions which are multi-valued so that a phase is aquired whenever one particle is exchanged with another. However, as we saw at the beginning of this chapter, we can make these wavefunctions single-valued by introducing a gauge field $\mathbf{a}$ (distinct from the electromagnetic field $\mathbf{A}$) which gives rise to a vanishing magnetic field but is not gauge-equivalent to zero, in the spirit of (19.2).

$$\mathbf{a}(\mathbf{x}) = \frac{\Phi}{2\pi} \sum_i \frac{\hat{\mathbf{z}} \times (\mathbf{x} - \mathbf{x}_i)}{|\mathbf{x} - \mathbf{x}_i|^2} \tag{19.8}$$

where $\mathbf{x}_i$ is the position of the i^{th} particle. When one particle winds around another, it aquires a phase. An exchange is half of a wind, so half of this phase is aquired during an exchange.

Such a gauge field is produced automatically if we add a *Chern-Simons term* to the action. Consider the addition of such a term to the action for a system of free fermions:

$$S = \int \left[\psi^\dagger \left(i\partial_t - a_0 \right) \psi + \frac{1}{2m} \psi^\dagger (i\nabla - \mathbf{a})^2 \psi \right] + \frac{1}{2\Phi} \int dt\, d^2x\, \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{19.9}$$

The action (19.9) is invariant under the gauge transformation

$$\begin{aligned} a_\mu(x) &\to a_\mu(x) - \partial_\mu \chi(x) \\ \psi(x) &\to e^{i\chi(x)} \psi(x) \end{aligned} \tag{19.10}$$

up to the boundary term

$$\delta S = \frac{1}{2\Phi} \int_{\partial R} d^2x \chi \epsilon_{ij} \partial_i a_j \tag{19.11}$$

In an infinite system or on a compact manifold, we can ignore this boundary term. When we consider a bounded region R of the plane, this term will be important, as we will discuss in the context of the quantum Hall effect.

Since no time derivative of a_0 appears in the Lagrange multiplier, it is a Lagrange multiplier. If we vary it, we obtain the constraint:

$$\nabla \times \mathbf{a} = \Phi\, \psi^\dagger \psi \tag{19.12}$$

This constraint completely fixes a_μ, up to gauge transformations. Hence, the gauge field a_μ has no independent dynamics of its own; it is completely

determined by $\psi(x)$. According to the constraint (19.12) a flux Φ is attached to each fermion.

Let us consider the Chern-Simons action in the gauge $a_0 = 0$. The action is

$$S = \int \left[\psi^\dagger \left(i\partial_t - a_0 \right) \psi + \frac{1}{2m} \psi^\dagger (i\nabla - \mathbf{a})^2 \psi \right] + \frac{1}{2\Phi} \int \left(a_1 \partial_0 a_2 - a_2 \partial_0 a_1 \right) \tag{19.13}$$

Thus, the Hamiltonian of the Chern-Simons gauge field vanishes. Note, however, that the Hamiltonian must be supplemented by the constraint (19.12).

Hence, the Chern-Simons term does what we want – i.e. implement anyonic braiding statistics – and it does nothing else.

19.4 Ground States on Higher-Genus Manifolds

Let us now imagine that the particles are all gapped, so that we can integrate them out. Let us further assume that the Chern-Simons coefficient is an integer m divided by 4π. We will return to this assumption below. Then, the effective action at low energies is simply

$$S = \frac{m}{4\pi} \int \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{19.14}$$

This theory would appear to be completely trivial. The gauge field is fixed by the constraint

$$\nabla \times \mathbf{a} = 0 \tag{19.15}$$

and the Hamiltonian vanishes. Thus, the effective action only describes the ground state – or states.

On the infinite plane or the sphere, the ground state is a unique, non-degenerate state. Pure Chern-Simons theory (i.e. without any other fields to it) has no other states. However, suppose that the theory is defined on the torus. Then $\nabla \times \mathbf{a} = 0$ can still give rise to non-trivial

$$e^{iA_\gamma} = e^{\oint_\gamma \mathbf{a} \cdot d\mathbf{l}} \tag{19.16}$$

if γ winds around one of the non-trivial cycles of the torus. According to the constraint, A_γ does not depend on the precise curve γ but only on how many times it winds around the generators of the torus. Furthermore, it is

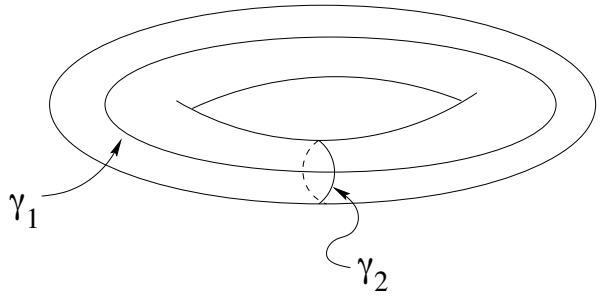


Figure 19.3: The basic operators A_1 and A_2 are constructed from the line integrals of $\mathbf{a}$ around γ_1 and γ_2.

clear that A_γ is additive in the sense that its value for a curve γ which winds twice around one of the generators of the torus is twice its value for a curve γ which winds once. Hence, we have only two independent variables, A_1, A_2 associated with the two generators of the torus. If we take coordinates $\theta_1, \theta_2 \in [0, 2\pi]$ on the torus, then

$$A_i = \int_0^{2\pi} a_i d\theta_i \tag{19.17}$$

From (19.13), we have the following equal-time commutation relations:

$$\big[a_1(x), a_2(x')\big] = i\frac{2\pi}{m}\delta^{(2)}(x - x') \tag{19.18}$$

from which it follows that

$$[A_1, A_2] = \frac{2\pi i}{m} \tag{19.19}$$

Since A_1, A_2 are not themselves gauge-invariant, we cannot simply use the analogy between their commutation relations and those of p, x for a single particle. We must work with the gauge invariant quantities e^{iA_i}. They have more complicated comutation relations. Since

$$e^{iA_1}\, e^{iA_2} = e^{[A_1,A_2]/2}\, e^{iA_1+iA_2} \tag{19.20}$$

we have the commutation relation

$$e^{iA_1}\, e^{iA_2} = e^{2\pi i/m}\, e^{iA_2}\, e^{iA_1} \tag{19.21}$$

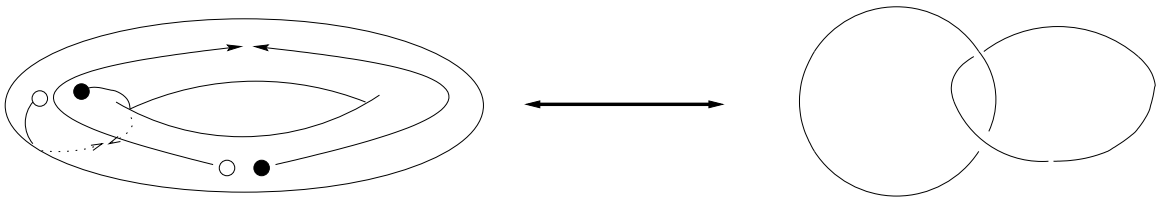


Figure 19.4: Creating a quasiparticle-quasihole pair, taking them around either of the generators of the torus and annihilating them leads yields two non-commuting operations which encode quasiparticle statistics in the ground state degeneracy.

This algebra can be implemented on a space of minimum dimension m:

$$\begin{aligned} e^{iA_1} |n\rangle &= e^{2\pi ni/m} |n\rangle \\ e^{iA_2} |n\rangle &= |n+1\rangle \end{aligned} \tag{19.22}$$

i.e. the ground state is m-fold degenerate. On a genus g manifold, this generalizes to m^g.

This has an interpretation in terms of the quasiparticle spectrum of the theory – about which we thought that we had lost all information by going to low energies below the quasiparticle energy gap. Imagine creating a quasihole-quasiparticle pair, taking them around one of the two non-trivial loops on the torus and annihilating them. Call the corresponding operators T_1, T_2. If the quasiparticles have statistics π/m, then

$$T_1 T_2 = e^{2\pi i/m} T_2 T_1 \tag{19.23}$$

because the particles wind around each other during such a process, as depicted on the right of figure 19.4. This is precisely the same algebra (19.21) which we found above, with a minimal representation of dimension m.

Hence, if we know that the ground state degeneracy of a system on a genus-g manifold is m^g, then one explanation of this degeneracy is that it has non-trivial quasiparticles of statistics $0, \pi/m, \ldots, (m-1)\pi/m$.

Why did we take the Chern-Simons coefficient to be an integer? This is required when we define Chern-Simons theory on compact manifolds or, equivalently, when we require invariance under large gauge transformations. On a compact manifold, the Chern-Simons action transforms under a gauge

transformation defined by a function $\chi(x)$ as:

$$\frac{m}{4\pi}\int \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \rightarrow \frac{m}{4\pi}\int \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho + 2\pi m\, N \tag{19.24}$$

where N is the winding number of the map from x to $e^{i\,\chi(x)} \in U(1)$. Hence, invariance of the functional integral mandates that we take m to be an integer.

CHAPTER 20

Introduction to the Quantum Hall Effect

20.1 Introduction

In 1879, E.H. Hall performed an experiment designed to determine the sign of the current-carrying particles in metals. If we suppose that these particles have charge e (with a sign to be determined) and mass m, the classical equations of motion of charged particles in an electric field, $\mathbf{E} = E_x\hat{\mathbf{x}} + E_y\hat{\mathbf{y}}$, and a magnetic field, $\mathbf{B} = B\hat{\mathbf{z}}$ are:

$$\begin{aligned} \frac{dp_x}{dt} &= eE_x - \omega_c p_y - p_x/\tau \\ \frac{dp_y}{dt} &= eE_y + \omega_c p_x - p_y/\tau \end{aligned} \tag{20.1}$$

where $\omega_c = eB/m$ and τ is a relaxation rate determined by collisions with impurities, other electrons, etc. Let us, following Hall, place a wire along the $\hat{\mathbf{x}}$ direction in the above magnetic fields and run a current, j_x, through it. In the steady state, $dp_x/dt = dp_y/dt = j_y = 0$, we must have $E_x = \frac{m}{ne^2\tau}\, j_x$ and

$$E_y = -\frac{B}{ne}\, j_x = \frac{-e}{|e|}\, \frac{h}{e^2}\, \frac{\Phi/\Phi_0}{N}\, j_x \tag{20.2}$$

where n and N are the density and number of electrons in the wire, Φ is the magnetic flux penetrating the wire, and $\Phi_0 =$ is the flux quantum. Hence, the sign of the charge carriers can be determined from a measurement of the transverse voltage in a magnetic field. Furthermore, according to (20.2),

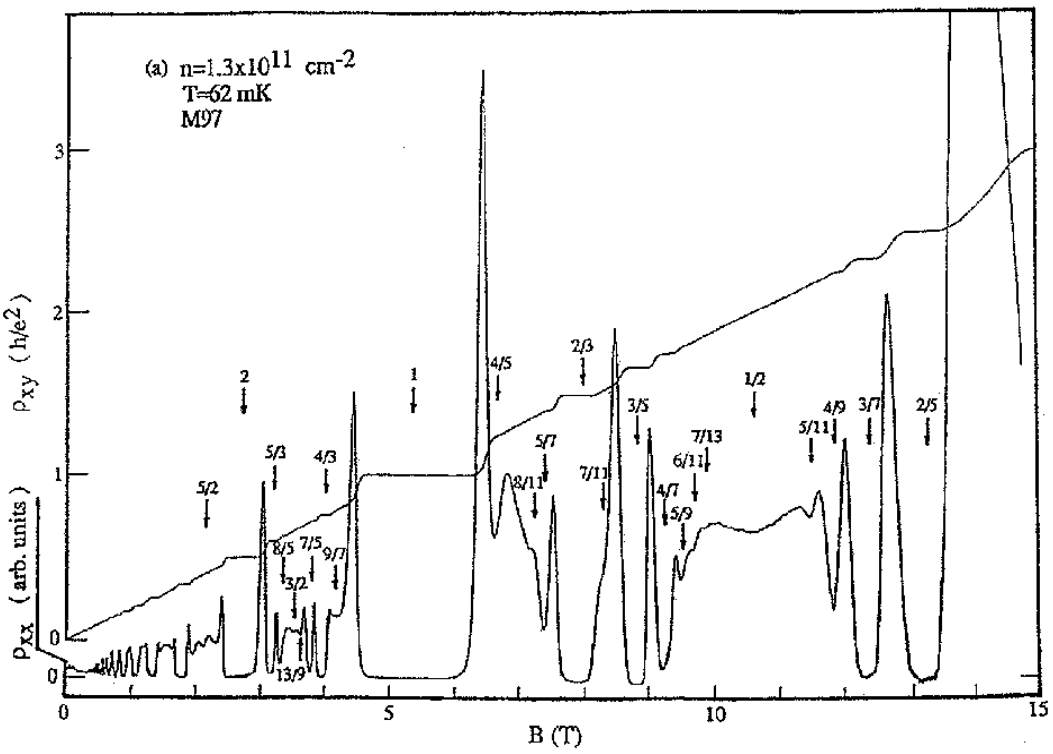


Figure 20.1: ρ_{xx} and ρ_{xy} vs. magnetic field, B, in the quantum Hall regime. A number of integer and fractional plateaus can be clearly seen. This data was taken at Princeton on a GaAs-AlGaAs heterostructure.

the density of charge carriers – *i.e.* electrons – can be determined from the slope of the $\rho_{xy} = E_y/j_x$ vs B. At high temperatures, this is roughly what is observed.

In the *quantum Hall regime*, namely at low-temperatures and high magnetic fields, very different behavior is found in two-dimensional electron systems. ρ_{xy} passes through a series of plateaus, $\rho_{xy} = \frac{1}{\nu}\frac{h}{e^2}$, where ν is a rational number, at which ρ_{xx} vanishes [**?**, **?**], as may be seen in Figure 20.1 (taken from [**?**]). The quantization is accurate to a few parts in 10^8, making this one of the most precise measurements of the fine structure constant, $\alpha = \frac{e^2}{\hbar c}$, and, in fact, one of the highest precision experiments of any kind.

Some insight into this phenomenon can be gained by considering the quantum mechanics of a single electron in a magnetic field. Let us sup-

pose that the electron's motion is planar and that the magnetic field is perpendicular to the plane. For now, we will assume that the electron is spin-polarized by the magnetic field and ignore the spin degree of freedom. The Hamiltonian,

$$H = \frac{1}{2m}\left(-i\hbar\nabla + \frac{e}{c}\,\mathbf{A}\right)^2 \tag{20.3}$$

takes the form of a harmonic oscillator Hamiltonian in the gauge $A_x = -By$, $A_y = 0$. (Here, and in what follows, I will take $e = |e|$; the charge of the electron is $-e$.) If we write the wavefunction $\psi(x,y) = e^{ikx}\,\phi(y)$, then:

$$H\psi = \left[\frac{1}{2m}\left(\left(\frac{eB}{c}\right)\,y + \hbar k\right)^2 + \frac{1}{2m}\left(-i\hbar\partial_y\right)^2\right]\,\phi(y)\,e^{ikx} \tag{20.4}$$

The energy levels $E_n = (n+\frac{1}{2})\hbar\omega_c$, called Landau levels, are highly degenerate because the energy is independent of k. To analyze this degeneracy (and in most of what follows) it will be more convenient to work in symmetric gauge, $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$ Writing $z = x + iy$, we have:

$$H = \frac{\hbar^2}{m}\,\left[-2\left(\partial - \frac{\bar{z}}{4\ell_0^2}\right)\left(\bar{\partial} + \frac{z}{4\ell_0^2}\right)\,+\frac{1}{2\ell_0^2}\,\right] \tag{20.5}$$

with (unnormalized) energy eigenfunctions:

$$\psi_{n,m}(z,\bar{z}) = z^m\,L_n^m(z,\bar{z})e^{-\frac{|z|^2}{4\ell_0^2}} \tag{20.6}$$

at energies $E_n = (n+\frac{1}{2})\hbar\omega_c$, where $L_n^m(z,\bar{z})$ are the Laguerre polynomials and $\ell_0 = \sqrt{\hbar/(eB)}$ is the magnetic length.

Let's concentrate on the lowest Landau level, $n = 0$. The wavefunctions in the lowest Landau level,

$$\psi_{n=0,m}(z,\bar{z}) = z^m\,e^{-\frac{|z|^2}{4\ell_0^2}} \tag{20.7}$$

are analytic functions of z multiplied by a Gaussian factor. The general lowest Landau level wavefunction can be written:

$$\psi_{n=0,m}(z,\bar{z}) = f(z)\,e^{-\frac{|z|^2}{4\ell_0^2}} \tag{20.8}$$

The state $\psi_{n=0,m}$ is concentrated on a narrow ring about the origin at radius $r_m = \ell_0\sqrt{2(m+1)}$. Suppose the electron is confined to a disc in the plane of area A. Then the highest m for which $\psi_{n=0,m}$ lies within the disc is

given by $A = \pi r_{m_{max}}$, or, simply, $m_{max} + 1 = \Phi/\Phi_0$, where $\Phi = BA$ is the total flux. Hence, we see that in the thermodynamic limit, there are Φ/Φ_0 degenerate single-electron states in the lowest Landau level of a two-dimensional electron system penetrated by a uniform magnetic flux Φ. The higher Landau levels have the same degeneracy. Higher Landau levels can, at a qualitative level, be thought of as copies of the lowest Landau level. The detailed structure of states in higher Landau levels is different, however.

Let us now imagine that we have not one, but many, electrons and let us ignore the interactions between these electrons. To completely fill p Landau levels, we need $N_e = p(\Phi/\Phi_0)$ electrons. Inverting the semi-classical resistivity matrix, and substituting this electron number, we find:

$$\sigma_{xy} = \frac{e^2}{h}\,\frac{N_e}{N_\Phi} = \frac{e^2}{h}\,p \tag{20.9}$$

for p filled Landau levels, where $N_\Phi = \Phi/\Phi_0$.

Suppose that we fix the chemical potential, μ. As the magnetic field is varied, the energies of the Landau levels will shift relative to the chemical potential. However, so long as the chemical potential lies between two Landau levels (see figure 20.2), an integer number of Landau levels will be filled, and we expect to find the quantized Hall conductance, (20.9).

These simple considerations neglected two factors which are crucial to the observation of the quantum Hall effect, namely the effects of impurities and inter-electron interactions.[8] The integer quantum Hall effect occurs in the regime in which impurities dominate; in the fractional quantum Hall effect, interactions dominate. [9]

20.2 The Integer Quantum Hall Effect

Let us model the effects of impurities by a random potential in which non-interacting electrons move. Clearly, such a potential will break the degeneracy of the different states in a Landau level. More worrisome, still, is the possibility that some of the states might be localized by the random potential and therefore unable to carry any current at all. The possible effects of impurities are summarized in the hypothetical density of states depicted in Figure 20.2.

Hence, we would be led to naively expect that the Hall conductance is less than $\frac{e^2}{h}\,p$ when p Landau levels are filled. In fact, this conclusion, though intuitive, is completely wrong. In a very instructive calculation (at least from a pedagogical standpoint), Prange [**?**] analyzed the exactly solvable model

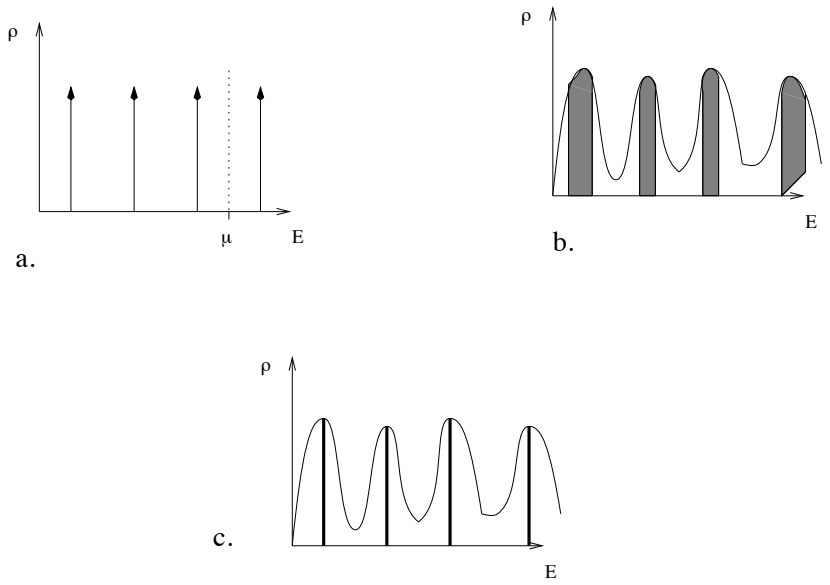


Figure 20.2: (a) The density of states in a pure system. So long as the chemical potential lies between Landau levels, a quantized conductance is observed. (b) Hypothetical density of states in a system with impurities. The Landau levels are broadened into bands and some of the states are localized. The shaded regions denote extended states. (c) As we mention later, numerical studies indicate that the extended state(s) occur only at the center of the band.

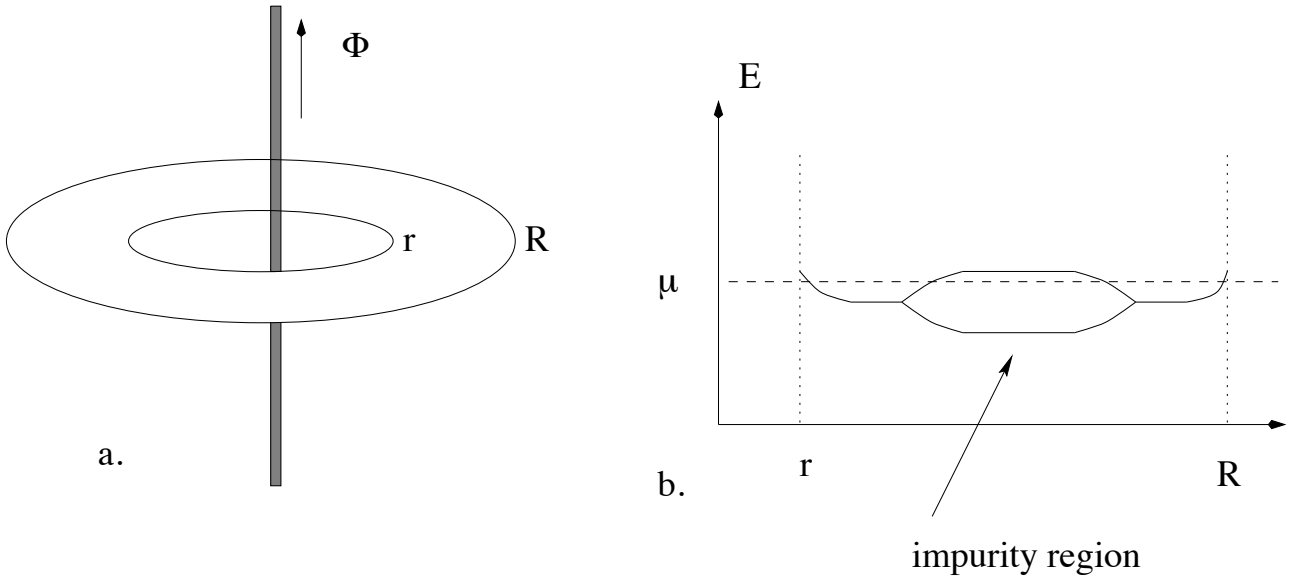


Figure 20.3: (a) The Corbino annular geometry. (b) Hypothetical distribution of energy levels as a function of radial distance.

of electrons in the lowest Landau level interacting with a single δ-function impurity. In this case, a single localized state, which carries no current, is formed. The current carried by each of the extended states is increased so as to exactly compensate for the localized state, and the conductance remains at the quantized value, $\sigma_{xy} = \frac{e^2}{h}$. This calculation gives an important hint of the robustness of the quantization, but cannot be easily generalized to the physically relevant situation in which there is a random distribution of impurities. To understand the quantization of the Hall conductance in this more general setting, we will turn to the beautiful arguments of Laughlin (and their refinement by Halperin [**?**]), which relate it to gauge invariance.

Let us consider a two-dimensional electron gas confined to an annulus such that all of the impurities are confined to a smaller annulus, as shown in Figure 20.3. Since, as an experimental fact, the quantum Hall effect is independent of the shape of the sample, we can choose any geometry that we like. This one, the Corbino geometry, is particularly convenient. States at radius r will have energies similar to to those depicted in Figure 20.3.

Outside the impurity region, there will simply be a Landau level, with energies that are pushed up at the edges of the sample by the walls (or a

smooth confining potential). In the impurity region, the Landau level will broaden into a band. Let us suppose that the chemical potential, μ, is above the lowest Landau level, $\mu > \hbar\omega_c/2$. Then the only states at the chemical potential are at the inner and outer edges of the annulus and, possibly, in the impurity region. Let us further assume that the states at the chemical potential in the impurity region – if there are any – are all localized.

Now, let us slowly thread a time-dependent flux $\Phi(t)$ through the center of the annulus. Locally, the associated vector potential is pure gauge. Hence, localized states, which do not wind around the annulus, are completely unaffected by the flux. Only extended states can be affected by the flux.

When an integer number of flux quanta thread the annulus, $\Phi(t) = p\Phi_0$, the flux can be gauged away everywhere in the annulus. As a result, the Hamiltonian in the annulus is gauge equivalent to the zero-flux Hamiltonian. Then, according to the adiabatic theorem, the system will be in some eigenstate of the $\Phi(t) = 0$ Hamiltonian. In other words, the single-electron states will be unchanged. The only possible difference will be in the occupancies of the extended states near the chemical potential. Localized states are unaffected by the flux; states far from the chemical potential will be unable to make transitions to unoccupied states because the excitation energies associated with a slowly-varying flux will be too small. Hence, the only states that will be affected are the gapless states at the inner and outer edges. Since, by construction, these states are unaffected by impurities, we know how they are affected by the flux: each flux quantum removes an electron from the inner edge and adds an electron to the outer edge. Then, $\int I\,dt = e$ and $\int V\,dt = \int \frac{d\Phi}{dt} = h/e$, so:

$$I = \frac{e^2}{h} V \tag{20.10}$$

Clearly, the key assumption is that there are no extended states at the chemical potential in the impurity region. If there were – and there probably are in samples that are too dirty to exhibit the quantum Hall effect – then the above arguments break down. Numerical studies [**?**] indicate that, so long as the strength of the impurity potential is small compared to $\hbar\omega_c$, extended states exist only at the center of the Landau band (see Figure 20.2). Hence, if the chemical potential is above the center of the band, the conditions of our discussion are satisfied.

The other crucial assumption, emphasized by Halperin [**?**], is that there are gapless states at the edges of the system. In the special setup which we assumed, this was guaranteed because there were no impurities at the

edges. In the integer quantum Hall effect, these gapless states are a one-dimensional chiral Fermi liquid. Impurities are not expected to affect this because there can be no backscattering in a totally chiral system. More general arguments, which we will mention in the context of the fractional quantum Hall effect, relate the existence of gapless edge excitations to gauge invariance.

One might, at first, be left with the uneasy feeling that these gauge invariance arguments are somehow too 'slick.' To allay these worries, consider the annulus with a wedge cut out, which is topologically equivalent to a rectangle (see the article by D.J. Thouless in the first reference in [**?**]). In such a case, some of the Hall current will be carried by the edge states at the two cuts (*i.e.* the edges which run radially at fixed azimuthal angle). However, probes which measure the Hall voltage between the two cuts will effectively couple these two edges leading, once again, to annular topology.

Laughlin's argument for exact quantization will apply to the fractional quantum Hall effect if we can show that the clean system has a gap. Then, we can argue that for an annular setup similar to the above there are no extended states at the chemical potential except at the edge. Then, if threading q flux quanta removes p electrons from the inner edge and adds p to the outer edge, as we would expect at $\nu = p/q$, we would have $\sigma_{xy} = \frac{p}{q}\frac{e^2}{h}$.

20.3 The Fractional Quantum Hall Effect: The Laughlin States

A partially filled Landau level of non-interacting electrons has a highly degenerate ground state in the absence of impurities. This degeneracy is broken by fairly generic interactions, including Coulomb repulsion. As we will see below, at special filling fractions, there is a non-zero gap between the ground state and the lowest excited state. In very clean samples, the impurity potential will be a weak perturbation which pins the quasiparticles but does not drastically affect the physics of the ground state. If the sample is too dirty, however, the fractional quantum Hall effect will be destroyed. In what follows, we will try to understand the physics of a partially filled Landau level of interacting electrons in a clean system. We will further assume that we can ignore all higher Landau levels. This assumption will be valid in the limit that the cyclotron energy is much larger than the Coulomb interaction energy, $\hbar\omega_c \gg \frac{e^2}{\ell_0}$. In a sample with density $1.25 \times 10^{-11}\text{cm}^{-2}$, the $\nu = \frac{1}{3}$ state is seen at $15T$, while $\hbar\omega_c = \frac{e^2}{\ell_0}$ at $6T$. Hence, higher Landau levels are probably unimportant qualitatively, but could lead to some

quantitative corrections.

Let us, following Haldane [**?**], consider the special interactions for which the Laughlin states are the exact ground states. To do this, let us first look at the two-electron problem in the lowest Landau level. We separate the center-of-mass and relative motions, $\psi = P(Z, z)$ (we will be sloppy and drop the Gaussian factors because they are unimportant for this analysis), where $Z = z_1 + z_2$, $z = z_1 - z_2$, and z_1 and z_2 are the coordinates of the two electrons. The Hamiltonian has no kinetic part in the lowest Landau level. Dropping the constant $\hbar\omega_c$, it is given simply by the interaction, V, which depends only on the relative motion

$$H = V\left(z, \frac{d}{dz}\right) \tag{20.11}$$

Let us now switch to a basis of relative (canonical) angular momentum eigenstates, $L_z|m\rangle = m|m\rangle$, which are given in position space by $\langle z|m\rangle = z^m$. Then, we can write:

$$H = \sum_{m\,\mathrm{odd}} V_m P_m \tag{20.12}$$

The restriction to odd m is due to Fermi statistics. $V_m = \langle m|V|m\rangle$; $\langle m|V|m'\rangle$ vanishes for $m \neq m'$ if V is rotationally invariant. P_m is the projection operator onto states of relative angular momentum m. Suppose we take $V_m > 0$ for $m < k$ and $V_m = 0$ for $m \geq k$. Then the states $\psi(z) = z^m$ are pushed up to energies $E_m = V_m$ for $m < k$ but the states $\psi(z) = z^m$, $m \geq k$ remain degenerate at $E = 0$.

The Hamiltonian for the N-electron problem is just:

$$H = \sum_{i>j} \sum_{m\,\mathrm{odd}} V_m P_m^{ij} \tag{20.13}$$

where P_m^{ij} projects the $i-j$ pair onto a state of relative angular momentum m. Let us consider the simple, but unrealistic interaction $V_1 \neq 0$, $V_m = 0$ for $m > 1$. Any wavefunction in the lowest Landau level, $\psi = P(z_1, z_2, \ldots, z_N)$ can be written:

$$\psi = \sum_{m\,\mathrm{odd}} (z_i - z_j)^m F_m(z_i + z_j; z_k, k \neq i, j) \tag{20.14}$$

If we take $F_m = 0$ for $m = 1$, then $H\psi = 0$. In this case, $(z_i - z_j)^3$ is a factor of ψ for all $i \neq j$. Hence, the following wavefunctions all have zero energy

$$\psi = \prod_{i>j} (z_i - z_j)^3 \; S(z_1, z_2, \ldots, z_N) \tag{20.15}$$

where $S(z_1, z_2, \ldots, z_N)$ is a symmetric polynomial. These states describe droplets of electrons. Of these wavefunctions, the Laughlin wavefunction [**?**],

$$\psi_3 = \prod_{i>j} (z_i - z_j)^3 \tag{20.16}$$

is the most spatially compact droplet. In a confining potential, this will be the ground state. The other symmetric polynomials correspond to quasiholes and edge excitations. Had we chosen $V_m > 0$ for $m < 2k+1$ and $V_m = 0$ for $m \geq 2k+1$, we would have found the wavefunction (20.16) with the power 3 replaced by $2k+1$:

$$\psi_{2k+1} = \prod_{i>j} (z_i - z_j)^{2k+1} \tag{20.17}$$

The maximum power of any z_i in the Laughlin state is $3(N-1)$. Since the single-electron state with canonical angular momentum m encloses area $2\pi\ell_0^2(m+1)$, the Laughlin state of N electrons occupies area $A = 2\pi\ell_0^2(3(N-1)+1) = 2\pi\hbar/(eB)(3(N-1)+1)$. The total flux piercing this area is $\Phi = BA = \Phi_0(3(N-1)+1)$. Hence, the filling fraction, ν is

$$\nu = \frac{N}{\Phi/\Phi_0} = \frac{N}{3(N-1)+1} \to \frac{1}{3} \tag{20.18}$$

in the thermodynamic limit.

To compress this state, that is, to get $\nu < 1/3$, at least one pair of particles will have relative angular momentum $m = 1$, which costs a finite amount of energy. A more precise and general way of stating this result involves calculating the compressibility, κ

$$\kappa = \frac{A}{N^2} \left(\frac{\partial N}{\partial \mu} \right)_L \tag{20.19}$$

at fixed angular momentum L (A is the area of the system). For our choice of interaction, $E_0(N) = E_0(N-1) = 0$ but $E_0(N+1) > 0$ for fixed total angular momentum $3N(N-1)$. Hence, $\mu_N^- = E_0(N) - E_0(N-1) = 0$ while $\mu_N^+ = E_0(N+1) - E_0(N) \neq 0$. The discontinuity in the chemical potential implies incompressibility according to (20.19). For more realistic potentials, it may no longer be true that $\mu_N^- = 0$, but the discontinuity will persist.

The Laughlin wavefunction (20.17) was initially proposed as a trial variational wavefunction for electrons interacting with Coulomb interactions. For small numbers of electrons, it has remarkably large overlap with the exact

ground state (see, for instance, the article by F.D.M. Haldane in the first reference in [**?**]). At filling fraction $\nu = 1/(2k+1)$, the wavefunction must be a homogeneous polynomial of degree $(2k+1)N(N-1)/2$. In other words, if we fix the coordinates $z_1, z_2, \ldots, z_{N-1}$ of $N-1$ of the electrons, then the wavefunction, considered as a function of the remaining electron, z_N, will have $(2k+1)$ zeroes for each of the $N-1$ electrons. Since the electrons are fermions, there must be at least one zero at the positions of the other electrons. A state at $\nu = 1/(2k+1)$ is specified by the positions of the other zeroes. In the Laughlin state, there is a $2k+1$-fold zero at the positions of the other electrons; all of the zeroes are at the electron locations. In the exact ground state of electrons with some other kind of interaction, say the Coulomb interaction, there are still $(2k+1)$ zeroes bound to each electron, but they are slightly displaced from the electron. The quantum Hall effect breaks down precisely when the zeroes dissociate from the electrons.

A particularly useful technique for obtaining many properties of the Laughlin states is the plasma analogy (see, for instance, the article by R.B. Laughlin in the first reference in [**?**]). Since $|\psi|^2$ is of the form of the Boltzmann weight for a classical finite-temperature plasma of charge $2k+1$ particles in a neutralizing background,

$$|\psi|^2 = e^{\frac{1}{2k+1}\left(2(2k+1)^2 \sum \ln|z_i - z_j| - (2k+1)\sum |z_i|^2/4\ell_0^2\right)} = e^{-\beta H_{\text{plasma}}} \tag{20.20}$$

the expectation value of many operators in the ground state is just given by the corresponding expectation values in the plasma. For instance, since the temperature $T = 2k+1$ is above the melting temperature for the plasma, we can conclude that the correlation functions of the density do not exhibit long-range positional order.[10] Combining this result with our earlier discussion of the compressibility, we can say that the Laughlin states describe incompressible quantum liquids.

To establish the quantum Hall effect in these states, we need to understand the excitation spectrum. In particular, we must show that there is a finite energy gap separating the ground state from excited states. If we imagine adiabatically inserting a flux tube at the origin in a Laughlin state at $\nu = 1/(2k+1)$, then, by arguments very similar to those used in the annulus geometry, we expect charge $e/(2k+1)$ to be transported from the insertion point to the outer edge of the system. The flux tube can be gauged away, leaving an eigenstate of the original Hamiltonian with a deficit of $1/(2k+1)$ of an electron at the origin [**?**].[11] Such an excitation is called a 'quasihole.' If the inserted flux were oppositely directed, an excitation with an excess charge of $-e/(2k+1)$ at the origin would be created, or a 'quasiparticle.'

Laughlin suggested the following quasihole state,

$$\psi^{\rm qh}_{2k+1} = \prod_i (z_i - \eta)\, \psi_{2k+1} \tag{20.21}$$

which is an exact zero-energy eigenstate of the Hamiltonian (20.13) and has a large overlap with the exact quasihole state of a system with a small number of electrons interacting through the Coulomb interactions. In this state, the angular momentum of each electron is increased by one and the net flux penetrating the electron droplet is increased by one flux quantum. The state:

$$\psi^{\rm qh}_{2k+1} = \prod_i (z_i - \eta)^{2k+1}\, \psi_{2k+1} \tag{20.22}$$

looks like the ground state of $N+1$ electrons, but with a deficit of one electron at the position η. Hence, the state (20.21) has charge $e/(2k+1)$ at η.

A quasiparticle wavefunction which is an exact eigenstate of the Hamiltonian (20.13) has not been found. The trial wavefunction:

$$\psi^{\rm qh}_{2k+1} = \prod_i \left(\frac{\partial}{\partial z_i} - \eta \right) \psi_{2k+1} \tag{20.23}$$

has reasonably good overlap with the exact quasihole state in systems with a small number of electrons. The quasiparticle has fractional charge $-e/(2k+1)$. As a general rule, exact quasiparticle eigenstates are more difficult to come by than quasihole states, so we will primarily discuss quasiholes. Most of the properties of quasiparticles can be inferred from those of quasiholes.

At $\nu = 1/(2k+1)$, the gap between the ground state and a state with a widely-separated quasihole-quasiparticle pair is just $(\mu^+ - \mu^-)/(2k+1)$. This follows from the definition of $\mu^\pm$ and the fact that a widely separated pair will have no interaction energy. $\Delta = (\mu^+ - \mu^-)/(2k+1)$ is the gap which is measured in transport experiments – for instance from $\rho_{xx} \sim e^{\Delta/T}$ – since a widely separated pair must be created to carry a longitudinal current. However, this is not the smallest gap in the system. A quasihole-quasiparticle pair at finite separation will have lower energy as a result of the Coulomb interaction between them. Suppose the distance between the quasihole and quasiparticle is parametrized by k so that the distance is $k\ell_0^2 \gg \ell_0$. Then, we can think of the quasihole and quasiparticle – which have core sizes on the order of a few magnetic lengths – as point charges; the energy of the pair will be $E(k) = \Delta - \frac{e^2}{k\ell_0^2}$ [**?**]. The pair will move in a straight line with velocity $v_k = \frac{\partial E(k)}{\partial k}$ perpendicular to the line connecting

them since the Coulomb force between them will exactly balance the Lorentz force due to the magnetic field. At low k, the quasihole-quasiparticle pair evolves continuously into a collective mode, the magneto-roton [**?**]. The name magneto-roton stems from the fact that this collective excitation is obtained in the single-mode approximation just as the roton was in Feynman's analysis of superfluid ^4He. As we will see later, the analogy between the quantum Hall effect and superfluidity can be further exploited.

To summarize, the Laughlin state has the following properties:

- It is a wavefunction describing electrons in a strong magnetic field. The electrons are assumed to be in the lowest Landau level.
- It is the non-degenerate ground state of a model repulsive Hamiltonian (20.13).
- It is an excellent approximation to the ground state of electrons in a magnetic field interacting through the Coulomb potential.
- The state is incompressible.
- The state does not break translational symmetry, i.e. it is a liquid.

In order to observe a fractional quantum Hall plateau with $\sigma_{xy} = \frac{1}{2k+1}\frac{e^2}{h}$, $\sigma_{xx} = 0$, we also need a small amount of impurities as well, in order to pin any quasiparticles which are produced by small changes of the magnetic field or electron density. However, we don't want too much disorder since this might simply pin the electrons and prevent them from forming a correlated state (20.16).

20.4 Fractional Charge and Statistics of Quasiparticles

Let us return to a discussion of the quantum numbers of the quasiholes and quasiparticles. We found earlier that these excitations carry fractional electric charge. This is remarkable, but has a precedent in polyacetylene; the statistics, to which we now turn, is perhaps even more exotic. If we suppose that the phase acquired by the wavefunction when one quasihole moves around another is $e^{i\phi}$, then the phase for taking one electron around another is $e^{i(2k+1)^2\phi}$, and the phase associated with taking an electron around a quasihole is $e^{i(2k+1)\phi}$, since m quasiholes is equal to a deficit of one electron. From the wavefunction (20.21), we see that $e^{i(2k+1)\phi} = e^{2\pi i}$ and

$e^{i(2k+1)^2\phi} = e^{2\pi(2k+1)i}$. This would lead us to conclude that $e^{i\phi} = e^{2\pi i/(2k+1)}$. Similar arguments would lead us to conclude that quasiparticles have the same statistics.[12]

These heuristic arguments for the charge and statistics of the quasiholes and quasiparticles are inadequate even though they give the correct answers. Fortunately, these quantum numbers can be determined directly. Following Arovas, Schrieffer, and Wilczek [**?**], we will calculate the Berry's phase [**?**] acquired when quasiholes are moved around loops. Recall that the adiabatic theorem deals with a family of Hamiltonians, $\mathbf{H}(\{\lambda_i\})$, parameterized by $\{\lambda_1, \lambda_2, \ldots, \lambda_k\}$, with non-degenerate eigenstates $|n(\{\lambda_i\})\rangle$:

$$\mathbf{H}(\{\lambda_i\})\,|n(\{\lambda_i\})\rangle = E_n(\{\lambda_i\})|n(\{\lambda_i\})\rangle \tag{20.24}$$

Suppose we vary the λ_i's slowly with time [13], $\lambda_i = \lambda_i(t)$, such that $\mathbf{H}(\lambda_i(0)) = \mathbf{H}(\lambda_i(T))$; then $|n(\{\lambda_i(0)\})\rangle = M\,|n(\{\lambda_i(T)\})\rangle$, where M is a phase. Often, we require $M = 1$, but this is unnecessary. A state $|\psi(t)\rangle$ satisfying $|\psi(0)\rangle = |n(\{\lambda_i(0)\})\rangle$ will evolve subject to Schrödinger's equation,

$$\mathbf{H}(\{\lambda_i(t)\})|\psi(t)\rangle = \frac{d}{dt}\,|\psi(t)\rangle \tag{20.25}$$

so that

$$|\psi(T)\rangle = M\,e^{i\gamma_n} e^{\frac{i}{\hbar}\int_0^T E(t)\,dt}|\psi(0)\rangle \tag{20.26}$$

Berry's phase, γ_n, is given, according to Schrödinger's equation (20.25), by

$$\gamma_n = i\int \langle n(\{\lambda_i(t)\})|\frac{d}{dt}|n(\{\lambda_i(t)\})\rangle \tag{20.27}$$

The integral (20.27) is reparameterization invariant, so Berry's phase depends only on the path in parameter space; in particular, γ_n remains finite in the adiabatic limit, unlike the dynamical phase, $\int_0^T E(t)\,dt$. One other point worth mentioning is that Berry's phase (20.27) only depends on the Hamiltonian implicitly. In what follows, we will be interested in the Berry's phase acquired by quasihole wavefunctions as the quasiholes are moved around. We will implicitly assume that there is some Hamiltonian with a pinning potential, say, for which the state with a quasihole at η is a non-degenerate eigenstate. As the location of the pinning potential is moved, this eigenstate evolves, and a Berry's phase will accumulate, but we need not be concerned with the details of the Hamiltonian to do this calculation.

We consider, then, the Laughlin quasihole

$$|\psi(t)\rangle = \prod_i (\eta(t) - z_i)\,\psi_{2k+1} \tag{20.28}$$

and take $\eta(t)$ to move slowly around some loop as a function of t. Since

$$\frac{d}{dt}\left|\psi(t)\right\rangle = \frac{d\eta}{dt}\sum_i \frac{1}{\eta(t)-z_i}\left|\psi(t)\right\rangle \tag{20.29}$$

we can rewrite

$$\begin{aligned}\left\langle\psi(t)\right|\frac{d}{dt}\left|\psi(t)\right\rangle &= \int d^2z\,\frac{1}{\eta(t)-z}\left\langle\psi(t)\right|\sum_i\delta(z-z_i)\left|\psi(t)\right\rangle \\ &= \int \frac{d^2z}{\eta(t)-z}\left\langle\psi(t)\right|\rho(z)\left|\psi(t)\right\rangle \end{aligned}\tag{20.30}$$

where $\rho(z)$ is the density. Then, the Berry's phase acquired in a circuit, C, bounding a region R of area A_R is:

$$\gamma_n = i\oint d\eta\int d^2z\,\frac{\langle\rho(z)\rangle}{\eta-z} \tag{20.31}$$

$\rho(z)=\rho_0$ except in the core of the quasihole. Since $\rho_0 A_R = N = \frac{1}{2k+1}\,\Phi_R/\Phi_0$, where Φ_R is the flux in the region R,

$$\begin{aligned}\gamma_n &= i\int d^2z\oint d\eta\,\frac{\langle\rho(z)\rangle}{\eta-z} \\ &= -2\pi\int d^2z\,\langle\rho(z)\rangle \\ &= -\frac{2\pi}{2k+1}\,(\Phi_R/\Phi_0)\end{aligned}\tag{20.32}$$

up to corrections of order r_c^2/A_R, where r_c is the size of the quasihole. This is just the phase that we would expect for a particle of charge $\nu e` = e/(2k+1)$ in a magnetic field.

Suppose, now, that we had considered a multi-quasihole wavefunction. If the loop C had enclosed another quasihole, $\rho(z)$ would no longer be given by $\rho_0 = \frac{1}{A_R}\frac{1}{2k+1}\Phi/\Phi_0$. There would be a charge deficit at the position of the second quasihole. Then, we would find:

$$\begin{aligned}\gamma_n &= i\int d^2z\oint d\eta\,\frac{\langle\rho(z)\rangle}{\eta-z} \\ &= -2\pi\int d^2z\,\langle\rho(z)\rangle \\ &= -\frac{2\pi}{2k+1}\,(\Phi_R/\Phi_0) + \frac{2\pi}{2k+1}\end{aligned}\tag{20.33}$$

Hence, there is an additional phase $2\pi/(2k+1)$ acquired when one quasihole winds around another. In other words, quasiholes in the Laughlin state at

$\nu = 1/(2k+1)$ have fractional statistics given by the statistics parameter $\alpha = 1/(2k+1)$, where bosons have $\alpha = 0$ and fermions, $\alpha = 1$. The fractional charge and statistics of the quasiholes are the characteristic features of fractional quantum Hall states.

In chapter 4, we will be interested in non-Abelian statistics, which can occur when there is a set of degenerate states, $|a; \{\lambda_i\}\rangle$, $a = 1, 2, \ldots, g$. In such a case, a state $|\psi(t)\rangle$ satisfying $|\psi(0)\rangle = |a; \{\lambda_i(0)\}\rangle$ evolves into:

$$|\psi(T)\rangle = e^{i\gamma_{ab}} e^{\frac{i}{\hbar}\int_0^T E(t)\,dt} M\,|b; \{\lambda_i(0)\}\rangle \tag{20.34}$$

The degenerate subspaces must be equivalent at $t = 0$ and $t = T$ since the Hamiltonians coincide, but the states $|a; \{\lambda_i(t)\}\rangle$ at $t = 0$ and $t = T$ can differ by an overall rotation; M is the matrix which implements this rotation. The Berry phase matrix, γ_{ab}, is given by:

$$\gamma_{ab} = i \int \langle a; \{\lambda_i(t)\}| \frac{d}{dt} |b; \{\lambda_i(t)\}\rangle \tag{20.35}$$

20.5 Fractional Quantum Hall States on the Torus

As we discussed in the last chapter the existence of anyonic quasiparticles in a system is reflected in its ground state degeneracy on higher-genus surfaces. By the arguments given there, we expect the Laughlin state for $\nu = 1/m$ to be m-fold degenerate on a torus. In this section, we will construct the m wavefunctions on a torus which are annihilated by the Hamiltonian with $V_1, \ldots, V_{m-1} \neq 0$, $V_m = V_{m+1} = \ldots = 0$.

In order to do so, we will make use of the Cauchy ϑ-functions, which are functions defined on the torus. Let us assume that z is a complex coordinate on the torus and that the torus is defined by $z \equiv z + 1$, $z \equiv z + \tau$, where τ is some complex number which is called the modular parameter of the torus. Then the ϑ-functions are defined by:

$$\vartheta_1(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i \left(n+\frac{1}{2}\right)^2 \tau} e^{2\pi i \left(n+\frac{1}{2}\right)\left(z+\frac{1}{2}\right)} \tag{20.36}$$

and

$$\vartheta_2(z|\tau) = \vartheta_1\left(z + \frac{1}{2}|\tau\right)$$
$$\vartheta_3(z|\tau) = e^{i\pi\tau/4}\, e^{i\pi z}\, \vartheta_1\left(z + \frac{1}{2}(1+\tau)|\tau\right)$$

$$\vartheta_4(z|\tau) = \vartheta_1\left(z + \frac{1}{2}\tau|\tau\right) \tag{20.37}$$

The following properties of ϑ_1 will be useful:

$$\begin{aligned}\vartheta_1(z+1|\tau) &= -\vartheta_1(z|\tau)\\ \vartheta_1(z+\tau|\tau) &= -e^{-i\pi\tau}e^{-2\pi iz}\vartheta_1(z|\tau)\\ \vartheta_1(-z|\tau) &= -\vartheta_1(z|\tau)\end{aligned} \tag{20.38}$$

Armed with these functions, we can generalize the Laughlin wavefunction to:

$$\psi = \prod_{i>j}\left[\vartheta_1(z_i - z_j|\tau)\right]^m \;\; e^{iKZ}\prod_{a=1}^{m}\vartheta_1(Z - Z_a|\tau) \tag{20.39}$$

At short distances, $\vartheta_1(z_i - z_j|\tau) \to z_i - z_j$, so this wavefunction is annihilated by the Hamiltonian which annihilates the Laughlin wavefunction on the plane. The only remaining requireent is that it be periodic under $z \to z+1$, $z \to z+\tau$. These will be satisfied if

$$\begin{aligned}(-1)^{mN}e^{iK} &= (-1)^{N_\phi}e^{iK} = 1\\ \left(-e^{i\pi\tau}\right)^{mN}e^{iK\tau}e^{2\pi\sum_a Z_a} &= (-1)^{N_\phi}e^{iK} = 1\end{aligned} \tag{20.40}$$

There are m different choices of K, Z_a. To see this, observe that the ratio between any two wavefunctions associated with two choices of K, Z_a is a meromorphic function of Z on the torus with m simple poles. By a special case of the Riemann-Roch theorem, there are m linearly independent such functions. (Haldane, 1984)

20.6 The Hierarchy of Fractional Quantum Hall States

Thus far, we have only explained the existence of the quantized Hall plateaus at $\nu = 1/(2k+1)$. From Figure (20.1), however, we can see that there are plateaus at several other odd-denominator fractions. These other states can be thought of as descending from the Laughlin states [**?**, **?**]. Following Halperin, let us consider a 'primary' state at $\nu = 1/(2k+1)$ with a finite density of quasiholes or quasiparticles. Since they are charged particles in a magnetic field, we might expect that the quasiholes or quasiparticles themselves would be in a primary state (e.g. a Laughlin state) at certain preferred

quasihole densities. At what densities would we expect this? Electrons form Laughlin states only at $\nu = 1/(2k+1)$ because these are the only filling fractions at which $\prod (z_i - z_i)^{1/\nu}$ is an acceptable fermionic wavefunction. A Laughlin state of bosonic particles would form at $\nu = 1/(2k)$. Following this reasoning, a Laughlin state of quasiparticles of statistics $-1/(2k+1)$ would be of the form $\prod (z_i - z_i)^{2p-\frac{1}{2k+1}}$, while a quasihole state would be of the form $\prod (\bar{z}_i - \bar{z}_i)^{2p+\frac{1}{2k+1}}$ since quasiholes have the opposite charge. Hence, the preferred filling fractions for quasiparticles and quasiholes are $1/(2p-\frac{1}{2k+1})$ and $1/(2p+\frac{1}{2k+1})$, respectively. However, we should remember that these particles are fractionally charged as well, so their Landau levels will have $(\Phi/(2k+1))/\Phi_0$ states rather than Φ/Φ_0. Hence, a 'descendent' of the $\nu = 1/(2k+1)$ primary state which has quasiholes or quasiparticles in, respectively, filling fraction $1/(2p \pm \frac{1}{2k+1})$ states has *electron* filling fraction:

$$\begin{aligned} \nu &= \frac{1}{2k+1} \mp \frac{1}{2k+1} \left(\frac{\frac{1}{2k+1}}{2p \pm \frac{1}{2k+1}} \right) \\ &= \frac{1}{(2k+1) \pm \frac{1}{2p}} \end{aligned} \tag{20.41}$$

If we now imagine the quasiholes or quasiparticles of this state forming a Laughlin state, and so on, we will get the continued fractions:

$$\nu = \cfrac{1}{2k+1 \pm \cfrac{1}{2p_1 \pm \cfrac{1}{2p_2 \pm \ddots}}} \tag{20.42}$$

Every odd-denominator fraction less than 1 can be obtained in this way. Of course, fractional quantum Hall states are not observed at all of these fractions. As we descend through this hierarchy of states, the energy gaps become smaller and hence more easily destroyed by impurities. Furthermore, even in a pure system, the quasiholes or quasiparticles could form Wigner crystal states at some filling fractions rather than quantum Hall states.

20.7 Flux Exchange and 'Composite Fermions'

Another perspective on the hierarchy of fractional quantum Hall states involves mapping a fractional quantum Hall state to an integer quantum Hall state. This can be accomplished by introducing an auxiliary Chern-Simons

gauge field which attaches an even number of flux tubes to each electron. The attachment of an even number of flux tubes has no physical effect since it will change the phase acquired under braiding or exchange by a multiple of 2π. However, approximations that would have seemed unnatural without the auxiliary gauge field appear quite sensible in its presence.

Let us consider the more general problem of anyons with statistics parameter θ in a magnetic field:

$$H = \frac{1}{2m}(\mathbf{p} - e\,(\mathbf{a} + \mathbf{A}))^2 + H_{\text{int}} \tag{20.43}$$

where

$$e\,\nabla \times \mathbf{a} = 2(\theta - \pi)\sum_i \delta(\mathbf{r} - \mathbf{r}_i) \tag{20.44}$$

Here, we have represented the anyons as fermions interacting with a Chern-Simons gauge field. If we now replace this field by its spatial average, $e\,\nabla \times \langle \mathbf{a} \rangle = 2(\theta - \pi)\rho$, then this mean field theory is just the problem of fermions in an effective magnetic field

$$eB_{\text{eff}} = e\,\nabla \times (\mathbf{a} + \mathbf{A}) = eB + 2(\theta - \pi)\rho \tag{20.45}$$

If there is a state of fermions in B_{eff} with a gap, then the fluctuations about mean-field theory can probably be ignored.

Suppose our anyons are actually fermions. Then, we can take $\theta = \pi$ and $eB_{\text{eff}} = eB$. However, we could, instead, take $\theta = (\pm 2k + 1)\pi$, since this will give fermionic statistics as well. In such a case, $eB_{\text{eff}} = eB \pm 2\pi(2k)\rho$, or $1/\nu_{\text{eff}} = 1/\nu \pm 2k$. Let us choose B_{eff} so that an integral number of Landau levels, n, are filled; this state will have a gap. Since $\nu_{\text{eff}} = n$,

$$\nu = \frac{n}{2kn \pm 1} \tag{20.46}$$

For $n = 1$, this is just the Laughlin sequence. By exchanging real magnetic flux for the fictitious statistical flux of an auxiliary Chern-Simons gauge field, we have related the Laughlin states to a single filled Landau level.

If we fix $k = 1$ and consider $\nu_{\text{eff}} = 1, 2, 3, \ldots, n$, we have $\nu = \frac{1}{3}, \frac{2}{5}, \frac{3}{7}, \ldots, \frac{n}{2n+1}$. These are the filling fractions of the hierarchical sequence descending from $\nu = \frac{1}{3}$,

$$\nu = \frac{1}{3}, \quad \frac{2}{5} = \frac{1}{3 - \dfrac{1}{2}}, \quad \frac{3}{7} = \frac{1}{3 - \dfrac{1}{2 - \dfrac{1}{2}}}, \ldots \tag{20.47}$$

Successive levels of the hierarchy are thereby related to states with additional filled Landau levels. In somewhat misleading, but ubiquitous, jargon, the fractional quantum Hall states of electrons are integer quantum Hall states of 'composite fermions' [**?**]. The term 'composite fermion' refers to a composite object formed by an electron and an even number of flux quanta. This object fills an integer number of Landau levels of the remaining, uncanceled magnetic field.

At this point, we have only shown that there are quantum Hall states obtained by the 'composite fermion' construction at the same filling fractions at which there are hierarchical states. It is not clear that the two different constructions yield states in the same universality class. That they do can be shown by demonstrating that both constructions lead to states with quasiparticles of the same charge and statistics and, hence, the same ground state degeneracy on a torus. We will show this in the next chapter using the field-theoretic descriptions of these states.

Here we have considered only the simplest 'composite fermion' states. More complicated states can be constructed by introducing Chern-Simons gauge fields which only interact with electrons in particular Landau levels. Similar constructions are also available for spin-unpolarized and multi-layer systems.

Jain [**?**] used the 'composite fermion' construction to motivate the following trial states for the filling fractions $\nu = \frac{n}{2kn+1}$:

$$\Psi_{\frac{n}{2kn+1}}(z_k) \;=\; \mathcal{P}_{LLL}\left(\prod_{i>j}(z_i - z_j)^{2k}\,\Psi_n(z_k)\right) \tag{20.48}$$

$\mathcal{P}_{LLL}$ indicates projection into the lowest Landau level. The wavefunction $\Psi_n(z_k)$ is the wavefunction of n filled Landau levels, so it has vanishing projection into the lowest Landau level, and will contain powers of $\bar{z}_i$. However, the factor $\prod_{i>j}(z_i - z_j)^{2k}$ will multiply this by many more powers of z_i. It may be shown that the resulting expression has large projection into the lowest Landau level. At an operational level, the lowest Landau level projection is accomplished by moving all of the factors of $\bar{z}_i$ to the left and making the replacement $\bar{z}_i \to \frac{\partial}{\partial z_i}$. These wavefunctions have large overlaps with the exact ground states of systems with small numbers of particles.

As we have seen, the mapping of an electron system at one filling fraction to a (presumably, weakly interacting) fermion system at a different filling fraction has shed considerable light on the fractional quantum Hall effect. This mapping has even proven to be useful starting point for a quantitative

analysis. This mapping is a special case of the flux exchange process [**?**]: if we change the braiding statistics of the particles in a system and, at the same time, change the magnetic field, in such a way that

$$2\rho\Delta\theta = e\Delta B \tag{20.49}$$

or, equivalently,

$$\Delta\left(\frac{\theta}{\pi}\right) = \Delta\left(\frac{1}{\nu}\right) \tag{20.50}$$

then the properties of the system will not change, *at the mean field level*. If we assume that the fluctuations about mean-field theory are small, then the phase diagram of Figure 20.4 holds, with properties qualitatively unchanged along the diagonals [**?**].

In this way, we can map electron systems to other fermion systems, to Bose systems, or even to systems whose basic constituents are anyons. In the next chapter, we will see that the mapping from a fractional quantum Hall state to a Bose superfluid is the starting point for effective field theories of the quantum Hall effect.

20.8 Edge Excitations

In our discussion of the integer quantum Hall effect, we saw that there were necessarily gapless excitations at the edge of the system. The same is true in the fractional quantum Hall effect. To see this, let us consider again our simple Hamiltonian which annihilates the Laughlin state. All of the states

$$\psi = S(z_1, z_2, \ldots, z_N)\prod_{i>j}(z_i - z_j)^m \tag{20.51}$$

are also annihilated by H. In a more realistic model, there will be a confining potential $V(r)$ which favors states of lower total angular momentum. The Laughlin state itself, with $S = 1$ is then the ground state, and the other states are *edge excitations*. They are spanned by:

$$\psi = \prod_n (s_n)^{p_n} \prod_{i>j}(z_i - z_j)^m \tag{20.52}$$

where

$$s_n = \sum_i z_i^n \tag{20.53}$$

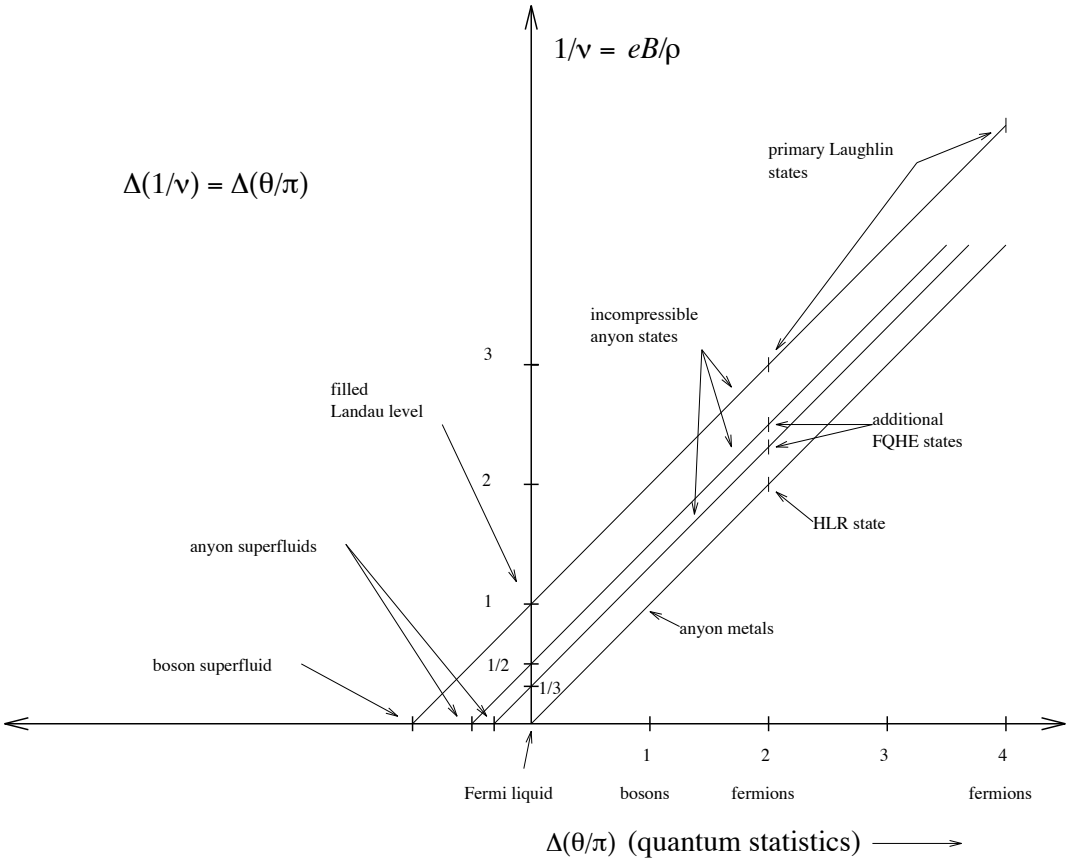


Figure 20.4: Systems at different points along the diagonals $\Delta\left(\frac{\theta}{\pi}\right) = \Delta\left(\frac{1}{\nu}\right)$ in the magnetic field-statistics plane have the same properties at the mean-field level.

Suppose that $H_{\rm disk} = H + V_{\rm conf}$. Then, to lowest order in the angular momentum, M, relative to the ground state:

$$\begin{aligned} H_{\rm disk}\,|p_1, p_2, \ldots\rangle &= f(M)\,|p_1, p_2, \ldots\rangle \\ &\approx (\text{const.})\,M\,|p_1, p_2, \ldots\rangle \\ &= \lambda \sum_n p_n n\,|p_1, p_2, \ldots\rangle \\ &= v\left(\sum_n p_n \frac{2\pi n}{L}\right)|p_1, p_2, \ldots\rangle \end{aligned}$$

where $v = 2\pi\lambda/L$. This is the spectrum of a free bosonic field, but a *chiral* one, since only $p_i > 0$ are allowed.

These bosonic excitations are simply the edge waves of an incompressible liquid. They will exist in any incompressible chiral fluid. To see how the edge excitations of a given quantum Hall state depend on the particular state, consider a quantum Hall state on an annulus, rather than a disk, so that there are inner and outer edges. The Laughlin state on an annulus can be described by:

$$\psi = \prod_i {z_i}^{tm} \prod_{i>j} (z_i - z_j)^m \tag{20.54}$$

where t is a large integer so that the inner radius of the annulus is $\ell_0\sqrt{2(tm+1)}$ while the outer radius is $\ell_0\sqrt{2m(N+t)}$. Essentially, we have carved out the inner hole of the annulus by removing t electrons from the center of the disk. If we take a quasiparticle in the bulk and move it along a trajectory encircling the origin, it will not aquire a phase $e^{2\pi i t} = 1$.

We can now create edge excitations generated by

$$s_n = \sum_i z_i^n \tag{20.55}$$

for *both positive and negative* n, so long as $n < t$. We will take $t \sim N$ so that both the inner and outer radii of the annulus are macroscopic; then for reasonable values of n, we will have excitations of both chiralities. Hence, the combined theory of both edges is a non-chiral boson.

This theory has free bosonic excitations which are divided into m sectors, corresponding to states which are built by acting with the s_n's on

$$\psi = \prod_i {z_i}^{tm+r} \prod_{i>j} (z_i - z_j)^m \tag{20.56}$$

where $r = 0, 1, \ldots, m-1$. Note that $r \equiv r + m$ by shifting t by one. These different sectors correspond to transferring a quasiparticle from the inner edge to the outer edge; sectors which differ by the transference of an ineteger number of electrons from on edge to the other are equivalent. The different sectors may be distinguished by the phases which are aquired when quasiparticles encircle the origin. $r = 0$ corresponds to periodic boundary conditions for quasiparticles. $r \neq 0$ corresponds to 'twisted' boundary conditions for quasiparticles; they aquire a phase $e^{2\pi ir/m}$ upon encircling the origin. The sectors of the edge theory correspond to the m-fold degenerate ground states of the theory on a torus, as may be seen by gluing the inner and outer edges of the annulus to form a torus..

CHAPTER 21

Effective Field Theories of the Quantum Hall Effect

21.1 Chern-Simons Theories of the Quantum Hall Effect

The preceding discussion has been heavily dependent on Laughlin's wavefunctions. However, these wavefunctions are not the exact ground states of any real experimental system. Their usefulness lies in the fact that they are representatives of a universality class of states, all of which exhibit the fractional quantum Hall effect. What has been missing to this point is a precise sense of which properties of these wavefunctions define the universality class, and which ones are irrelevant perturbations. We alluded earlier to the binding of zeroes to electrons. We will formalize this notion and use it to find low-energy, long-wavelength effective field theories for the fractional quantum Hall effect. One formulation of these effective field theories is in the form of a Landau-Ginzburg theory which is strongly reminiscent of superfluidity or superconductivity. One important difference, however, is that the order parameter is not a local function of the electron variables. This is not a trivial distinction, and it is, ultimately, related to the conclusion that a novel type of ordering is present in the quantum Hall states, namely 'topological ordering.'

Recall that in a superfluid or a superconductor the off-diagonal entries

of the density matrix:

$$\rho(\mathbf{r},\mathbf{r}') = \langle \psi^\dagger(\mathbf{r})\psi(\mathbf{r}')\rangle = \int d\mathbf{r}_2 \ldots d\mathbf{r}_N\, \psi^*(\mathbf{r},\mathbf{r}_2,\ldots,\mathbf{r}_N)\,\psi(\mathbf{r}',\mathbf{r}_2,\ldots,\mathbf{r}_N) \tag{21.1}$$

exhibit off-diagonal long-range order,

$$\rho(\mathbf{r},\mathbf{r}') \to \phi^*(\mathbf{r})\phi(\mathbf{r}') \tag{21.2}$$

for some non-zero function $\phi(\mathbf{r})$. Feynman argued that the ground state wavefunction of a Bose fluid would have no zeroes, so it can be chosen everywhere real and positive. In the absence of phase variations, (21.2) will hold. As a result of (21.2), we can choose states of indefinite particle number such that $\langle\psi(\mathbf{r})\rangle = \phi(\mathbf{r})$. $\phi(\mathbf{r})$ can be treated as a classical field and used to analyze interference phenomena such as the Josephson effect. More importantly, off-diagonal long-range order is the hallmark of superfluidity.

What happens if we calculate (21.2) in a Laughlin state state at $\nu = 1/(2k+1)$?

$$\begin{aligned}\rho(\mathbf{r},\mathbf{r}') &= \langle\psi^\dagger(\mathbf{r})\psi(\mathbf{r}')\rangle \\ &= \int d^2z_2\ldots d^2z_2 \prod_i (z-z_i)^{2k+1}(\bar{z}'-\bar{z}_i)^{2k+1}\prod_{k>l}|z_k-z_l|^{2(2k+1)}\, e^{-\sum|z_i|^2/2\ell^2}\end{aligned} \tag{21.3}$$

This correlation function does not show any signs of long-range order. The fluctuating phases of the first two terms in the integral lead to exponential falloff. On the other hand, if we consider correlation functions of:

$$\phi^\dagger(z) = e^{-i(2k+1)\int d^2z' \mathrm{Im}\ln(z-z')}\psi^\dagger(z) \tag{21.4}$$

this phase is removed and we find algebraic falloff of correlation functions, or quasi-long-range order,

$$\langle\phi^\dagger(z')\phi(z)\rangle \sim \frac{1}{|z-z'|^{(2k+1)/2}} \tag{21.5}$$

as may be shown using the plasma analogy [**?**]. The drawback of this order parameter is that it is not an analytic function of the z's, and, hence, is not a lowest Landau level operator. We could, instead, take:

$$\phi^\dagger(z) = e^{-im\int d^2z' \ln(z-z')}\psi^\dagger(z) \tag{21.6}$$

which not only has true long-range order, but also remains strictly within the lowest Landau level [**?**]. However, the field theory of this operator is more complicated, so we will use (21.4).

A Landau-Ginzburg theory may be derived for the order parameter (21.4) [**?**] in the following way. Begin with the Lagrangian for interacting electrons in a magnetic field such that $\nu = 1/(2k+1)$:

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \psi^* \left(i\partial_0 - A_0\right)\psi + \frac{\hbar^2}{2m^*}\,\psi^*(i\nabla - \mathbf{A})^2\psi - \mu\psi^\dagger\psi \\ + V(x-x')\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x')\end{aligned} \tag{21.7}$$

We now rewrite this in terms of a bosonic field, $\phi(x)$, interacting with a gauge field. The gauge field is given a Chern-Simons action of the type discussed two chapters ago, so that its only role is to transform the bosons ϕ into fermionic electrons.

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \phi^* \left(i\partial_0 - (a_0 + A_0)\right)\phi + \frac{\hbar^2}{2m^*}\,\phi^*(i\nabla - (\mathbf{a}+\mathbf{A}))^2\phi \\ -\mu\phi^\dagger\phi + V(x-x')\phi^\dagger(x)\phi(x)\phi^\dagger(x')\phi(x') \\ + \frac{1}{2k+1}\,\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\end{aligned} \tag{21.8}$$

Note that the coefficient of the Chern-Simons term is $2k+1$. We could have chosen any odd integer in order to obtain the correct statistics; the coefficient $2k+1$ is chosen for reasons which will become clear momentarily. To see that the correct statistics are obtained, note that the a_0 equation of motion is:

$$\nabla \times \mathbf{a}(\mathbf{r}) = 2\pi(2k+1)\,\rho(\mathbf{r}) \tag{21.9}$$

The Chern-Simons gauge field equation attaches $2k+1$ flux tubes to each ϕ boson. As one boson is exchanged with another, it acquires an Aharonov-Bohm phase of $(-1)^{2k+1} = -1$ as a result of these flux tubes.

As in the Landau-Ginzburg theory of a superconductor, long-range order in the bosonic field ϕ – *i.e.* $|\phi|^2 = \rho$ – breaks a $U(1)$ symmetry. The Meissner effect results, *i.e.* $\mathbf{a} + \mathbf{A} = 0$, since a non-zero constant effective magnetic field $\nabla \times (\mathbf{a} + \mathbf{A})$ would lead to badly divergent energy in (21.8) if $|\phi|^2 = \rho$. This implies that $B = -\nabla \times \mathbf{a} = 2\pi(2k+1)\rho$, or $\nu = 1/(2k+1)$.

Furthermore, excitations about this mean-field solution are gapped, as in the case of a superconductor, as a result of the Anderson-Higgs effect. This is, of course, what we expect for a quantum Hall state: there is a gap to all excitations.

If B is increased or decreased from this value, vortices are created, as in a type II superconductor:

$$\phi(r,\theta) = |\phi(r)|\, e^{i\theta}$$

$$b + B = f(r)$$

with $|\phi(0)| = 0$, $|\phi(\infty)| = \sqrt{\rho_0}$, $f(\infty) = 0$. These vortices are the Laughlin quasiholes and quasiparticles. They have one flux quantum of the **a** gauge field and very little real magnetic flux. As a result of the flux quantum of **a** which they carry, they have charge $1/(2k+1)$, according to (21.9).

Essentially, the electrons have become bound to $2k+1$ flux tubes – or $2k+1$ zeroes as we put it earlier – thereby transmuting them into bosons in zero field. These bosons undergo Bose condensation; the fractional quantum Hall liquids are these Bose condensed states. Said slightly differently, the Chern-Simons gauge field, which satisfies $\nabla \times \mathbf{a}(\mathbf{r}) = (2k+1)\rho(\mathbf{r})$, has been replaced by its spatial average, $\nabla \times \langle \mathbf{a}(\mathbf{r}) \rangle = (2k+1)\langle \rho \rangle$. The average field cancels the magnetic field so that the bosons can condense. The fluctuations of a around its average value could, in principle, destabilize the Bose condensed state, but they do not because there is an energy gap.

At finite temperature, there will always be some thermally excited quasiparticles, with a density $\sim e^{-\Delta/T}$, where Δ is the (finite) energy cost of a quasiparticle. The presence of these quasiparticles means that the quantum Hall effect is destroyed at any finite temperature: $\rho_{xx} \sim e^{-\Delta/T}$. Howver, the deviation from the zero-temperature behavior is small for small T.

Note that the Chern-Simons term results in an important difference between the Landau-Ginzburg theory of the quantum Hall effect and the Landau-Ginzburg theory of a superconductor. The Chern-Simons term attaches flux to charges, so both particles (electrons) and vortices (quasiparticles) carry charge *and* flux. As a result, they are very much on the same footing; this can be made even more explicit, as we will see later. In a superconductor, on the other hand, particles (Cooper pairs) carry charge while vortices carry flux; they are thereby differentiated.

21.2 Duality in $2+1$ Dimensions

The Landau-Ginzburg theory which we have just discussed has a dual formulation which will prove useful in much of the following discussion. We will first consider duality more generally for a $U(1)$ theory in $2+1$ dimensions and then consider the particular case of the quantum Hall effect.

Consider a Landau-Ginzburg theory for a $U(1)$ symmetry:

$$\mathcal{L} = \psi^* i\partial_0 \psi \; + \; \frac{\hbar^2}{2m^*} \psi^* \nabla^2 \psi \; + \; V\left(|\psi|^2\right) \tag{21.10}$$

In chapter 11, we showed that such a theory could, in its broken symmetry phase, be simplified by writing $\psi = \sqrt{\rho_s} e^{i\theta}$ and integrating out the gapped fluctuations of ρ_s:

$$\mathcal{L} = \frac{\rho_s}{2m} (\partial_\mu \theta)^2 \tag{21.11}$$

This Lagrangian has a conserved current, $\partial_\mu j_\mu = 0$ given by

$$j_\mu = \rho_s \, \partial_\mu \theta \tag{21.12}$$

We have assumed that there are no fluctuations in the amplitude. However, we can allow one type of amplitude fluctuations, namely vortices, if we allow θ to have singularities. Then the vortex current takes the form:

$$j^v_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu \partial_\lambda \theta \tag{21.13}$$

The conservation law (21.12) can be automatically satisfied if we take

$$j_\mu = \epsilon_{\mu\nu\lambda} \partial_\nu a_\lambda \tag{21.14}$$

Note that a_λ is not uniquely defined, but is subject to the gauge transformation $a_\lambda \to a_\lambda + \partial_\mu \chi$. Equation (21.14) can be used to solve for θ and substituted into equation (21.13):

$$\partial_\nu f_{\mu\nu} = \rho_s \, j^v_\mu \tag{21.15}$$

where $f_{\mu\nu}$ is the field strength associated with the gauge field a_λ:

$$f_{\mu\nu} = \partial_\nu a_\mu - \partial_\mu a_\nu \tag{21.16}$$

If we introduce a vortex annihilation operator, Φ_v, then (21.15) is the equation of motion of the *dual Lagrangian*:

$$\mathcal{L}_{\text{Dual}} = \frac{1}{2} \kappa \, |(\partial_\mu - i a_\mu) \Phi_v|^2 \; + \; V_\Phi \left(|\Phi_v| \right) \; + \; \frac{1}{2\rho_s} f_{\mu\nu} f_{\mu\nu} \tag{21.17}$$

where κ is a vortex stiffness and $V_\Phi \left(|\Phi_v| \right)$ is the vortex-vortex interaction. The vortex density and current are given by

$$j^v_\mu = \text{Im} \left\{ \kappa \Phi^*_v (\partial_\mu - i a_\mu) \Phi_v \right\} \tag{21.18}$$

The final term in the dual Lagrangian is a Maxwell term for the gauge field,

$$\begin{aligned} f_{\mu\nu} f_{\mu\nu} &= e^2 - b^2 \\ e_i &= \partial_0 a_i - \partial_i a_0 \end{aligned}$$

$$b = \epsilon_{ij}\partial_i a_j \tag{21.19}$$

which is of the same form as the action for the electromagnetic gauge field, $F_{\mu\nu}F_{\mu\nu} = E^2 - B^2$.

Notice that the conservation law (21.12) which followed from the equations of motion in the original representation is a trivial, topological identity in the dual representation, following from (21.14). The definition (21.13) of the vortex current in the original representation is the equation of motion (21.15) in the dual representation.

The broken symmetry phase of our original theory (21.11) is the phase in which $\langle \Phi_v \rangle = 0$. Vortices are gapped; the low-energy effective action in the dual language is simply

$$\mathcal{L}_{\rm Dual} = \frac{1}{2\rho_s} f_{\mu\nu} f_{\mu\nu} \tag{21.20}$$

The gauge field a_μ is the dual formulation of the Goldstone boson. However, when the symmetry is restored by the proliferation and condensation of vortices, $\langle \Phi_v \rangle = \Phi_0 \neq 0$, the dual action is in its Higgs phase:

$$\mathcal{L}_{\rm Dual} = \frac{\kappa}{2} |\Phi_0|^2 \, a_\mu a_\mu + \frac{1}{2\rho_s} f_{\mu\nu} f_{\mu\nu} \tag{21.21}$$

and the gauge field a_μ becomes massive. Hence, it is possible, by a duality transformation, to pass between an XY theory and a $U(1)$ Higgs theory.

The degrees of freedom of the scalar field θ have a dual representation in terms of a gauge field a_μ. In $2+1$ dimensions, a gauge field has one transverse component – i.e. one gauge-invariant degree of freedom per point – i.e. the same number of degrees of freedom as a scalar field. In $1+1$ dimensions, a guage field has no local degrees of freedom. Duality in $1+1$ dimensions connects two scalar field theories, as we will see later.

Topological defects in the vortex order parameter, Φ, carry one quantum of a_μ flux. Hence, they are simply charges – e.g. Cooper pairs if the Landau-Ginzburg theory describes a superconductor. Hence, the duality operation exchanges particles and vortices. In the original representation (21.11), the Cooper pairs are the fundamental quanta while vortices are topological defects. In the dual representations, the fundamental quanta are vortices while the topological defects are Cooper pairs.

Let us now extend this to transformation to the Chern-Simons theory of the quantum Hall effect. Suppose we consider this theory

$$\mathcal{L}_{\rm eff} = \phi^* \left(i\partial_0 - (a_0 + A_0) \right) \phi + \frac{\hbar^2}{2m^*} \, \phi^* (i\nabla - (\mathbf{a} + \mathbf{A}))^2 \phi$$

$$+ V\left(|\phi|^2\right) + \frac{1}{2k+1}\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \tag{21.22}$$

in its fractional quantized Hall phase. We write $\phi = \sqrt{\rho}e^{i\theta}$ and integrate out the gapped fluctuations of ρ:

$$\mathcal{L}_{\text{eff}} = \frac{1}{2}\rho\left(\partial_\mu\theta - a_\mu - A_\mu\right)^2 + \frac{1}{2k+1}\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \tag{21.23}$$

This theory has a conserved current which is simply the electrical current:

$$j_\mu = \rho\left(\partial_\mu\theta - a_\mu - A_\mu\right) \tag{21.24}$$

We construct the dual representation of this current with a gauge field α:

$$j_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\alpha_\lambda \tag{21.25}$$

As in the derivation above, we consider vortices in the order parameter. As we saw at the beginning of this chapter, they are simply Laughlin quasiparticles and quasiholes. Their current is given by:

$$j^{\text{qp}}_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\theta \tag{21.26}$$

Using the dual expression for the current to eliminate $\partial_\mu\theta$ from the right-hand-side of this equation, we have:

$$\frac{1}{\rho_s}\partial_\nu f_{\mu\nu} = j^v_\mu - \epsilon_{\mu\nu\lambda}\partial_\nu\left(a_\mu + A_\mu\right) \tag{21.27}$$

This is the equation of motion of the dual Lagrangian:

$$\begin{aligned}\mathcal{L}_{\text{Dual}} = &\frac{1}{2}\kappa\left|(\partial_\mu - i\alpha_\mu)\Phi^{\text{qp}}\right|^2 + V_\Phi\left(|\Phi^{\text{qp}}|\right) + \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} \\ &+ \frac{1}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu\left(a_\lambda + A_\lambda\right) + \frac{1}{2k+1}\frac{1}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho\end{aligned} \tag{21.28}$$

where Φ is the vortex annihilation operator. Integrating out a_μ, which appears quadratically, we find:

$$\begin{aligned}\mathcal{L}_{\text{Dual}} = &\frac{1}{2}\kappa\left|(\partial_\mu - i\alpha_\mu)\Phi^{\text{qp}}\right|^2 + V_\Phi\left(|\Phi^{\text{qp}}|\right) + \\ &\frac{2k+1}{4\pi}\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho + \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \frac{1}{2\pi}\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu\end{aligned} \tag{21.29}$$

Since the Maxwell term for α_μ has one extra derivative compared to the Chern-Simons term, it is irrelevant in the long-wavelength limit. Let us

drop the Maxwell term and consider the effect of the Chern-Simons term. Since the quasiparticle annihilation operator is coupled to the Chern-Simons gauge field, α_μ, each quasiparticle has flux $1/(2k+1)$ attached to it. Hence, quasiparticles have statistics $\pi/(2k+1)$. According to the last term in (21.29), the external electromagnetic potential A_0 is coupled to the fictitious flux $\epsilon_{ij}\partial_i\alpha_j$. Since each quasiparticle has flux $1/(2k+1)$ attached to it, it has charge $e/(2k+1)$.

In the phase in which $\langle\Phi^{\rm qp}\rangle = 0$, we can integrate out the quasiparticles, thereby renormalizing the Maxwell term. We can then integrate out α_μ,

$$\begin{aligned}\mathcal{L}_{\rm eff} &= \frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho \;+\; \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} \;+\; \frac{1}{2\pi}\,\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu \\ &= \frac{1}{2k+1}\,A_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu \end{aligned} \tag{21.30}$$

which leaves us with an effective action for the electromagnetic field which incorporates the Hall conductance $1/(2k+1)$.

Hence, this duality transformation has transformed an action (21.22) in which the basic field ϕ represents a charge e fermionic electron and the basic soliton is a charge $e/(2k+1)$, statistics $\pi/(2k+1)$ quasiparticle into an action (21.29) in which the basic field $\Phi^{\rm qp}$ represents a charge $e/(2k+1)$, statistics $\pi/(2k+1)$ quasiparticle. To complete the correspondence, we must show that the basic soliton in (21.29) is a charge e fermionic electron. To do this, we must consider the state in which $\Phi^{\rm qp}$ condenses.

When $\langle\Phi^{\rm qp}\rangle = \Phi_0^{\rm qp} \neq 0$, there are solitons in this state

$$\begin{aligned}\Phi^{\rm qp}(r,\theta) &= |\Phi^{\rm qp}(r)|\,e^{i\theta} \\ \beta + B &= f(r)\end{aligned}$$

with $|\Phi^{\rm qp}(0)| = 0$, $|\Phi^{\rm qp}(\infty)| = \sqrt{\rho_0}$, $f(\infty) = 0$. They carry one flux quantum of the gauge field, so they are fermionic, charge e particles – i.e. electrons are the solitonic excitations of the state in which $\Phi^{\rm qp}$ condense.

When $\langle\Phi^{\rm qp}\rangle = \Phi_0^{\rm qp} \neq 0$, we have the effective action:

$$\mathcal{L}_{\rm Dual} = \frac{\kappa}{2}\,|\Phi_0^{\rm qp}|^2\,\alpha_\mu\alpha_\mu + \frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho \;+\; \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu \tag{21.31}$$

The first term gives a Higgs mass to the gauge field α_μ, which can now be integrated out. In doing so, we can neglect the Chern-Simons term, which is irrelevant compared to the Higgs mass. The resulting effective action for the electromagnetic gauge field is then

$$\mathcal{L}_{\rm eff} = \frac{1}{2}\kappa_0\,F_{\mu\nu}F_{\mu\nu} \tag{21.32}$$

In other words, the system is an insulator. The quantum Hall state with $\sigma_{xy} = \nu e^2/h$ is dual to the insulating state with $\sigma_{xy} = 0$ which is formed when quasiparticles condense.

Note that this insulating state is not the only state into which quasiparticles can condense. As we saw earlier in our construction of the hierarchy, quasiparticles can also condense into fractional quantum Hall states, thereby leading to $\sigma_{xy} = (2k + 1 \pm 1/p)^{-1} e^2/h$. Hence, the hierarchy construction is simply a variant of duality in $2 + 1$ dimensions, as we discuss in the next section. In this section, we have considered the 'usual' case, $p = 0$.

21.3 The Hierarchy and the Jain Sequence

To construct the hierarchy, let us begin with the dual theory (21.29).

$$\begin{aligned}\mathcal{L}_{\text{Dual}} = &\frac{1}{2}\kappa\,|(\partial_\mu - i\alpha_\mu)\Phi^{\text{qp}}|^2 \;+\; V_\Phi\left(|\Phi^{\text{qp}}|\right) \;+ \\ &\frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho \;+\; \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \frac{1}{2\pi}\,\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu\end{aligned} \tag{21.33}$$

Let us now introduce another gauge field, f, which attaches an even number of flux tubes to the quasiparticles and, therefore, has no effect. In other words, (21.29) is equivalent to the effective action

$$\begin{aligned}\mathcal{L}_{\text{Dual}} = &\frac{1}{2}\kappa\,|(\partial_\mu - i\alpha_\mu - if_\mu)\tilde{\Phi}^{\text{qp}}|^2 \;+\; V_{\tilde{\Phi}}\left(|\tilde{\Phi}^{\text{qp}}|\right) \;+ \\ &\frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}\alpha_\mu\partial_\nu\alpha_\rho \;+\; \frac{1}{2\rho}f_{\mu\nu}f_{\mu\nu} + \frac{1}{2\pi}\,\alpha_\mu\epsilon_{\mu\nu\lambda}\partial_\nu A_\mu \\ &-\frac{1}{4\pi}\,\frac{1}{2p}\,\epsilon^{\mu\nu\rho}f_\mu\partial_\nu f_\rho\end{aligned} \tag{21.34}$$

Let's suppose that $\tilde{\Phi}$ condenses. Then the corresponding Meissner effect requires that:

$$\nabla \times \alpha + \nabla \times \mathbf{f} = 0 \tag{21.35}$$

Meanwhile, the α_0 and f_0 equations of motion are:

$$\begin{aligned}(2k+1)\nabla \times \alpha + \nabla \times \mathbf{A} &= 2\pi\,\rho_{\text{qp}} \\ \nabla \times \mathbf{f} &= -2\pi(2p)\rho_{\text{qp}}.\end{aligned} \tag{21.36}$$

Combining these three equations,

$$\left(2k + 1 - \frac{1}{2p}\right)\nabla \times \alpha + \nabla \times \mathbf{A} = 0 \tag{21.37}$$

Since $\nabla \times \alpha equiv 2\pi\rho$, we find $\nu = 1/(2k + 1 - \frac{1}{2p})$. Continuing in this way, we can find the Landau-Ginzburg theories for all of the hierarchy states.

Let's now consider the 'composite fermion' construction for the 'Jain sequence' $\nu = n/(2pn \pm 1)$. We represent the electrons as fermions interacting with a Chern-Simons gauge field c_μ which attaches $2p$ flux tubes to each fermion. The Lagrangian can then be written:

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \psi^* \left(i\partial_0 - c_0 - A_0\right)\psi + \frac{\hbar^2}{2m^*}\,\psi^*(i\nabla - \mathbf{c} - \mathbf{A})^2\psi \\ -\mu\psi^\dagger\psi + V(x-x')\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x') \\ -\frac{1}{2p}\,\frac{1}{4\pi}\,\epsilon^{\mu\nu\rho}c_\mu\partial_\nu c_\rho \end{aligned} \tag{21.38}$$

The flux of c_μ is anti-aligned with the magnetic field so the effective magnetic field seen by the fermions is $\nabla \times (\mathbf{c} + \mathbf{A})$, which is such that the ψ's – the composite fermions – fill n Landau levels in the effective magnetic field.

To derive the effective theory for this state, we must now construct the effective theory for n Landau levels. At $\nu = 1/m$, we introduced a Chern-Simons gauge field so that we could represent each electron as a boson attached to m flux quanta. At $\nu = n$, it is not useful to introduce a single Chern-Simons gauge field, which would allow us to represent each electron as an anyon attached to $1/n$ flux quanta. Instead, it is more useful introduce n gauge fields, each of which is coupled to the electrons in one of the Landau levels. We can then represent each electron as a boson attached to one flux quantum. The problem with such an approach is that we can only introduce n gauge fields if there are n different conserved quantities, namely the charge in each Landau level. These charges are not, in general, conserved: only the total charge is conserved. We will come back to this point later, and assume for now that this will not make a difference.

Then, the Lagrangian takes the form:

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \phi_I^* \left(i\partial_0 - a_0^I - c_0 - A_0\right)\phi_I + \frac{\hbar^2}{2m^*}\,\phi_I^*\big(i\nabla - \mathbf{a}^I - \mathbf{c} - \mathbf{A}\big)^2\phi_I \\ -\mu\phi_I^*\phi_I + V(x-x')\phi_I^*(x)\phi_I(x)\phi_J^*(x')\phi_J(x') \\ -\frac{1}{2p}\,\frac{1}{4\pi}\,\epsilon^{\mu\nu\rho}c_\mu\partial_\nu c_\rho \;+\; \frac{1}{4\pi}\,\epsilon^{\mu\nu\rho}a_\mu^I\partial_\nu a_\rho^I \end{aligned} \tag{21.39}$$

where ϕ_I annihilates a boson corresponding to an electron in the I^{th} Landau level.

21.4 K-matrices

A compact summary of the information in the Landau-Ginzburg theory is given by the dual theory. Consider the Landau-Ginzburg theory for $\nu = 1/(2k+1)$:

$$\mathcal{L}_{\text{eff}} = \phi^* \left(i\partial_0 - (a_0 + A_0)\right) \phi + \frac{\hbar^2}{2m^*} \phi^* (i\nabla - (\mathbf{a} + \mathbf{A}))^2 \phi + u|\phi|^4 + \frac{1}{2k+1} \frac{1}{4\pi} \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho \tag{21.40}$$

Let's apply $2+1$-dimensional duality to this Lagrangian, following (21.22)-(21.29). We find the dual theory:

$$\begin{aligned} \mathcal{L}_{\text{Dual}} = & \frac{1}{2} \kappa \left|(\partial_\mu - i\alpha_\mu)\Phi^{\text{qp}}\right|^2 + V_\Phi \left(|\Phi^{\text{qp}}|\right) + \\ & \frac{2k+1}{4\pi} \epsilon^{\mu\nu\rho} \alpha_\mu \partial_\nu \alpha_\rho + \frac{1}{2\rho} f_{\mu\nu} f_{\mu\nu} + \alpha_\mu \epsilon_{\mu\nu\lambda} \partial_\nu A_\mu \end{aligned} \tag{21.41}$$

or, keeping only the most relevant terms, simply

$$\mathcal{L}_{\text{dual}} = \frac{2k+1}{4\pi} \epsilon^{\mu\nu\rho} \alpha_\mu \partial_\nu \alpha_\rho + A_\mu \epsilon^{\mu\nu\rho} \partial_\nu \alpha_\rho + \alpha_\mu j^\mu_{\text{vortex}} + \mathcal{L}_{\text{vortex}} \tag{21.42}$$

where $\mathcal{L}_{\text{vortex}}$ is the quasiparticle effective Lagrangian. Let's assume that vortices are gapped, but allow for the possibility that the magnetic field is not quite commensurate with the density so that there is some fixed number of pinned vortices. Then, we can drop the last term, $\mathcal{L}_{\text{vortex}}$.

This generalizes to an arbitrary abelian Chern-Simons theory:

$$\mathcal{L}_{\text{dual}} = \frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\rho} \alpha^I_\mu \partial_\nu \alpha^J_\rho + t_I A_\mu \epsilon^{\mu\nu\rho} \partial_\nu \alpha^I_\rho + \alpha^I_\mu j^\mu_{\text{vortex } I} \tag{21.43}$$

The Hall conductance of such a state can be obtained by integrating out the Chern-Simons gauge fields, which appear quadratically:

$$\sigma_H = \sum_{I,J} t^I t^J (K^{-1})_{IJ} \tag{21.44}$$

The charge of a vortex (*i.e.* a quasiparticle) of type i is:

$$q_I = \sum_J t^J (K^{-1})_{IJ} \tag{21.45}$$

and the braiding statistics between vortices of types i and j is:

$$\theta_{IJ} = (K^{-1})_{IJ} \ . \tag{21.46}$$

Implicit in the normalizations is the assumption that the charges associated with the $j^I_{\mu\,\mathrm{vortex}}$ are quantized in integers. Distinct quantum Hall states are therefore represented by equivalence classes of (K,t) pairs under $SL(\kappa,Z)$ basis changes where κ is the rank of the K-matrix.

Let's now construct the K-matrices associated with the hierarchy and the Jain sequence. First, consider the Landau-Ginzburg theory (21.34) of a hierarchy state.

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = &\frac{1}{2}\,\kappa\,|(\partial_\mu - ic^1_\mu - if^\mu)\Phi^{\mathrm{qp}}|^2 \;+\; V\left(\Phi^{\mathrm{qp}}\right)\\ &+\;\frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu c^1_\rho + \;\frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu A_\rho - \frac{1}{4\pi}\,\frac{1}{2p}\,\epsilon^{\mu\nu\rho}f_\mu\partial_\nu f_\rho\end{aligned} \tag{21.47}$$

where $c^1_\mu \equiv \alpha_\mu$. We write $\tilde{\Phi}^{\mathrm{qp}} = |\tilde{\Phi}^{\mathrm{qp}}|e^{i\varphi_{\mathrm{qp}}}$, integrate out the gapped fluctuations of $|\phi_{\mathrm{qp}}|$, and apply steps (21.22)-(21.29) to (21.34) by introducing a gauge field, c^2_μ:

$$\epsilon^{\mu\nu\lambda}\partial_\nu c^2_\lambda = J^\mu_{\mathrm{qp}} \equiv |\phi_{\mathrm{qp}}|^2\left(\partial^\mu\varphi_{\mathrm{qp}} - c^\mu - f^\mu\right) \tag{21.48}$$

We use the $2+1$-dimensional duality transformation to substitute this into (21.34).

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = &\frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}\left(c^1_\mu + f_\mu\right)\partial_\nu c^2_\rho \;-\; \frac{1}{4\pi}\,\frac{1}{2p}\,\epsilon^{\mu\nu\rho}f_\mu\partial_\nu f_\rho\\ &+\;\frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu c^1_\rho \;+\; \frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu A_\rho\end{aligned} \tag{21.49}$$

Finally, we integrate out f_μ:

$$\begin{aligned}\mathcal{L}_{\mathrm{eff}} = &\frac{2k+1}{4\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu c^1_\rho \;+\; \frac{2p}{4\pi}\;\epsilon^{\mu\nu\rho}c^2_\mu\partial_\nu c^2_\rho\\ &+\;\frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu c^2_\rho \;+\; \frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}c^1_\mu\partial_\nu A_\rho\end{aligned} \tag{21.50}$$

Hence, a state at the first level of the hierarchy has

$$K = \begin{pmatrix} 2k+1 & 1 \\ 1 & p_1 \end{pmatrix} \tag{21.51}$$

and $t^I = (1,0)$.

Continuing in this fashion, we find the K-matrix of an arbitrary hierarchy state (20.42):

$$K^h = \begin{pmatrix} 2k+1 & 1 & 0 & 0 & \dots \\ 1 & p_1 & 1 & 0 & \\ 0 & 1 & p_2 & 1 & \\ 0 & 0 & 1 & p_3 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{21.52}$$

and $t^I = \delta^I_1$.

Let's now consider the flux-exchange construction of the Jain sequence. Starting with (21.39), we write $\phi_I = |\phi_I| e^{\varphi_I}$, integrate out the gapped fluctuations of $|\phi_I|$, and apply steps (21.22)-(21.29) to (21.39) by introducing gauge fields, α^I_μ:

$$\epsilon^{\mu\nu\lambda}\partial_\nu \alpha^I_\lambda = J^\mu_I \equiv |\phi_I|^2 \left(\partial^\mu \varphi_I - a^{I\,\mu} - c^\mu - A^\mu\right) \tag{21.53}$$

Using $2+1$-dimensional duality, we re-write this as

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \frac{1}{2\rho}\left(f^I_{\mu\nu}\right)^2 \,+\, \frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}\left(\sum_I \alpha^I_\mu\right)\partial_\nu\left(c_\rho + A_\rho\right) \,+\, \frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}\alpha^I_\mu\partial_\nu a^I_\rho \\ + \frac{1}{4\pi}\,\epsilon^{\mu\nu\rho}a^I_\mu\partial_\nu a^I_\rho \,-\, \frac{1}{4\pi}\,\frac{1}{2p}\,\epsilon^{\mu\nu\rho}c_\mu\partial_\nu c_\rho\end{aligned} \tag{21.54}$$

Integrating out a^I_μ and c_μ, and dropping the subleading Maxwell terms we find

$$\begin{aligned}\mathcal{L}_{\text{eff}} = \frac{1}{4\pi}\,\epsilon^{\mu\nu\rho}\alpha^I_\mu\partial_\nu\alpha^I_\rho \,+\, \frac{2p}{2\pi}\,\epsilon^{\mu\nu\rho}\left(\sum_I \alpha^I_\mu\right)\partial_\nu\left(\sum_J \alpha^J_\mu\right) \\ + \frac{1}{2\pi}\,\epsilon^{\mu\nu\rho}\left(\sum_I \alpha^I_\mu\right)\partial_\nu A_\rho\end{aligned} \tag{21.55}$$

In other words, the flux exchange construction of the Jain sequence is summarized by the K-matrix:

$$K^{\text{Jain}} = \begin{pmatrix} 2p+1 & 2p & 2p & 2p & \dots \\ 2p & 2p+1 & 2p & 2p & \\ 2p & 2p & 2p+1 & 2p & \\ 2p & 2p & 2p & 2p+1 & \\ \vdots & & & & \ddots \end{pmatrix} \tag{21.56}$$

and $t^I = (1, 1, \dots, 1)$.

If we make the change of basis $K^h = W^T K^{\text{Jain}} W$ and $t^h = W^{-1} t^{\text{Jain}}$, with $W = \delta_{IJ} - \delta_{I+1,J}$, then (21.56) is transformed into (21.52) with $2k+1 = 2p+1$ and $p_1 = p_2 = \ldots = 2$. Meanwhile, t^{Jain} is transformed into t^h. Hence, the two constructions are identical for the corresponding filling fractions.

The K-matrix formalism also applies to some quantum Hall states which we have not yet discussed. These include double-layer quantum Hall states – in which there are two parallel layers of electrons – and spin-unpolarized

systems. Although, we have thus far assumed that the electrons are spin-polarized by the magnetic field, band mass and g factor corrections make the ratio of Zeeman to cyclotron energies $\sim 7/400$, so that it may be necessary to include both spins when describing electrons in moderately strong magnetic fields, even when the filling fraction is less than unity [**?**]. An example of a wavefunction which can describe spin-polarized electrons in a double-layer system or spin-unpolarized electrons in a single-layer system is the (m, m, n) wavefunction:

$$\Psi_{(m,m,n)}(w_i, z_j) \;=\; \prod_{\mathrm{i<j}} (w_i - w_j)^m \prod_{\mathrm{i<j}} (z_i - z_j)^m \prod_{\mathrm{i,j}} (w_i - z_j)^n \;. \qquad (21.57)$$

The w_i's and z_j's are, respectively, the coordinates of the up and down spin electrons, or upper- and lower-layer electrons and the filling fraction is $\nu = \frac{2}{m+n}$. The notation of (21.57) is sloppy; (21.57) should be multiplied by the spin part of the wavefunction and antisymmetrized with respect to exchanges of up- and down-spin electrons. The K-matrix for this state is

$$K = \begin{pmatrix} m & n \\ n & m \end{pmatrix} \qquad (21.58)$$

and $t^1 = t^2 = 1$. By considering hierarchies built on the (m, m, n) states or states of unpolarized electrons in multi-layer systems, we can imagine a cornucopia of fractional quantum Hall states specified by K matrices.

What exactly do we mean when we say that a Chern-Simons theory such as (21.43) is the low-energy effective field theory of a quantum Hall state? Let us first imagine that our quantum Hall liquid is on a compact surface such as a sphere or a torus, rather than in some bounded region of the plane as it would be in a real experiment. The Hamiltonian of (21.43) vanishes, so every state in the theory has vanishing energy. In other words, the Chern-Simons theory is a theory of the ground state(s). This includes states with – essentially non-dynamical – quasiholes and quasiparticles at fixed positions, since they are the lowest energy states at a given filling fraction. This theory is only valid at energies much smaller than the gap since it ignores all of the physics above the gap. The leading irrelevant corrections to (21.43) are Maxwell terms of the form $\left(\partial^\nu \alpha^\lambda - \partial^\lambda \alpha^\nu\right)^2$ which, by dimensional analysis, must have a coefficient suppressed by the inverse of the gap. The quasiparticle charges (21.45) and statistics (21.46) are the essential physics of the ground state which is encapsulated in this theory. This is not all, however. On a surface of genus g, even the state with no quasiparticles is degenerate. Two chapters ago, we saw that a Chern-Simons

with coefficient $2k+1$ has a $2k+1$-fold degenerate ground state on the torus. This is precisely the ground state degeneracy which we obtained in the previous chapter by adapting the Laughlin wavefunctions to the torus. This can be generalized to an arbitrary quantum Hall state by diagonalizing its K-matrix and multipying the degeneracies of the resulting decoupled Chern-Simons terms or; in other words, the degeneracy is simply $\det K$. On a genus-g surface, this becomes $(\det K)^g$ [**?**]. Since numerical studies can be – and usually are – done on the sphere or torus, the degeneracy is an important means of distinguishing distinct quantum Hall states with different K-matrices at the same filling fraction.

21.5 Field Theories of Edge Excitations in the Quantum Hall Effect

If, instead, we look at the Chern-Simons theory (21.43) on a bounded region of the plane [**?**], then the variation of the action $S = \int \mathcal{L}$ is:

$$\begin{aligned}\delta S = &\frac{1}{2\pi}\int d^3x\, K_{IJ}\,\delta\alpha^I_\mu \epsilon^{\mu\nu\rho}\partial_\nu\alpha^J_\rho \\ &+\frac{1}{2\pi}\int_{\text{boundary}} dt\,dx\,n_\nu\left(K_{IJ}\,\epsilon^{\mu\nu\rho}\alpha^I_\mu\,\delta\alpha^J_\rho + t_J\epsilon^{\mu\nu\rho}A_\mu\delta\alpha^J_\rho\right)\end{aligned} \tag{21.59}$$

if we set $A_\mu = j_\mu = 0$. The action is extremized if we take $\epsilon^{\mu\nu\rho}\partial_\nu\alpha^J_\rho = 0$ subject to boundary conditions such that $K_{IJ}\alpha^I_\mu\epsilon^{\mu\nu\rho}\,\delta\alpha^J_\rho = 0$ at the boundary. Let us suppose that x and y are the coordinates along and perpendicular to the boundary. Then, the most general such boundary condition is $K_{IJ}\alpha^I_0 + V_{IJ}\alpha^I_x = 0$. V_{IJ} is a symmetric matrix which will depend on the details of the boundary such as the steepness of the confining potential. Clearly, $K_{IJ}\alpha^I_0 + V_{IJ}\alpha^I_x = 0$ would be a sensible gauge choice since it is compatible with the boundary condition. In this gauge, the equation of motion following from the variation of $K_{IJ}\alpha^I_0 + V_{IJ}\alpha^I_x$ in (21.43) is a constraint, which can be satisfied if $a^I_i = \partial_i\phi^I$ for some scalar field ϕ. Substituting this into the Lagrangian and integrating by parts, we find that all of the action is at the edge:

$$S = \frac{1}{2\pi}\int dt\,dx\,K_{IJ}\,\partial_t\phi^I\,\partial_x\phi^J - V_{IJ}\,\partial_x\phi^I\,\partial_x\phi^J + A_\mu\epsilon_{\mu\nu}\partial_\nu\phi^I t^I \tag{21.60}$$

The Chern-Simons theory of the bulk has been reduced to a theory of (chiral) bosons at the edge of the system. These are precisely the excitations which

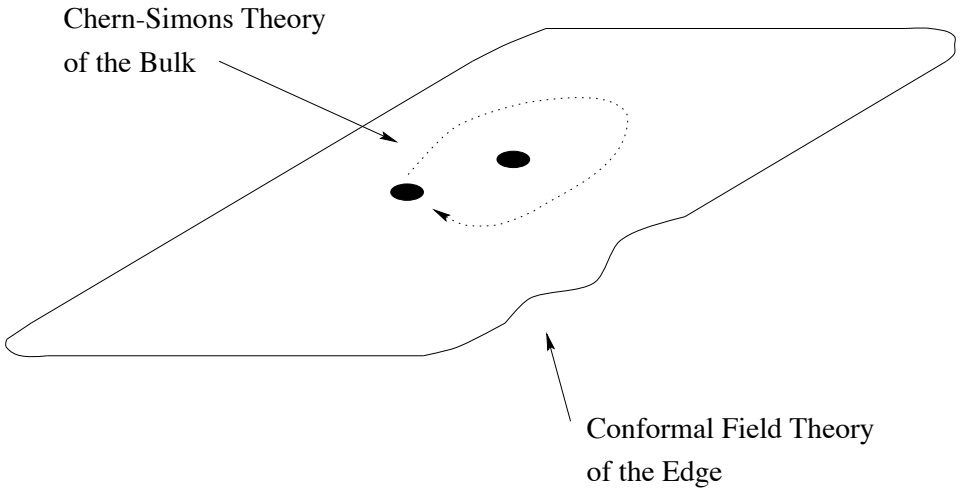


Figure 21.1: The Chern-Simons theory which describes the braiding of quasiparticles in the bulk is associated with a Conformal Field Theory which describes the gapless excitations at the edge.

we derived by multiplying the Laughlin state by symmetric polynomials in the previous chapter.

Let's consider the simplest case, $\nu = 1/m$

$$S = \frac{m}{2\pi} \int dt\, dx\ \partial_t \phi\, \partial_x \phi - v\, \partial_x \phi\, \partial_x \phi + \frac{1}{2\pi} \int dt\, dx\, A_\mu \epsilon_{\mu\nu} \partial_\nu \phi^I t^I \quad (21.61)$$

This is the action for a free chiral boson. The equations of motion (for $A_\mu = 0$ for simplicity)

$$(\partial_t - v\partial_x)\, \partial_x \phi = 0 \quad (21.62)$$

are satisfied if the field is chiral, $\phi(x,t) = \phi(x+vt)$.

The equal-time commutations relations which follow from this action are:

$$\left[\partial_x \phi(x), \phi(x')\right] = i\, \frac{2\pi}{m}\, \delta(x - x') \quad (21.63)$$

By varying the electromagnetic field, we derive the charge and current operators:

$$\rho = \frac{1}{2\pi} \partial_x \phi$$

$$j = \frac{1}{2\pi}\partial_t \phi \tag{21.64}$$

Hence, the operators $e^{i\phi}$ and $e^{im\phi}$ create excitations of charge $1/m$:

$$\begin{aligned}\left[\rho(x), e^{i\phi(x')}\right] &= \frac{1}{m}\, e^{i\phi(x')}\, \delta(x-x') \\ \left[\rho(x), e^{im\phi(x')}\right] &= e^{i\phi(x')}\, \delta(x-x')\end{aligned} \tag{21.65}$$

These operators create quasiparticles and electrons respectively.

To compute their correlation functions, we must first compute the $\phi - \phi$ correlation function. This is most simply obtained from the imaginary-time functional integral by inverting the quadratic part of the action. In real-space, this gives:

$$\begin{aligned}\langle \phi(x,t)\, \phi(0,0)\rangle - \langle \phi(0,0)\, \phi(0,0)\rangle &= \int \frac{dk}{2\pi}\, \frac{d\omega}{2\pi}\, \frac{2\pi}{m}\, \frac{1}{k(i\omega - vk)}\, \left(e^{i\omega\tau - ikx} - 1\right) \\ &= \frac{2\pi}{m} \int \frac{dk}{2\pi}\, \frac{1}{k}\left(e^{-ik(x+iv\tau)} - 1\right) \\ &= -\frac{1}{m}\int_{\frac{1}{x+iv\tau}}^{\Lambda} \frac{dk}{2\pi}\, \frac{1}{k} \\ &= -\frac{1}{m}\, \ln\left[(x+iv\tau)/a\right]\end{aligned} \tag{21.66}$$

where $a = 1/\Lambda$ is a short-distance cutoff. Hence, the quasiparticle correlation function is given by:

$$\begin{aligned}\left\langle e^{i\phi(x,\tau)} e^{i\phi(0,0)}\right\rangle &= e^{\langle \phi(x,t)\, \phi(0,0)\rangle - \langle \phi(0,0)\, \phi(0,0)\rangle} \\ &= \frac{1}{(x+iv\tau)^{1/m}}\end{aligned} \tag{21.67}$$

while the electron correlation function is:

$$\begin{aligned}\left\langle e^{im\phi(x,\tau)} e^{im\phi(0,0)}\right\rangle &= e^{m^2\langle \phi(x,t)\, \phi(0,0)\rangle - m^2\langle \phi(0,0)\, \phi(0,0)\rangle} \\ &= \frac{1}{(x+iv\tau)^{m}}\end{aligned} \tag{21.68}$$

Hence, the quasiparticle creation operator has dimension $2/m$ while the electron creation operator has dimension $m/2$.

Let us suppose that a tunnel junction is created between a quantum Hall fluid and a Fermi liquid. The tunneling of electrons from the edge of

the quantum Hall fluid to the Fermi liquid can be described by adding the following term to the action:

$$S_{\text{tun}} = t \int d\tau \, e^{im\phi(0,\tau)} \, \psi(0,\tau) \, + \, \text{c.c.} \tag{21.69}$$

Here, $x = 0$ is the point at which the junction is located and $\psi(x,\tau)$ is the electron annihilation operator in the Fermi liquid. As usual, it is a dimension $1/2$ operator. This term is irrelevant for $m > 1$:

$$\frac{dt}{d\ell} = \frac{1}{2}\,(1-m)\,t \tag{21.70}$$

Hence, it can be handled perturbatively at low-temperature. The finite-temperature tunneling conductance varies with temperature as:

$$G_t \sim t^2 \, T^{m-1} \tag{21.71}$$

while the current at zero-temperature varies as:

$$I_t \sim t^2 \, V^m \tag{21.72}$$

A tunnel junction between two identical quantum Hall fluids has tunneling action:

$$S_{\text{tun}} = t \int d\tau \, e^{im\phi_1(0,\tau)} \, e^{-im\phi_2(0,\tau)} \, + \, \text{c.c.} \tag{21.73}$$

Hence,

$$\frac{dt}{d\ell} = (1-m)\,t \tag{21.74}$$

and the tunneling conductance varies with temperature as:

$$G_t \sim t^2 \, T^{2m-2} \tag{21.75}$$

while the current at zero-temperature varies as:

$$I_t \sim t^2 \, V^{2m-1} \tag{21.76}$$

Suppose we put a constriction in a Hall bar so that tunneling is possible from the top edge of the bar to the bottom edge. Then quasiparticles can tunnel across the interior of the Hall fluid. The tunneling Hamiltonian is:

$$S_{\text{tun}} = v \int d\tau \, e^{i\phi_1(0,\tau)} \, e^{-i\phi_2(0,\tau)} \, + \, \text{c.c.} \tag{21.77}$$

The tunneling of quasiparticles is relevant

$$\frac{dt}{d\ell} = \left(1 - \frac{1}{m}\right) v \tag{21.78}$$

where ϕ_1 and ϕ_2 are the edge operators of the two edges. Hence, it can be treated perturbatively only at high-temperatures or large voltages. At low voltage and high temperature, the tunneling conductance varies with temperature as:

$$G_t \sim v^2 \, T^{\frac{2}{m}-2} \tag{21.79}$$

while the current at zero-temperature varies as:

$$I_t \sim v^2 \, V^{\frac{2}{m}-1} \tag{21.80}$$

so long as V is not too small. If we measure the Hall conductance of the bar by running current from the left to the right, then it will be reduced by the tunneling current:

$$G = \frac{1}{m} \frac{e^2}{h} - (\text{const.}) v^2 \, T^{\frac{2}{m}-2} \tag{21.81}$$

When T becomes low enough, the bar is effectively split in two so that all that remains is the tunneling of electrons from the left side to the right side:

$$G \sim t^2 \, T^{2m-2} \tag{21.82}$$

In other words, the conductance is given by a scaling function of the form:

$$G = \frac{1}{m} \frac{e^2}{h} \; Y\left(v^2 \, T^{\frac{2}{m}-2}\right) \tag{21.83}$$

with $Y(x) - 1 \sim -x$ for $x \to 0$ and $Y(x) \sim x^{-m}$ for $x \to \infty$.

For a general K-matrix, the edge structure is more complicated since there will be several bosonic fields, but they can still be analyzed by the basic methods of free field theory. Details can be found in .

21.6 Duality in $1 + 1$ Dimensions

At the end of the previous section, we saw that a problem which could, in the weak-coupling limit, be described by the tunneling of quasiparticles was, in the strong coupling limit, described by the tunneling of electrons. This is an example of a situation in which there are two dual descriptions of the same problem. In the quantum Hall effect, there is one description in which

electrons are the fundamental objects and quasiparticles appear as vortices in the electron fluid and another description in which quasiparticles are the fundamental objects and electrons appear as aggregates of three quasiparticles. We have already discussed this duality in the $2+1$-dimensional Chern-Simons Landau-Ginzburg theory which describes the bulk. In this section, we will examine more carefully the implementation of this duality in the edge field theories. As we will see, it essentially the same as the duality which we used in our analysis of the Kosterlitz-Thouless transition. In the next section, we will look at the analogous structure in the bulk field theory.

Let us consider a free non-chiral boson φ. It can be expressed in terms of chrial fields ϕ_L and ϕ_R:

$$\begin{aligned} \varphi &= \phi_L + \phi_R \\ \tilde{\varphi} &= \phi_L - \phi_R \end{aligned} \tag{21.84}$$

Here, we have defined the dual field $\tilde{\varphi}$. Observe that:

$$\partial_\mu \tilde{\varphi} = \epsilon_{\mu\nu}\, \partial_\nu \varphi \tag{21.85}$$

The free action takes the form:

$$S_0 = \frac{g}{8\pi} \int dx\, d\tau\, \left[(\partial_\tau \varphi)^2 + v^2 (\partial_x \varphi)^2 \right] \tag{21.86}$$

where φ is an agular variable: $\varphi \equiv \varphi + 2\pi$. Let us rescale $\varphi \to \varphi/\sqrt{g}$ so that the action is of the form:

$$S_0 = \frac{1}{8\pi} \int dx\, d\tau\, \left[(\partial_\tau \varphi)^2 + v^2 (\partial_x \varphi)^2 \right] \tag{21.87}$$

As a result of the rescaling of φ which we performed in going from (21.86) to (21.87), φ now satisfies the identification $\varphi \equiv \varphi + 2\pi\sqrt{g}$.

Note that this theory has a conserved current, $\partial_\mu j_\mu = 0$

$$j_\mu = \partial_\mu \varphi \tag{21.88}$$

which is conserved by the equation of motion. It also has a current

$$j^D_\mu = \partial_\mu \tilde{\varphi} \tag{21.89}$$

which is trivially conserved.

Let us consider the Fourier decomposition of $\phi_{R,L}$:

$$\phi_R(x,\tau) = \frac{1}{2}\, x_0^R + p_R\,(i\tau - x) + i\sum_n \frac{1}{n}\alpha_n e^{-n(\tau+ix)}$$
$$\phi_L(x,\tau) = \frac{1}{2}\, x_0^L + p_L\,(i\tau + x) + i\sum_n \frac{1}{n}\tilde{\alpha}_n e^{-n(\tau-ix)} \qquad (21.90)$$

Hence,

$$\varphi(x,\tau) = \varphi_0 \;+\; i\,(p_L + p_R)\,\tau \;+\; (p_L - p_R)\,x \;+\; i\sum_n \frac{1}{n}\left[\alpha_n e^{-n(\tau+ix)} + \tilde{\alpha}_n e^{-n(\tau-ix)}\right]$$
$$\tilde{\varphi}(x,\tau) = \tilde{\varphi}_0 \;+\; i\,(p_L - p_R)\,\tau \;+\; (p_L + p_R)\,x \;-\; i\sum_n \frac{1}{n}\left[\alpha_n e^{-n(\tau+ix)} - \tilde{\alpha}_n e^{-n(\tau-ix)}\right] \qquad (21.91)$$

where $\varphi_0 = x_0^L + x_0^R$ and $\tilde{\varphi}_0 = x_0^L - x_0^R$. From the identification $\varphi \equiv \varphi + 2\pi\sqrt{g}$, it follows that $\varphi_0 \equiv \varphi_0 + 2\pi\sqrt{g}$. From the canonical commutation relations for φ, it follows that φ_0 and $(p_L + p_R)/2$ are canonical conjugates. The periodicity condition satisfied by φ_0 imposes the following quantization condition on $(p_L + p_R)$:

$$p_L + p_R = \frac{M}{\sqrt{g}}\,, \;\; M \in Z \qquad (21.92)$$

Furthermore, physical operators of the theory must respect the periodicity condition of φ. The allowed exponential operators involving φ are of the form:

$$\{M, 0\} \equiv e^{i\frac{M}{\sqrt{g}}\varphi(x,\tau)} \qquad (21.93)$$

and they have dimension M^2/g.

Let us assume that our edges are closed loops of finite extent, and rescale the length so that $x \in [0, \pi]$ with $\varphi(\tau, x) \equiv \varphi(\tau, x + \pi) + 2\pi N\sqrt{g}$ for some integer N. Then, from (21.91), we see that we must have

$$(p_L - p_R) = 2N\sqrt{g}\,, \;\; N \in Z \qquad (21.94)$$

These degrees of freedom are called 'winding modes'. Hence, we have:

$$p_L = \frac{M}{2\sqrt{g}} + N\sqrt{g}$$
$$p_R = \frac{M}{2\sqrt{g}} - N\sqrt{g} \qquad (21.95)$$

Note that this is reversed when we consider $\tilde{\varphi}$. Momentum modes are replaced by winding modes and vice-versa. Following our earlier steps, but taking this reversal into account, the allowed exponentials of $\tilde{\varphi}$ are of the form:

$$\{0, N\} \equiv e^{i2N\sqrt{g}\,\tilde{\varphi}(x,\tau)} \tag{21.96}$$

Hence, the most general exponential operator is of the form:

$$\{M, N\} \equiv e^{i\left(\frac{M}{2\sqrt{g}}\varphi + N\sqrt{g}\,\tilde{\varphi}\right)} = e^{i\left[\left(\frac{M}{2\sqrt{g}}+N\sqrt{g}\right)\phi_L + \left(\frac{M}{2\sqrt{g}}-N\sqrt{g}\right)\phi_R\right]} \tag{21.97}$$

with scaling dimension:

$$\begin{aligned}\dim(M,N) &= \frac{1}{2}\left(\frac{M}{2\sqrt{g}} + N\sqrt{g}\right)^2 + \frac{1}{2}\left(\frac{M}{2\sqrt{g}} - N\sqrt{g}\right)^2 \\ &= \frac{M^2}{4g} + N^2 g \\ &= \frac{(M/2)^2}{g} + N^2 g\end{aligned} \tag{21.98}$$

These dimensions are invariant under the transformation $g \leftrightarrow 1/4g$, $M \leftrightarrow N$. In fact the entire theory is invariant under this transformation. It is simply the transformation which exchanges φ and $\tilde{\varphi}$.

When we couple two identical non-chiral bosons, φ_1 and φ_2, we form $\varphi_\pm = (\varphi_1 \pm \varphi_1)/\sqrt{2}$. The factor of $\sqrt{2}$ is included so that both $\varphi_\pm$ have the same coefficent, $1/8\pi$, in front of their actions. However, this now means that $\varphi_\pm \equiv \varphi_\pm + 2\pi\sqrt{g/2}$. When we couple two bosons though exponential tunneling operators, only φ_- is affected. Hence, the appropriate duality is that for φ_-: $(g/2) \leftrightarrow 1/[4(g/2)]$ or, simply $g \leftrightarrow 1/g$. This duality exchanges $\cos\varphi_-/\sqrt{g/2}$ and $\cos\tilde{\varphi}_-\sqrt{g/2}$, which transfer, repectively, a pair of solitons (i.e. electrons) and a particle-hole pair from system 1 to system 2.

Let us now apply these considerations to quantum Hall edges. In order to apply the above duality – which applies to non-chiral bosons – to a quantum Hall edge, which is chiral, we must 'fold' the edge in order to define a non-chiral field, as depicted in figure 21.2.

If we fold the edge at $x = 0$, we can define $\varphi = (\phi(x) + \phi(-x))/\sqrt{2}$ and $\tilde{\varphi} = (\phi(x) - \phi(-x))/\sqrt{2}$. The latter vanishes at the origin; only the former is important for edge tunneling. The allowed operators are:

$$\left[e^{iN\varphi/\sqrt{m}}\right] = \frac{N^2}{2m} \tag{21.99}$$

Figure 21.2: An infinite chiral edge mode with a tunnel junction at one point can be folded into a semi-infinite nonchiral mode with a tunnel junction at its endpoint.

The factor of $1/2$ on the right-hand-side comes from the $\sqrt{2}$ in the definition of φ. If we couple two edges, we can now define φ_-, which has allowed operators

$$\left[e^{iN\varphi_-/\sqrt{m/2}}\right] = \frac{N^2}{m} \tag{21.100}$$

and dual operators

$$\left[e^{2iM\tilde{\varphi}_-\sqrt{m/2}}\right] = M^2 m \tag{21.101}$$

which are dual under $M \leftrightarrow N$, $m \leftrightarrow 1/m$. The description in terms of φ is equivalent to the description in terms of $\tilde{\varphi}$. However, as we saw in the previous section, the tunneling of quasiparticles between the two edges of a quantum Hall droplet is most easily discussed in terms of φ when the tunneling is weak, i.e. in the ultraviolet, when the tunneling operator can be written $e^{i\varphi/\sqrt{m}}$. However, when the tunneling becomes strong, in the infrared, the dual description in terms of $\tilde{\varphi}$ is preferable, since the corresonding tunneling operator is $e^{i\tilde{\varphi}\sqrt{m}}$.

CHAPTER 22

Frontiers in Electron Fractionalization

22.1 Introduction

At present, the only topological phases which are know to occur in nature are those which are observed in the fractional quantum Hall regime. As described in the previous chapter, a number of experimentally-observed phases are associated with various Abelian topological states. There is some reason for believing that there are also non-Abelian topological phases lurking in some relatively weak quantum Hall plateaus in the second Landau level. Furthermore, there is nothing about topological phases which is intrinsic to the quantum Hall regime. In principle, they can occur in a number of different physical contexts and, in fact, an even wider variety of phases (such as those which are P, T-invariant) might appear as we explore the full freedom of the phase diagram of electrons in solids. This is a frontier topic, so our discussion will necessarily be tentative. Since very little is known on the experimental side, our discussion will be rather speculative, but we will try to stick to topics where at least the mathematical and theoretical issues are clear. In particular, we will focus on the effective field theories which describe various topological phases. We will also try to briefly address the question of which particular Hamiltonians of electrons in solids will actually give rise, but this is still very much an open question.

22.2 A Simple Model of a Topological Phase in P, T-Invariant Systems

Consider the following simple model of spins on a honeycomb lattice. Each $s = 1/2$ spin lies on a link of the lattice. The spins interact through the Hamiltonian:

$$H = J_1 \sum_i A_i - J_2 \sum_p F_p \tag{22.1}$$

where

$$A_i \equiv \Pi_{k \in \mathcal{N}(i)} \sigma_k^z \,, \qquad F_p \equiv \Pi_{k \in p} \sigma_k^x \tag{22.2}$$

and $\sigma_k^z = \pm$. These operators all commute,

$$\left[F_p, F_{p'}\right] = [A_i, A_j] = [F_p, A_j] = 0 \tag{22.3}$$

so the model can be solved exactly by diagonalizing each term in the Hamiltonian: the ground state $|0\rangle$ satisfies $A_i|0\rangle = -|0\rangle, F_p|0\rangle = |0\rangle$. If we represent $\sigma_z = 1$ by colored bonds and $\sigma_z = -1$ by uncolored bonds, then $A_i|0\rangle = -|0\rangle$ requires chains of bonds to never end, while $F_p|0\rangle = |0\rangle$ requires the ground state to contain an equal superposition of any configuration with one obtained from it by flipping all of the spins on any plaquette (i.e. switching colored and uncolored bonds).

To understand this Hamiltonian better, it is useful to consider the $J_1 \to \infty$ limit and to introduce the following representation for the low-energy states of this model. We consider wavefunctions $\Psi[\alpha]$ which assign a complex amplitude for any 'multi-loop' α on the honeycomb lattice. By 'multi-loop', we mean simply the disjoint union of a number of closed loops which do not share any links of the lattice. The multi-loop α simply represents the locations of the $\sigma_k^z = +1$ spins, so this representation is just the σ_k^z basis.

The ground state condition, $F_p|0\rangle = |0\rangle$ is the statement that $\Psi[\alpha]$ is invariant under various geometrical manipulations of the multi-loop α. If we draw the multi-loops as if they were in the continuum, but remember that they are really on the lattice, then we can draw the relations imposed by $F_p|0\rangle = |0\rangle$ in the following pictorial form as three distinct relations. Depending on whether the plaquette is empty

The first follows from F_p acting on a empty plaquette. It flips this plaquette into one which contains a small, contractible loop. Hence, the ground state wavefunction is invariant under such an operation, as depicted in figure 22.1. Suppose now that a loop passes through a single link of a

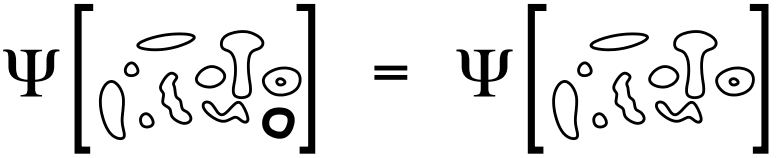


Figure 22.1: The ground state wavefunction is invariant under the removal of a small contractible loop.

plaquette. When F_p acts on this plaquette, it deforms the loop so that it passes through the other 5 links of the plaquette and now encloses it. The action of F_p causes similar deformation of a loop which passes through a plaquette along $2, 3, \ldots, 5$ consecutive links. Hence, the ground state must be invariant under such a deformation of any loop, as depicted in figure 22.2. If two loops touch a plaquette, then F_p cuts the loops and rejoins them so

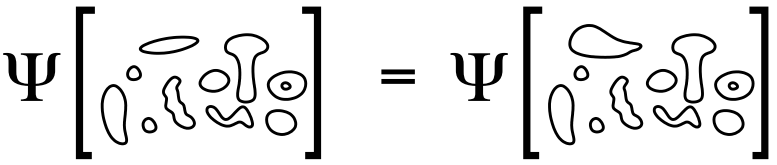


Figure 22.2: The ground state wavefunction is invariant under smooth deformation of any loop. In the figure, the loop in the upper left has been deformed.

that they form one big loop. Conversely, if the same loop passes through a plaquette twice, then F_p breaks it into two loops. The ground state must be invariant under such surgery operations, as depicted in figure 22.3. If three

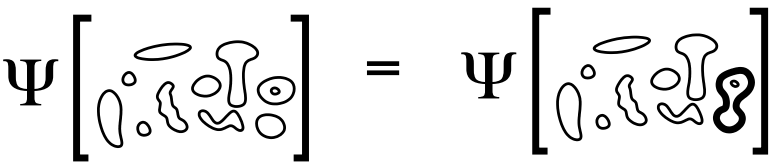


Figure 22.3: The ground state wavefunction is invariant under a surgery procedure which cuts and rejoins two loops which come near each other.

loops touch a plaquette, then F_p performs a surgery which is equivalent to two pairwise surgeries, which can be performed in any order.

Now consider this model on an annulus. It is now possible for loops

to wind around the annulus. However, the surgery relation implies that ground state wavefunctions must have the same value on configurations with winding numbers $0, 2, 4, \ldots$. Similarly, they must have the same value on configurations with winding numbers $1, 3, 5, \ldots$. Thus, there are two ground states, corresponding to even and odd winding numbers of the loops around the annulus. On the torus, there are four ground states, corresponding to even/odd winding numbers around the two generators of the torus. Thus, the ground state degeneracy depends on the topology of the manifold on which the system is defined – on a genus g surface, it is 4^g. Note, further, that the different ground states are locally indistinguishable. They all have $A_i = -1$, $F_p = 1$ at every vertex and plaquette, so correlation functions of local operators are all the same (and vanish unless the two operators share a plaquette). The only way to distinguish the various ground states is to do a measurement which is sensitive to the global topology of the system. Thus, it is in a topological phase.

Consider now the excited states of the system. They are collections of localized excitations which come in two varieties: vertices at which $A_i = 1$ and plaquettes at which $F_p = -1$. In other words, we have excitations which are endpoints of broken loops, which we can think of as 'electric' particles. Clearly, they can only be created in pairs. We also have excitations which are frustrated plaquettes: the state acquires a minus sign whenever a loop moves through this plaquette. We can think of these excitations as vortices or 'magnetic' particles.

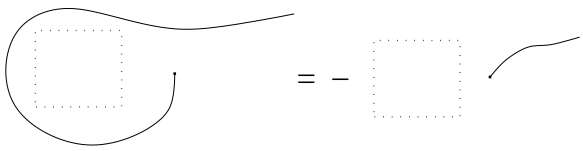


Figure 22.4: When an $A_v = -1$ electric excitation is taken around an $F_p = -1$ magnetic excitation, the wavefunction acquires a -1.

Now, observe that when an electric particle is taken around a magnetic particle, we must move the curve attached to it though the excited plaquette, so the wavefunction acquires a phase π, as depicted in figure 22.4. Hence, electric and magnetic particles have non-trivial mutual braiding statistics. On the other hand, electric particles have bosonic statistics with themselves, as do magnetic particles. A composite formed by an electric and a magnetic particle is fermionic.

22.3 Effective Field Theories

The basic physics of the Hamiltonian (22.1) is the relative statistics between the 'electric' and 'magnetic' particles. This is encapsulated in the following topological field theory:

$$S_{\rm BF} = \frac{1}{2\pi}\int d^2x\, d\tau\, \epsilon^{\mu\nu\lambda} e_\mu f_{\nu\lambda} - j_\mu^{\rm electric} a_\mu - j_\mu^{\rm magnetic} e_\mu \tag{22.4}$$

where $f_{\nu\lambda} = \partial_\nu a_\lambda - \partial_\lambda a_\nu$, as usual. This theory is commonly known as 'BF theory' because the field e_μ is usually called b_μ. However, e_μ is better notation because it is the canonical conjugate of a_μ, as may be seen from (22.4):

$$\frac{\partial \mathcal{L}}{\partial a_i} = \epsilon_{ij}\, e_j \qquad \Rightarrow \qquad [a_i(x), \epsilon_{kj}\, e_j(0)] = i\,\delta_{ik}\,\delta(x) \tag{22.5}$$

The time derivatives of the fields e_0 and a_0 do not appear in this action. They are Lagrange multipliers which enforce the constraints:

$$\begin{aligned} f_{12} &= \pi\,\rho^{\rm magnetic} \\ \partial_1 e_2 - \partial_2 e_1 &= \pi\,\rho^{\rm electric} \end{aligned} \tag{22.6}$$

Thus, each magnetic particle is accompanied by π-flux of the a_μ gauge field. Electric particles 'see' the gauge field a_μ according to their coupling $j_\mu^{\rm electric} a_\mu$, so when an electric particle goes around a magnetic particle, the wavefunction changes sign. Conversely, each electric particle is accompanied by π-flux of the e_μ field, so when a magnetic particle goes around an electric particle, the wavefunction changes sign.

We can quantize this theory along the lines of our quantization of Chern-Simons theory, to which it is related according to

$$2S_{\rm BF}[e,a] = S_{\rm CS}[a+e] - S_{\rm CS}[a-e] \tag{22.7}$$

By introducing Wilson loop operators for a_μ, e_μ and quantizing their algebra, we can see that the theory (22.4) has ground state degeneracy 4 on the torus. Rather than recapitulating this, we will see how this structure arises in another version of the theory below.

The transition into this phase can be understood as the deconfinement transition of Z_2 gauge theory, whose deconfined phase is described at low energies by the U(1) BF theory above. There are two different ways of realizing such a theory. We could begin with a $U(1)$ gauge theory with Maxwell action which is coupled to a charge-2 matter field. When this matter field condenses, the $U(1)$ symmetry is broken to Z_2. This construction can be

done directly in the continuum. Alternatively, one can work with Z_2 gauge fields from the beginning. However, one must, in such a case, work on a lattice. Let us follow the latter avenue. We consider a $2+1$ dimensional space-time lattice on which there is an Ising gauge field degree of freedom $\sigma_z = \pm 1$ on each *link* of the lattice. We will label them by a lattice site, $\mathbf{x}$, and a direction $i = x, y, \tau$ so that there are three links associated with each site. The action is the sum over all plaquettes of the product of σ_zs around a plaquette:

$$S = -K \sum_{\text{plaq.}} \sigma_z \sigma_z \sigma_z \sigma_z \tag{22.8}$$

To quantize this theory, it is useful to choose temporal gauge, $\sigma_z(\mathbf{x}, \tau) = 1$ for all $\mathbf{x}$. In this gauge, the Hamiltonian takes the form:

$$H = -\sum_{\mathbf{x},i} \sigma_x(\mathbf{x}, i) - K \sum_{\text{spatial plaq.}} \sigma_z \sigma_z \sigma_z \sigma_z \tag{22.9}$$

In temporal gauge, there are residual global symmetries generated by the operators

$$G(\mathbf{x}) = \sigma_x(\mathbf{x}, x)\sigma_x(\mathbf{x}, y)\sigma_x(\mathbf{x} - \hat{\mathbf{x}}, x)\sigma_x(\mathbf{x} - \hat{\mathbf{y}}, y). \tag{22.10}$$

The extreme low-energy limit, in which this theory becomes topological, is the $K \to \infty$ limit. In this limit, $\sigma_z \sigma_z \sigma_z \sigma_z = 1$ for every spatial plaquette.

It is useful to define operators $W[\gamma]$ associated with closed curves γ on the lattice:

$$L[\gamma] = \prod_{\mathbf{x}, i \in \gamma} \sigma_z(\mathbf{x}, i) \tag{22.11}$$

We also need operators $Y[\alpha]$ associated with closed curves on the dual lattice, i.e. closed curves which pass through the centers of a sequence of adjacent plaquettes.

$$Y[\alpha] = \prod_{\mathbf{x}, i \perp \alpha} \sigma_x(\mathbf{x}, i) \tag{22.12}$$

The product is over all links which α intersects. $L[\gamma]$ is analogous to a Wilson loop operator while $Y[\gamma]$ creates a Dirac string.

Let us consider the space of states which are annihilated by the Hamiltonian; this is the Hilbert space of the $K \to \infty$ limit. When restricted to states within this Hilbert space, $L[\gamma]$ and $Y[\alpha]$ satisfy the operator algebra

$$\begin{aligned} L[\gamma]\, Y[\alpha] &= (-1)^{I(\gamma,\alpha)}\, Y[\alpha]\, L[\gamma] \\ [L[\gamma], L[\alpha]] &= [Y[\gamma], Y[\alpha]] = 0 \end{aligned} \tag{22.13}$$

Now, it is clear that such an operator algebra can be represented on a vector space of the form derived in the previous section:

$$\begin{aligned} L[\gamma]\,\Psi[\{\alpha\}] &= (-1)^{I(\gamma,\alpha)}\,\Psi[\{\alpha\}] \\ Y[\gamma]\,\Psi[\{\alpha\}] &= \Psi[\{\alpha\cup\gamma\}] \end{aligned} \tag{22.14}$$

The notable difference is that the allowed states must now satisfy the constraints

$$\begin{aligned} \Psi[\{\alpha\}] &= \Psi[\{\alpha\cup\bigcirc\}] \\ \Psi[\{\alpha\}] &= \Psi[\{\tilde{\alpha}\}] \end{aligned} \tag{22.15}$$

Again, $\tilde{\alpha}$ is obtained from α by performing the surgery operation $)(\rightarrow\asymp$ on any part of α.

If α is contractible, then $Y[\alpha]$ commutes with all other operators in the theory, so its effect on any wavefunction should be multiplication by a scalar. If we take this scalar to be 1, then we have the first constraint above. The second constraint is necessary in order to realize the operator algebra (22.14) and is also required by consistency with the first. As a result of the second line of (22.15), we can resolve crossings of α and γ in either way since they are equivalent in the low-energy Hilbert space.

22.4 Other P,T-Invariant Topological Phases

It is clear that the structure which we have described above is rather general. Any system whose intermediate-scale degrees of freedom are fluctuating loops can give rise to such a phase. This includes, for instance, domain walls between Ising spins. If the Ising spins lie on the vertices of a triangular lattice, the domain walls will lie on the honeycomb lattice, as depicted in figure 22.5.

Suppose we wish to generalize this structure. We can modify the relations and/or change the degrees of freedom to e.g. directed or colored loops. Consider the first approach. The relation $\Psi[\alpha] = \Psi[d\alpha]$ where $d\alpha$ is any continuous deformation of the multi-curve α must presumably be satisfied by in topological phase. However, the other two relations can be modified. Suppose $\Psi[\alpha] = \Psi[\alpha\cup\bigcirc]$ is modified to $\Psi[\alpha] = d\,\Psi[\alpha\cup\bigcirc]$. Then the surgery relation must be modified as well since the surgery relation of figure 22.3 is in conflict with $d \neq 1$, as may be seen in figure 22.6. Thus, for any $d \neq 1$, a new surgery relation which is consistent with it must be found. It turns out that for most d, such a relation cannot be found. Only for the

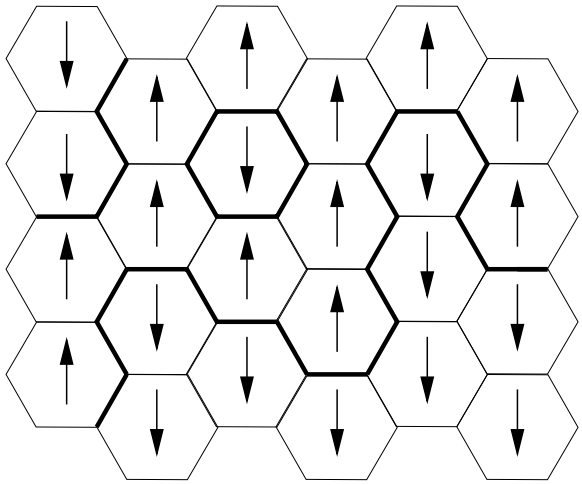


Figure 22.5: A set of basis states of a system of Ising spins on a triangular lattice is equivalent to the possible loop configurations of a honeycomb lattice.

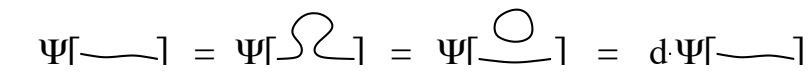


Figure 22.6: If $d \neq 1$, the surgery relation of figure 22.3 cannot hold since the above contradiction would follow.

special values

$$d = \cos\left(\frac{\pi}{k+2}\right) \tag{22.16}$$

do such relations exist. The $k = 1$ case is the $d = 1$ phase which we discussed above. The next phase is the $k = 2$ phase, which has $d = \sqrt{2}$ and the surgery relation The surgery relations become more complex with increasing k.

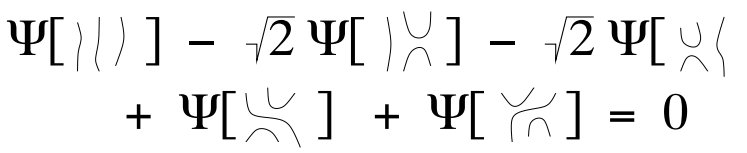


Figure 22.7: The surgery relation for $d = \sqrt{2}$.

22.5 Non-Abelian Statistics

The basic feature of these generalizations is that their quasiparticles exhibit *non-Abelian braiding statistics*. This is a possibility which we neglected in our discussion of exotic statistics in chapter **??**. Suppose we have g degenerate states, ψ_a, $a = 1, 2, \ldots, g$ of particles at $x_1, x_2, \ldots, x_n$. Exchanging particles 1 and 2 might do more than just change the phase of the wavefunction. It might rotate it into a different one in the space spanned by the ψ_as, so that:

$$\psi_a \to M^{12}_{ab}\, \psi_b \tag{22.17}$$

On the other hand, exchanging particles 2 and 3 leads to:

$$\psi_a \to M^{23}_{ab}\, \psi_b \tag{22.18}$$

If M^{12}_{ab} and M^{23}_{ab} do not not commute, $M^{12}_{ab} M^{23}_{bc} \neq M^{23}_{ab} M^{12}_{bc}$, the particles obey *non-Abelian braiding statistics*.

To see how such a phenomenon might occur, consider one of the topological phases mentioned at the end of the previous section. Because such a phase will not have a surgery relation on two strands, there are multiple linearly-independent states with four particles, as may be seen in figure 22.8

Figure 22.8: These two states are linearly independent, except for $k = 1$. In the $k = 1$ case, the surgery relation of figure 22.3 implies that these two states are the same.

Thus, we have satisfied the first condition necessary for non-Abelian statistics: a degenerate set of states of quasiparticles at fixed positions. We further observe that braiding particles rotates states in this degenerate subspace into each other. For instance, taking particle 2 around 3 and 4 transforms the first state in figure 22.8 into the second. The transformations enacted by braiding operations are such that the order is important, as may be seen in figure 22.9.

The number of states with n-quasiparticles grows very rapidly, in fact exponentially $\sim d^n$ in the states with $d = 2\cos\frac{\pi}{k+2}$ alluded to in the previous section.

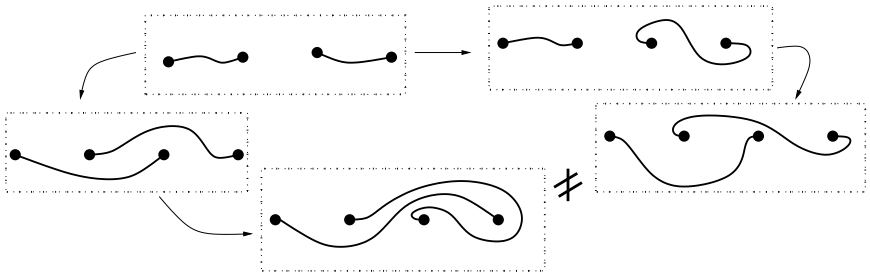


Figure 22.9: By switching the order of two exchanges, we obtain a different state. Starting from the state in the upper left, we can exchange quasiparticles 3 and 4 first and then 2 and 3 (depicted on the right). Alternatively, we can exchange 2 and 3 first and then 3 and 4 (depicted on the left).

Non-Abelian braiding statistics can also occur in the quantum Hall regime. The likeliest candidate for such a state is the quantized Hall plateau observed at $\nu = \frac{5}{2}$. From the perspective of our earlier discussion of the hierarchy of Abelian states, this plateau is strange because $\nu = 2 + \frac{1}{2}$ has a fractional part which does not have an odd denominator. Thus, it cannot arise in the hierarchy. Thus, we must consider states outside of the Abelian hierarchy, such as the 'Pfaffian state', which we discuss below. There is some numerical evidence that this particular state is a good description of the ground state at $\nu = \frac{5}{2}$.

The Pfaffian (ground) state takes the form

$$\Psi(z_j) \;=\; \prod_{j<k}(z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\,\left(\frac{1}{z_j - z_k}\right) . \tag{22.19}$$

In this equation the last factor is the Pfaffian: one chooses a specific ordering $z_1, z_2, ...$ of the electrons, chooses a pairing, takes the product of the indicated factor for all pairs in the chosen pairing, and finally takes the sum over all pairings, with the overall sign determined by the evenness or oddness of the order in which the zs appear. The result is a totally antisymmetric function. For example for four electrons the Pfaffian takes the form

$$\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} + \frac{1}{z_1 - z_3}\frac{1}{z_4 - z_2} + \frac{1}{z_1 - z_4}\frac{1}{z_2 - z_3} \; . \tag{22.20}$$

When the wavefunction is expanded in this way, in each term the electrons are grouped in pairs. Indeed, the Pfaffian state is reminiscent of the real-space form of the BCS pairing wavefunction; it is the quantum Hall incarnation of a p-wave superconducting state.

This wavefunction may be considered as a variational ansatz for electrons at filling fraction $\nu = \frac{1}{2}$ in the first excited Landau level (with both spins of the lowest Landau level filled) interacting through Coulomb interactions. This is the approach used in determining the relevance of this wavefunction to experiments, but for a theoretical study of quasiparticle statistics, it is more useful to consider this wavefunction (and quasihole excitations in it) as the exact zero-energy states of the three-body Hamiltonian below. Working with this Hamiltonian has the great advantage of making the entire discussion quite explicit and tractable.

$$H = \sum_i \sum_{j \neq i} \sum_{k \neq i,j} \delta'(z_i - z_j)\delta'(z_i - z_k) \tag{22.21}$$

This Hamiltonian annihilates wavefunctions for which every triplet of electrons i, j, k satisfies the condition that if i and j have relative angular momentum 1 then i and k must have relative angular momentum ≥ 2. The Pfaffian state (22.19) satisfies this condition since i and j have relative angular momentum 1 only when they are paired, but if i is paired with j, then it cannot be paired with k. Since the distance between particles is proportional to their relative angular momentum, this roughly translates into the following: by pairing up and and getting near particle j, particle i is able to stay further away from all of the other particles, thereby minimizing its interaction energy.

As in a superconductor, there are half-flux quantum excitations. The state

$$\Psi_{2\,\mathrm{qh}} = \prod_{j<k} (z_j - z_k)^2 \prod_j e^{-|z_j|^2/4} \cdot \mathrm{Pf}\, \left(\frac{(z_j - \eta_1)(z_k - \eta_2) + (z_j - \eta_2)(z_k - \eta_1)}{z_j - z_k} \right) . \tag{22.22}$$

has half-flux quantum quasiholes at η_1 and η_2. These excitations have charge $e/4$.

One includes $2n$ quasiholes at points η_α by modifying the Pfaffian in the manner

$$\mathrm{Pf}\, \left(\frac{1}{z_j - z_k} \right) \rightarrow \mathrm{Pf}\, \left(\frac{(z_j - \eta_\alpha)(z_j - \eta_\beta)...(z_k - \eta_\rho)(z_k - \eta_\sigma)... + (j \leftrightarrow k)}{z_j - z_k} \right) . \tag{22.23}$$

In this expression, the $2n$ quasiholes have been divided into two groups of n each (*i. e.* here $\alpha, \beta, ...$ and $\rho, \sigma, ...$), such that the quasiholes within each group always act on the same electron coordinates within an electron pair. There are apparently

$$\frac{(2n)!}{2\, n! n!} \tag{22.24}$$

ways of making such a division; the factor 1/2 arising from the possibility to swap the two groups of n as wholes. In fact, not all of these wavefunctions are linearly independent: the true dimension of this space of wave functions is actually 2^{n-1}.

Consider first the case of four quasiholes. The basic identity that has to be taken into account is, in its most primitive form,

$$\begin{aligned}(z_1-\eta_1)(z_1-\eta_2)(z_2-\eta_3)(z_2-\eta_4)-(z_1-\eta_1)(z_1-\eta_3)(z_2-\eta_4)(z_2-\eta_2)+(z_1\leftrightarrow z_2)\\ =(z_1-z_2)^2(\eta_1-\eta_4)(\eta_2-\eta_3)\ .\end{aligned}\tag{22.25}$$

It will be convenient to abbreviate the left-hand side to $(12)(34)-(13)(24)$. Then we have as an immediate consequence of (22.25) the relation

$$\frac{(12)(34)-(13)(24)}{(12)(34)-(14)(23)}\ =\ \frac{(\eta_1-\eta_4)(\eta_2-\eta_3)}{(\eta_1-\eta_3)(\eta_2-\eta_4)}\ .\tag{22.26}$$

It is interesting that on the right-hand side the basic projective invariant of four complex numbers, the cross-ratio, appears. For present purposes, however, the important point simply that it is independent of the zs. An immediate consequence is that for two electrons and four quasiholes the three apparently different ways of constructing quasihole states are reduced to two through the relation

$$(12)(34)(\eta_1-\eta_2)(\eta_3-\eta_4)+(13)(42)(\eta_1-\eta_3)(\eta_4-\eta_2)+(14)(23)(\eta_1-\eta_4)(\eta_2-\eta_3)\ =\ 0\,.\tag{22.27}$$

Now we want to argue that (22.26) and (22.27) still hold good for any even number of electrons, N_e. To see this we insert (22.25) into the Pfaffian of (22.23):

$$\begin{aligned}\mathrm{Pf}_{(13)(24)}&=\mathcal{A}\,(\frac{(13)(24)}{z_1-z_2}\ \frac{(13)(24)}{z_3-z_4}\ \frac{(13)(24)}{z_5-z_6}\ \ldots)\\ &=\mathcal{A}\,(\frac{(12)(34)-(z_1-z_2)^2\eta_{14}\eta_{23}}{z_1-z_2}\ \frac{(12)(34)-(z_3-z_4)^2\eta_{14}\eta_{23}}{z_3-z_4}\ldots)\end{aligned}\tag{22.28}$$

where $\eta_{ij}\equiv\eta_i-\eta_j$ and $\mathcal{A}$ denotes the instruction to antisymmetrize on the zs. If we expand,

$$\begin{aligned}&\mathcal{A}(\frac{(12)(34)-(z_1-z_2)^2\eta_{14}\eta_{23}}{z_1-z_2}\ \frac{(12)(34)-(z_3-z_4)^2\eta_{14}\eta_{23}}{z_3-z_4}\ \ldots)\\ &=\ \mathcal{A}\,(\frac{(12)(34)}{z_1-z_2}\ \frac{(12)(34)}{z_3-z_4}\ \ldots)\\ &\quad-\mathcal{A}\,((z_1-z_2)\eta_{14}\eta_{23}\ \frac{(12)(34)}{z_3-z_4}\ \ldots)\end{aligned}$$

$$+\mathcal{A}\left((z_1-z_2)\eta_{14}\eta_{23}\times(z_3-z_4)\eta_{14}\eta_{23}\times\frac{(12)(34)}{z_5-z_6}\ldots\right)+\ldots \quad (22.29)$$

there will be terms on the right hand side of (22.29) with zero, one, two, ..., N_e factors of $(z_i - z_j)$. Upon antisymmetrization, however, a term with k factors of $(z_i - z_j)$ would have to antisymmetrize $2k$ variables with a polynomial that is linear in each. Since this is impossible for $k > 1$, such terms vanish. Hence

$$\begin{aligned}\mathcal{A}\Big(&\frac{(12)(34)-(z_1-z_2)^2\eta_{14}\eta_{23}}{z_1-z_2}\ \frac{(12)(34)-(z_3-z_4)^2\eta_{14}\eta_{23}}{z_3-z_4}\ \ldots\Big)\\ &=\mathcal{A}\Big(\frac{(12)(34)}{z_1-z_2}\ \frac{(12)(34)}{z_3-z_4}\ \ldots\Big)\\ &\quad-\mathcal{A}\left((z_1-z_2)\eta_{14}\eta_{23}\ \frac{(12)(34)}{z_3-z_4}\ \ldots\right). \end{aligned} \quad (22.30)$$

Similarly, one has

$$\begin{aligned}\mathrm{Pf}_{(14)(23)}=&\mathcal{A}\Big(\frac{(12)(34)}{z_1-z_2}\ \frac{(12)(34)}{z_3-z_4}\ \ldots\Big)\\ &+\mathcal{A}\left((z_1-z_2)\eta_{13}\eta_{24}\ \frac{(12)(34)}{z_3-z_4}\ \ldots\right). \end{aligned} \quad (22.31)$$

From these we deduce the many-electron generalization of (22.25):

$$\mathrm{Pf}_{(12)(34)}-\mathrm{Pf}_{(14)(23)}=\frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}}\left(\mathrm{Pf}_{(12)(34)}-\mathrm{Pf}_{(13)(24)}\right). \quad (22.32)$$

This is a linear relation among the three pairing possibilities for two quasiholes. It depends on their coordinates but – remarkably – takes the same form for any number of electrons.

Thus, we have shown that there are two four-quasihole states. In the same way, it can be shown that there are 2^{n-1} $2n$-quasihole states. Braiding operations cause these states to be rotated into linear combinations of each other, as further analysis shows [].

Part VII

Localized and Extended Excitations in Dirty Systems

CHAPTER 23

Impurities in Solids

23.1 Impurity States

In the previous parts of this book, we have discussed the low-energy excitations which result from broken symmetry, criticality, or fractionalization. In this final part of the book, we discuss the low-energy excitations which result from the presence of 'dirt' or 'disorder' in a solid. By 'dirt' or 'disorder', we mean impurities which are frozen into the solid in some random way. Consider phosphorous impurities in silicon. Presumably, the true ground state of such a mixture is one in which the phosphorus atoms form a superlattice within the silicon lattice. However, this equilibrium is never reached when the alloy is made: it is cooled down before equilibrium is reached, and the phosphorus impurities get stuck (at least on time scales which are relevant for experiments) at random positions. These random static spatial fluctuations have interesting effects on the electronic states of the system: they can engender low-lying excitations and they can dramatically change the nature of such excitations. To see the significance of this, recall that, in the absence of a broken continuous symmetry, a system will generically form a gap between the ground state and all excited states. By tuning to a critical state – such as a Fermi liquid – we can arrange for a system to have low-energy excitations. However, in the presence of disorder, it will have low-energy excitations even without any tuning. To get a sense of why this should be so, suppose that, in the absence of disorder, there were a gap

to the creation of a quasiparticle. In the presence of disorder, the potential varies from place to place, and, in the thermodynamic limit, there will be some place in the system where the potential energy offsets the energy required to create a qasiparticle. In this region, it won't cost any energy to create a quasiparticle. Note that such an excitation, though low in energy, may not have large spatial extent. By the same token, if the system were gapless in the absence of disorder, as in the case of a Fermi liquid, then disorder can extend the critical state into a stable phase and it can cause the low-lying exciations of the system to become spatially localized.

Consider the simple example of a single phosphorus impurity in silicon. The phosphorus takes the place of one of the silicon atoms. Without the phosphorus, silicon is an insulator with a band gap E_g. Phosphorus has an extra electron compared to silicon and an extra positive charge in its nucleus. Let us neglect electron-electron interactions and write Schrödinger's equation for the extra electron in the form:

$$(H_{\text{lattice}} + H_{\text{impurity}})\,\psi(\mathbf{r}) = E\,\psi(\mathbf{r}) \tag{23.1}$$

where H_{lattice} is the Hamiltonian of a perfect lattice of silicon ions and H_{impurity} is the potential due to the extra proton at the phosphorus site. Let us write $\psi(\mathbf{r})$ in the form

$$\psi(\mathbf{r}) = \chi(\mathbf{r})\,u_{\mathbf{k}_0}(\mathbf{r})\,e^{i\mathbf{k}_0\cdot\mathbf{r}} \tag{23.2}$$

where $u_{\mathbf{k}_0}(\mathbf{r})\,e^{i\mathbf{k}_0\cdot\mathbf{r}}$ is the eigenstate of H_{lattice} at the conduction band minimum. Then $\chi(\mathbf{r})$ satisfies the equation:

$$\left(-\frac{\hbar^2}{2m^*}\nabla^2 - \frac{e^2}{\epsilon r}\right)\chi(\mathbf{r}) = E_b\,\chi(\mathbf{r}) \tag{23.3}$$

where m^* is the effective mass in the conduction band, E_b is measured from the bottom of the conduction band, and ϵ is the dielectric constant of silicon.

Hence, so long as we can neglect electron-electron interactions, the electron will be trapped in a bound state at the impurity. If the binding energy is much less than the band gap, $\frac{e^2}{2\epsilon a_B} \ll E_g$, then our neglect of electron-electron interactions is justified because the interaction will be too weak to excite electrons across the gap. (Note that a_B is the effective Bohr radius in silicon, $a_B = \frac{\epsilon\hbar^2}{m^*e^2} \approx 20\mathring{A}$). Hence, in the presence of impurities, there are states within the band gap. If there is a random distribution of phosphorus impurities, we expect a distribution of bound state energies so that the gap will be filled in. In other words, there will generically be states at

the chemical potential (although there will be a small density of states when the impurity density is small), unlike in a pure system, where it is possible for the chemical potential to lie within an energy gap.

23.2 Localization

23.2.1 Anderson Model

What is the nature of these electronic states when there are several impurities? Presumably, there is some mixing between the states at different impurities. One can imagine that this mixing leads to the formation of states which are a superposition of many impurity states so that they extend across the system. As we will see, this naive expectation is not always correct. Consider, first, the case of a high density of electrons and impurities. One would expect the kinetic energy to increase with the density, n, as $n^{2/d}$ while the potential energy should increase as $n^{1/d}$, so that the kinetic energy should dominate for large n. When the kinetic energy dominates, we might expect some kind of one-electron band theory to be valid. This would lead us to expect the system to be metallic. How about the low-density limit? In the case of a single impurity, the electron is trapped in a hydrogenic bound state, as we saw in the previous section. What happens when there is a small, finite density of impurities? One might expect exponentially small, but finite overlaps between the hydrogenic bound states so that a metal with very small bandwidth forms. This is not the case in the low-density limit, as we will see in this section.

Consider the Anderson model:

$$H = \sum_i \epsilon_i c_i^\dagger c_i - \sum_{ij} t_{ij} c_i^\dagger c_j + \text{h.c.} \tag{23.4}$$

In this model, we ignore electron-electron interactions. Spin is an inessential complication, so we take the electrons to be spinless. ϵ_i is the energy of an electron at impurity i, and t_{ij} is the hopping matrix element from impurity i to impurity j. These are random variables which are determined by the positions of the various impurities.

One could imagine such a model arising as an effective description of the many-impurity problem. Since the impurities are located at random positions, there tunneling matrix elements between their respective bound states will be random t_{ij}. One might also suppose that their different locations will lead to different effective environments and therefore different bound state energies ϵ_i. Anderson simplified the problem by arranging the

sites i on a lattice and setting $t_{ij} = t$ for nearest neighbor sites and zero otherwise. The effect of the randomness is encapsulated in the ϵ_i's, which are taken to be independent random variables which are equally likely to take any value $\epsilon_i \in [-W/2, W/2]$. This is a drastic simplification, but it already contains rich physics, as we will see.

For $W = 0$, there is no randomness, so the electronic states are simply Bloch waves. The system is metallic. For $W \ll t$, the Bloch waves are weakly scattered by the random potential. Now consider the opposite limit. For $t = 0$, all of the eigenstates are localized at individual sites. In other words, the eigenstates are $|i\rangle$, with eigenenergies ϵ_i. The system is insulating. Where is the transition between the $t/W = \infty$ and the $t/W = 0$ limits? Is it at finite $t/W = \infty$? The answer is yes, and, as a result, for t/W small, the electronic states are localized, and the system is insulating. (We say that a single-particle state is *localized* if it falls off as $e^{-r/\xi}$. ξ is called the *localization length*)

To see why this is true, consider perturbation theory in t/W. The perturbed eigenstates will, to lowest order, be of the form:

$$|i\rangle + \sum_j \frac{t}{\epsilon_i - \epsilon_j} |j\rangle \tag{23.5}$$

Perturbation theory will be valid so long as the second term is small. Since the ϵ_i's are random, one can only make probabilistic statements. The typical value of $\epsilon_i - \epsilon_j$ is $W/2$. The typical smallest value for any given i is $W/2z$, where z is the coordination number of the lattice. Hence, we expect corrections to be small – and, hence, perturbation theory to be valid – if $2tz/W < 1$. On the other hand, this is not a foolproof argument because there is always some probability for $\epsilon_i - \epsilon_j$ small. Nevertheless, it can be shown (Anderson; Frohlich and Spencer) that perturbation theory converges with a probability which approaches 1 in the thermodynamic limit. Hence, there is a regime at low density (small t) where the electronic states are localized and the system is insulating, by which we mean that the DC conductivity vanishes at $T = 0$. From our discussion of the hydrogenic bound state of an electron at an impurity in a semiconductor, it is not surprising that there are localized states in a disordered system. What *is* surprising is that if the disorder strength is sufficiently strong, then *all* states will be localized, i.e. the entire band will consist of localized states.

If the disorder strength is weaker than this limit, $2tz/W \gg 1$, then we can do perturbation theory in the random potential. In perturbation theory, the states will be Bloch waves, weakly-scattered by impurities as

we discuss in the next section. Such states are called *extended states*. The perturbative analysis is correct for states in the center of the band. Near the band edges, however, the perturbative analysis breaks down and states are localized: the correct measure of the electron kinetic energy, against which the disorder strength should be measured, is the energy relative to the band edge. Thus, for weak disorder, there are extended states near the band center and localized states near the band edges. When it lies in the region of extended states, it is metallic.

When the chemical potential lies in the region of localized states, the system is insulating. To see this, we write the DC conductance, g, of s system of non-interacting fermions of size L in the following form:

$$g(L) = \frac{\overline{\delta E}}{\overline{\Delta E}} \tag{23.6}$$

δE is the disorder-avergaged energy change of eigenstates at E_F when the boundary conditions are changed from periodic to antiperiodic. ΔE is the mean level spacing of eigenstates. (In order to derive such a relation analytically, one must consider disorder-averaged quantities.) If the states at E_F are localized with localization length ξ, then $\delta E \sim e^{-L/\xi}$, and the conductance will vanish in the thermodynamic limit.

This form for the conductance can be motivated as follows. The conductance is the response of the current to a change in the electromagnetic potential (which can be applied entirely at the boundary). However, the potential couples to the phase of the wavefunction, so the conductance can be viewed as the sensitivity of the wavefunction to changes of the boundary conditions. A systematic derivation proceeds from the Kubo formula.

23.2.2 Lifschitz Tails

As noted above, the states near the band edge are localized. The asymptotic density of such states can be obtained by the following argument due to Lifschitz. If the on-site disorder lies in the range $[-W/2, W/2]$ and the hopping parameter is t then there are no states below the band edge, $E_{\min} = -2dt - W/2$ on the d-dimensional hypercubic lattice. Consider a state near $E_{\min}$ which is spread over N sites. Then each of these sites has on-site energy in the range $[-W/2, -W/2 + I]$. The probability of having such a configuration of neighboring sites is $\sim (I/W)^N$. The kinetic energy of such a state is $\delta E \sim t\, N^{-2/d}$, so a state which is at energy $E_{\min} + \delta E$ is spread over N sites with $N \sim (\delta E/t)^{-d/2}$. Hence,

$$N(E_{\min} + \delta E) \sim e^{-c(t/\delta E)^{d/2}} \tag{23.7}$$

where c is a constant.

Thus, there isn't a van Hove singularity at the band edge at which $\frac{dN}{dE}$ diverges, but rather an essential singularity at which $\frac{dN}{dE}$ vanishes.

23.2.3 Anderson Insulators vs. Mott Insulators

The preceding analysis shows that a disorder-induced insulating state of non-interacting fermions is stable over a finite range of hopping strengths. What about the effect of electron-electron interactions? At high density, we expect the electrons to screen the impurity potentials, so that they are actually of the form $\frac{e^2}{\epsilon r}\, e^{-\lambda r}$. In the high-density limit, $\lambda \to \infty$, we expect the impurities to be extremely well-screened so that there won't be any impurity bound states. Hence, electron-electron interactions should, naively, enhance metallic behavior at high density. At low density, $\lambda \to 0$, so the impurities are not effectively screened, and we expect the preceeding analysis to hold. To summarize, electron-electron interactions tend to stabilize the metallic state by screening the random potential due to impurities. However, when the density is sufficiently low, this screening effect is too weak to prevent localization. As we will discuss in a later section, unscreened electron-electron Coulomb interactions do strongly influence the properties of the insulating state.

Of course, electron-electron interactions can do other things besides screen the impurities. They can lead to ordered states such as superconductivity or ferromagnetism, which would, of course, require a re-examination of the above analysis. In fact, sufficiently strong interactions can also cause insulating behavior even in an extremely clean system. At low densities where Coulomb interactions dominate over the kinetic energy, electrons will form a Wigner crystal. A tiny amount of disorder will pin this crystal, thereby making it insulating. Hence, it would be more accurate to say that electron-electron interactions enhance metallic behavior in the high-density limit but cause insulating behavior in the low-density limit.

The term 'Anderson insulator' is usually used for an insulator which would be metallic in the absence of impurities. The implications is that it can be more or less continuously deformed into a non-interacting fermion system which is insulating as a result of Anderson localization. A 'Mott insulator' is an insulator which would be metallic in the absence of electron-electron interactions. The Wigner crystal is an example of a Mott insulator. Another canonical example is given by the Hubbard model:

$$H = -\sum_{<i,j>} t c^\dagger_{i\alpha} c_{j\alpha} + \text{h.c.} U \sum_i n_{i\uparrow} n_{i\uparrow} \tag{23.8}$$

For $U = 0$, it would be metallic at half-filling. However, for sufficiently large U there is precisely one electron. Because the energy cost for having two electrons on any site is prohibitive, the system is insulating.

Doped semiconductors and many other real materials do not fall so neatly into one class or the other, however. While they might be metallic in the absence of both disorder and electron-electron interactions, their insulating behavior cannot be entirely ascribed to one or the other. Let us reconsider, momentarily, the formation of hydrogenic bound states in a doped semiconductor. If we were to neglect localization, we would still expect a sharp crossover between highly-conducting and poorly-conducting regimes at, respectively, high and low densities. The screened Coulomb interaction, $V(r) \sim \frac{e^2}{\epsilon r} e^{-\lambda r}$ has no bound states in the high-density limit, where λ is large. Therefore, as the dopant density is decreased, bound states will begin to appear when λ falls below a certain value $\sim a_B$. This would lead to small conductivity. (But not zero yet because the exponentially-small overlap between these bound states would still be expected to lead to metallic behavior, albeit with a very narrow band, were it not for localization.) Anderson localization of the impurity bound states will make the crossover from a metal to a poor metal into a sharp metal-insulator transition. Thus, it is more accurate to say that insulating behavior is due to the interplay between localization and interactions.

23.3 Physics of the Insulating State

At $T > 0$, any system will have finite conductivity since there will always be some probability that thermally-excited carriers can transport charge across the system. The conductivity of a finite-sized system will also always be non-vanishing. For instance, consider a non-interacting Fermi system whose single-particle states at the Fermi energy are localized with localization length ξ. Then the conductivity of the system will be $\sigma \sim L^{2-d} e^{-L/\xi}$ in a system of size L.

Thus, we define an insulator as any state in which the DC conductivity vanishes at $T = 0$ in the thermodynamic limit. In this section, we discuss the properties of a disorder-driven insulating state.

When a solid is driven into an insulating phase as a result of impurities, it is as far from a perfect crystalline lattice as possible, as far as its electronic properties are concerned. Not only is translational symmetry absent, but physical properties are dominated by localized excitations so that it is difficult to construct even an averaged description which is translationally

invariant. However, some of these properties – the DC and AC conductivities, magnetic response, and specific heat – can be deduced by simple arguments, which we present in this section.

23.3.1 Variable Range Hopping

One of the most celebrated results in the study of disordered insulators is Mott's formula for the DC conductivity due to *variable range hopping*. Let us consider an insulator at low temperatures. We imagine that the electrons are all in single-particle localized states. However, at finite-temperature, an electron can be excited by, say, a phonon so that it hops to another nearby localized state. Through a sequence of such hops, it can conduct electricity. One might imagine that electrons will only hop to the nearest neighboring states since the matrix element to hop to a more distant state will be exponentially (in distance) smaller. However, a state which is further away might be closer in energy, and therefore exponentially (in temperature) easier to reach. The competition between these competing effects leads to variable range hopping.

Suppose that an electron which is in a localized electronic state ψ hops to another one a distance R away. The number of available states per unit energy within such a distance is $\frac{4}{3}\pi R^3 N_F$, where N_F is the density of states at the Fermi energy. Hence, the typical smallest energy difference between the state ψ and a state within a distance R is the inverse of this:

$$W(R) \equiv \left(\begin{array}{c} \text{typical smallest energy difference between} \\ \text{states separated by a distance } R \end{array} \right) = \frac{3}{4\pi N_F R^3} \tag{23.9}$$

The conductivity associated with such a hopping process depends on the matrix element (squared) between the two states, which we take to be exponentially decaying $\sim e^{-2\alpha\overline{R}}$ (where $\overline{R} = 3R/4$ is the average hopping distance) and on the probability that the electron receives the activation energy needed to hop, which follows the Boltzmann distribution $\sim e^{-\beta W(R)}$ Hence, the conductivity due to such a hopping process varies as

$$\sigma_R \sim e^{-2\alpha\overline{R} - \beta W(R)} \tag{23.10}$$

The prefactor of the exponential contains various comparatively weak dependences on R; for instance, there should be a factor of R corresponding to the fact that a longer hop constitutes a larger contribution to the current.

There are many such hopping processes available to our electron in state ψ. The electron could hop a short distance with relatively large matrix

element, but the activation energy would probably be high in such a case. It could instead hop a long distance and find a state with low activation energy, but the matrix element would be very small. The optimal route is for the electron to hop a distance which minimizes the exponential in (23.10):

$$-2\alpha - \beta \frac{dW}{dR} = 0 \tag{23.11}$$

or

$$R_{\rm opt} = \left(\frac{3}{2\pi\alpha N_F T} \right)^{1/4} \tag{23.12}$$

Hence, as the temperature is decreased, the electron makes longer and longer hops. The resulting temperature dependence of the conductivity is:

$$\sigma(T) = \sigma_0(T)\, e^{-(T_0/T)^{1/4}} \tag{23.13}$$

where $\sigma_0(T)$ is relatively weakly-dependent on T, e.g. power-law. In d dimensions, the preceeding analysis immediately generalizes to

$$\sigma(T) = \sigma_0(T)\, e^{-(T_0/T)^{\frac{1}{d+1}}} \tag{23.14}$$

23.3.2 AC Conductivity

There are many situations in which the $\omega \gg k_B T$ frequency dependence of a quantity can be obtained from its DC temperature dependence by simply replacing $k_B T$ with ω. The conductivity of an insulator does not fall within this class. The AC conductivity is entirely different in form.

At frequency ω, an electron will oscillate between states separated in energy by ω. Let us think about these resonant states in terms of hydrogenic bound states. There are two possibilities. (1) R is large, so that they are essentially individual, independent bound states separated in energy by the difference between their respective potential wells. Then the dipole matrix element is $\sim R\, e^{-\alpha R}$; the current matrix element is $\sim \omega R\, e^{-\alpha R}$. Since the matrix element will be exponentially suppressed, the contribution of such resonant pairs will be small, and we turn to the second possibility. (2) The distance R is not very large, so that the energy eigenstates are actually linear combinations of the hydrogenic bound states. Then the energy difference between the eigenstates is essentially the matrix element of the Hamiltonian between the hydrogenic bound states $\sim I_0 e^{-\alpha R}$. The dipole matrix element is $\sim R$; the current matrix element is $\sim \omega R$. There is no exponential

suppression because the two eigenstates are different linear combinations of the same two bound states.

Then the conductivity due to this resonant pair is:

$$\sigma(\omega) \sim (\text{dipole matrix element})^2 \cdot (\text{available phase space}) \quad (23.15)$$

The available phase space is the density of states per unit volume multiplied by the volume in which the states can lie, given that they are a distance R apart: $\sim N_F^2 R^2 \Delta R$. ΔR is the size ξ of a localized state. Hence,

$$\sigma(\omega) \sim (\omega R)^2 R^2 \sim \omega^2 R^4 \quad (23.16)$$

However, by the arguments of the preceding paragraph,

$$R \sim \ln(I_0/\omega) \quad (23.17)$$

Thus, we have the following formula for the conductivity:

$$\sigma(\omega) \sim \omega^2 \ln^4(I_0/\omega) \quad (23.18)$$

This formula also holds at finite temperatures, so long as $\omega \gg T$. However, when the frequency is decreased, the conductivity will cross over to the variable-range hopping form.

23.3.3 Effect of Coulomb Interactions

In an insulator, Coulomb interactions are not screened at long-wavelengths and low-frequencies because there aren't mobile charges. Thus, their effects are particularly strong. One effect, pointed out by Efros and Shklovskii is a suppression of the single-particle density-of-states at the Fermi energy. The basic physics is as follows. Suppose that there were two (strongly-localized) single-particle states, one just below the Fermi energy and one just above. The 'single-particle energy cost' associated with promoting an electron from the lower one to the higher one would be at least partially offset by the negative Coulomb energy associated with the resulting particle-hole pair. If the distance between the states is small, this Coulomb energy will be large and will overcompensate the single-particle energy. Hence, the states must be far apart and, therefore, the single-particle density of states (per unit volume) must be small.

More quantitatively, suppose the single-particle energy separation between an un-occupied state above the Fermi energy and an occupied one below the Fermi energy is $\Delta E_{\text{unocc}-\text{occ}}$ while the distance between the states

is r. The 'single-particle energies' include the Coulomb interaction energy between the two states under consideration and all of the other occupied states in the system. Then it must be the case that

$$\Delta E_{\mathrm{unocc-occ}} > \frac{e^2}{r} \tag{23.19}$$

or else the energy would be lowered by transferring an electron from the occupied state to the unoccupied one, contradicting our assumption. The right-hand-side is the Coulomb energy between the electron and the hole which results when such a transfer occurs.

Hence, the number of states per unit volume which are within energy ΔE of the Fermi surface is

$$n(\Delta E) \sim \frac{1}{r^3} < (\Delta E)^3 \tag{23.20}$$

from which we see that the density of states is

$$N(\Delta E) \sim (\Delta E)^2 \tag{23.21}$$

Thus, the single-particle density of states vanishes at the Fermi energy, and is suppressed near it. This effect is called the *Coulomb gap*.

Suppose we define energy, length scales Δ, r_Δ by

$$\begin{aligned} \frac{e^2}{r_\Delta} &= \Delta \\ N_0 r_\Delta^3 \Delta &= 1 \end{aligned} \tag{23.22}$$

where N_0 is the density of states in the absence of long-range Coulomb interactions, to which the density of states returns when the distance from the Fermi surface exceeds $\Delta = e^3\sqrt{N_0}$, as shown in figure 23.1.

One could measure the single-particle density of states by performing a tunneling experiment in which current tunnels into a disordered insulator from a metallic lead to which it is contacted. The differential conductance dI/dV will be proportional to the density of states. According to the above arguments, dI/dV will be suppressed at zero voltage and will increase with voltage as V^2. However, when we change the applied voltage on our disordered insulator, the Fermi energy moves with it. Shouldn't the differential conductance remain zero? The answer is no. As we change the voltage, dI/dV increases with voltage as V^2, as if the chemical potential had not moved. However, if we raise the temperature of the insulator, change the applied voltage to V_0, and cool it down again, we find that dI/dV varies

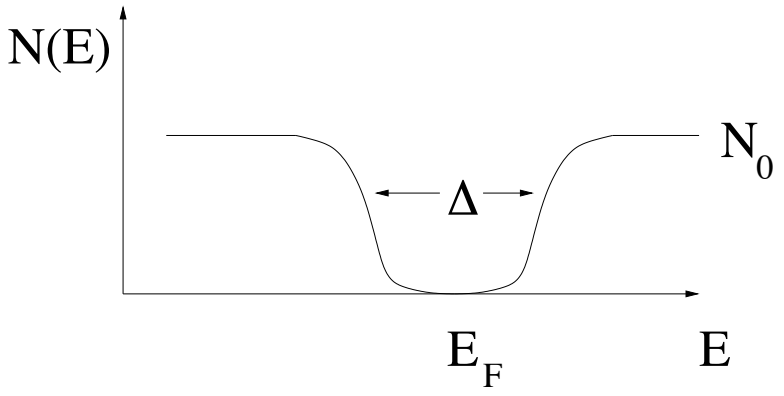


Figure 23.1: The single-particle density-of-states is suppressed at the Fermi energy. For $|E_{E_F}| > \Delta$, the single-particle density-of-states returns to N_0.

as $(V - V_0)^2$, so that the Fermi energy has moved to V_0. Essentially, the suppression of the density of states remains wherever it was when the system last equilibrated. The non-equilibrium nature of the Coulomb gap is related to the assumption above that the only Coulomb interaction energy which must be accounted for is that between the excited electron and the hole left behind, $-e^2/r$. However, when an electron is excited from an occupied state to an unoccupied one, all of the other electrons can and will rearrange themselves. This will occur in such a way as to allow a state near the Fermi surface. Thus, if an experiment were done slowly enough to allow such re-equilibration, the suppression calculated above would not occur.

The suppression of the density-of states near the Fermi energy has consequences for variable-range hopping. Recall that we estimated that within a region of size R, the typical smallest energy separation ΔE is $\Delta\, E \sim 1/(N_0 R^3)$. However, $N(\Delta E) \sim (\Delta E)^2$, or $\Delta E \sim 1/R$, as expected if the energy is determined by Coulomb interactions. Following the minimization procedure of subsection 23.3.1, we find that the variable-range-hopping formula in the presence of Coulomb interactions (in arbitrary dimension) is:

$$\sigma \sim e^{-(T_0/T)^{1/2}} \tag{23.23}$$

23.3.4 Magnetic Properties

In the absence of electron-electron interactions, there will be two electrons in each localized single-particle state, and the ground state will be para-

magnetic. However, it is clearly unrealistic to ignore electron-electron interactions. We expect them to prevent two electrons from occupying the same state. Thus, we must consider the spin-spin interactions between the electrons in different localized states.

An isolated impurity state, occupied by a single electron will have a Curie form for the magnetic susceptibility:

$$\chi_c(T) \sim \frac{1}{T} \tag{23.24}$$

If we have many such spins but ignore the interactions between them, we will have a susceptibility of the same form, proportional to the density of spins.

If, however, we include the interactions between spins, the susceptibility will have this form at high-temperatures, but once T becomes comparable to or less than the typical spin-spin interaction strength J, $\chi(T)$ will exhibit different behavior. There are several possibilities. One, which occurs in, for instance, Cu:Mn and Au:Fe alloys is *spin glass* behavior, in which the susceptibility has a cusp at a temperature $T_{\rm SG}$, as shown in figure 23.2. The non-linear susceptibility actually diverges at $T_{\rm SG}$. Spin glasses are rather incompletely understood, and many of their properties are still the subject of controversy. However, the basic caricature is that the spins order, but unlike in a ferromagnet, in which they all point in the same direction, the spins point in a random (but fixed) direction which varies from point to point. At the freezing temperature, $T_{\rm SG}$, the system undergoes a transition into such a state in which $\langle \mathbf{S}(\mathbf{x}) \rangle = \mathbf{s_0}(\mathbf{x})$ where $\mathbf{s_0}(\mathbf{x})$ is a randomly-varying function of $\mathbf{x}$. There is some disagreement about whether or not such a state has infinitely-many distinct degenerate ground states (over and above the degeneracy due to broken rotational symmetry) and the related issue of whether or not there is a sharp phase transition in non-zero magnetic field. These issues are discussed in (refs.).

Spin-glass order is believed to require spin-spin interactions of both ferromagnetic and antiferromagnetic signs which are sufficiently disordered as to frustrate either of these ordering tendencies of clean systems. As a result, the spins order in a novel type of ordered state which can occur only in a dirty system. An alternate possibility which can occur in a system which has predominantly antiferromagnetic interactions (but of randomly-varying strengths) is that the spins do not order but rather form a *random singlet phase*. In such a phase, the system essentially breaks up into a set of pairs of spins which interact with strong antiferromagnetic exchange couplings J_i.

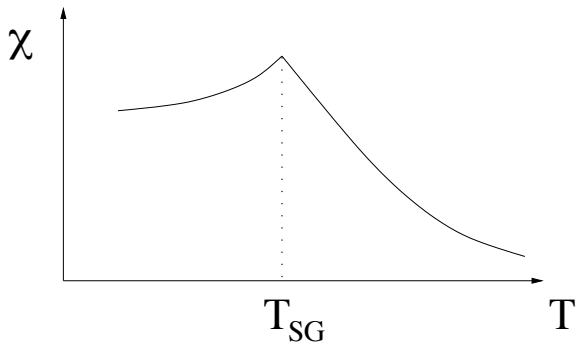


Figure 23.2: The susceptibility of a spin-glass has a kink at $T_{\rm SG}$.

Each of these pairs has susceptibility

$$\chi_{i^{\rm th}\,{\rm pair}}(T) = \frac{1}{T}\,\frac{2}{3+e^{\beta J_i}} \tag{23.25}$$

The susceptibility of the system must be summed over all pairs with some probability distribution, $P(J)$, for the values of J:

$$\chi(T) = \frac{1}{T}\int dJ\; P(J)\,\frac{2}{3+e^{\beta J}} \tag{23.26}$$

At a given temperature, T, there will be some spins whose J is smaller than T. Such spins will not have formed singlets yet, so they will give a Curie-like contribution to the susceptibility, which will be the dominant contribution:

$$\chi_{\rm unpaired}(T) \sim \frac{\rho(T)}{T} \tag{23.27}$$

where $\rho(T)$ is the number of unpaired spins at temperature T.

The function $\rho(T)$ depends on the distribution of exchange couplings, $P(J)$. Let us suppose that there is a small density n of randomly-situated localized spins, and suppose that the coupling between spins separated by a distance r is $J(r) = J_0\,e^{-2r/a}$ for some constant a which is roughly the size of the localized states. With this functional form for J, we know $P(J)$ if we know $P(R)$ such that $P(R)dR$ is the probability that a spin has its nearest neighbor lying in a thin shell between R and $R+dR$. We can compute $P(R)$ in the following way. Define $p(R)$ as the probability that the nearest neighbor is a distance R or less. Then

$$P(R) = \frac{dp}{dR} \tag{23.28}$$

The probability that the nearest neighbor is between R and $R+dR$ is given by the number of impurities in a thin shell at R, $4\pi nR^2 dR$, multiplied by the probability that there is no closer neighbor, $1-p(R)$

$$P(R) = 4\pi nR^2(1-p(R)) \tag{23.29}$$

or, simply,

$$\frac{dp}{dR} = 4\pi nR^2(1-p) \tag{23.30}$$

Solving this differential equation, we have

$$p(R) = 1 - e^{-4\pi nR^3} \quad \Rightarrow \quad P(R) = 4\pi nR^2\, e^{-4\pi nR^3} \tag{23.31}$$

We substitute this expression into (23.26),

$$\chi(T) = \frac{1}{T}\int dR\, 4\pi nR^2\, e^{-\frac{4}{3}\pi nR^3}\, \frac{2}{3+e^{\beta J(R)}} \tag{23.32}$$

In the low-temperature limit, this is simply

$$\chi(T) \approx \frac{1}{T}\int dR\, 8\pi nR^2\, e^{-\frac{4}{3}\pi nR^3}\, e^{-\beta J_0\, \exp(-2R/a)} \tag{23.33}$$

The integrand has a saddle-point at $R \approx \frac{a}{2}\ln(J_0/T)$, dropping $\ln\ln T$ terms which are subleading at low T. Hence, the integral can be evaluated by saddle-point approximation:

$$\chi(T) \sim \frac{1}{T}\, e^{-\frac{1}{6}\pi na^3 \ln^3(J_0/T)} \tag{23.34}$$

The physical interpretation of this result is simple. Unpaired spins at temperature T are those spins whose largest coupling to the other spins is less than T:

$$J(R) \sim T \quad \Rightarrow \quad R \sim \frac{a}{2}\ln\left(J_0/T\right) \tag{23.35}$$

The density of such spins should therefore be:

$$\rho(T) \sim e^{-\frac{1}{6}\pi na^3 \ln^3(J_0/T)} \tag{23.36}$$

as we found above.

The approximate form which we used for $P(R)$ underestimates the number of unpaired spins, thereby underestimating the susceptibility. We took $P(R)$ to be the probability of having a neighboring spin at R, namely $4\pi nR^2$, multiplied by the probability that there were no closer neighbors, $1-p(R)$.

However, the neighboring spin at R might itself have already paired up with a third spin, in which case our spin must look for an even further neighbor with which to form a singlet pair. Thus, we should multiply by the probability that the neighbor at distance R doesn't have a closer neighbor of its own, which is roughly $1 - p(R)$. (It should really be a fraction of this because we should exclude the region which is less than a distance R from both spins since this region has already been accounted for by the first factor of $1 - p(R)$.) Hence,

$$P(R) = 4\pi n R^2 (1 - p(R))^2 \tag{23.37}$$

or

$$P(R) = \frac{4\pi n R^2}{\left(1 + \frac{4}{3}\pi n R^3\right)^2} \tag{23.38}$$

Substituting this into (23.26), we find the low-temperature form

$$\chi(T) \approx \frac{1}{T} \int dR \, \frac{8\pi n R^2}{\left(1 + \frac{4}{3}\pi n R^3\right)^2} \, e^{-\beta J_0 \, \exp(-2R/a)} \tag{23.39}$$

The saddle-point is again at $R \approx \frac{a}{2} \ln(J_0/T)$, up to $\ln \ln T$ terms. Hence, the saddle-point approximation for the integral is:

$$\chi(T) \sim \frac{1}{T \, \ln^3 (J_0/T)} \tag{23.40}$$

Hence, the number of unpaired spins,

$$\rho(T) \sim \frac{1}{\ln^3 (J_0/T)} \tag{23.41}$$

goes to zero very slowly; so slowly, in fact, that the susceptibility diverges at $T \to 0$.

In $1D$, the asymptotic low-temperature behavior can be found exactly (refs.).

Exercise: Compute the specific heat at low T in a random singlet phase.

23.4 Physics of the Metallic State

23.4.1 Disorder-Averaged Perturbation Theory

A metallic state in a disordered system can be approached perturbatively from the clean metallic state. In order to do so, we will need to handle

the perturbation caused by impurity scattering. In this section, we describe the simplest way to do so. Let us imagine breaking our system into a large number N cells, each of which is macroscopic. Then the configuration of impurities in each cell will be different. If we compute an extensive quantity, such as the free energy, then it will be essentially (up to boundary terms) the sum of the free energies of each of the cells. Since each cell will have a different impurity configuration, the sum over cells will be an average over impurity configurations. Hence, the free energy in a disordered system can be computed by computing the free energy for a fixed realization of the disorder and then averaging over all realizations of the disorder (with respect to some probability distribution for the disorder). Such an approximation (which we take to be exact in the thermodynamic limit) for the free energy is called the *quenched approximation* and such an average is called a *quenched average*.

Let us consider a perturbative computation of the free energy. For the sake of concreteness, suppose that the action is

$$S[\psi, \psi^\dagger] = S_{\text{clean}}[\psi, \psi^\dagger]] + \int d\tau \, d^3x \, V(x)\psi^\dagger(x,\tau)\psi(x,\tau) \tag{23.42}$$

Then there is a vertex in the theory of the form depicted in figure 23.3a. To each such vertex, we assign a factor of the Fourier transform of the potential, $V(q)$ and integrate over $\mathbf{q}$. We can therefore organize the free energy in powers of $V(q)$:

$$F\left[V(q)\right] = \sum_n \frac{1}{n!} \int \ldots \int V(q_1) \ldots V(q_n) f(q_1, \ldots q_n) \tag{23.43}$$

If we think about computing the free energy perturbatively in some coupling constant in S_{clean} (which includes the simplest case, namely that S_{clean} is a free theory), then the free energy will only contain *connected diagrams*.

Now suppose that we average over $V(q)$. Let us consider, for simplicity, a distribution for $V(q)$ of the form

$$\begin{aligned} \overline{V(\mathbf{q})} &= 0 \\ \overline{V(\mathbf{q})V(\mathbf{q}')} &= n_i v^2 \delta(\mathbf{q} + \mathbf{q}') \end{aligned} \tag{23.44}$$

where n_i is the impurity concentration and v is a measure of the strength of the scattering potential due to each impurity. All higher moments are taken to be factorizable into products of $\overline{VV}$. In other words, we will average

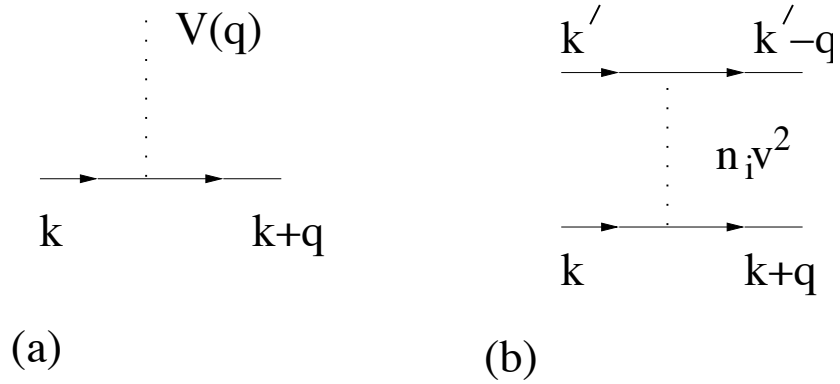


Figure 23.3: (a) The impurity vertex for a fixed $V(q)$. (b) The induced 'interaction' which results from averaging over the propability distribution for $V(q)$.

$F[V(q)]$ over the distribution:

$$\overline{F} = \int \mathcal{D}V \, F[V] \, e^{-\int d^d x V^2(x)/2n_i v^2} \tag{23.45}$$

Integrating over V ties together different impurity vertices with impurity lines in all possible ways, according to Wick's theorem. To each such line, we attach a factor of $n_i v^2 \delta(\mathbf{q} + \mathbf{q}')$, as shown in figure 23.3b.

In essence, we have a new Feynman rule associated with impurity lines connecting impurity vertices. There are two important differences with most of the Feynman rules which we have considered thus far. (1) No energy flows along these lines, only momentum. (2) The diagrams must be connected even if all of the impurity lines are cut. In particular, if the clean theory is a non-interacting theory, then the disorder-averaged perturbation theory can have no closed electron loops. For example, figure 23.4a is allowed, but 23.4b is not.

Correlation functions are obtained from functional derivatives of the free energy with respect to sources, so the same rules apply to them.

23.4.2 Lifetime, Mean-Free-Path

The properties of a dirty metal can be derived neatly with the use of an effective field theory which we will discuss in the next chapter. Therefore,

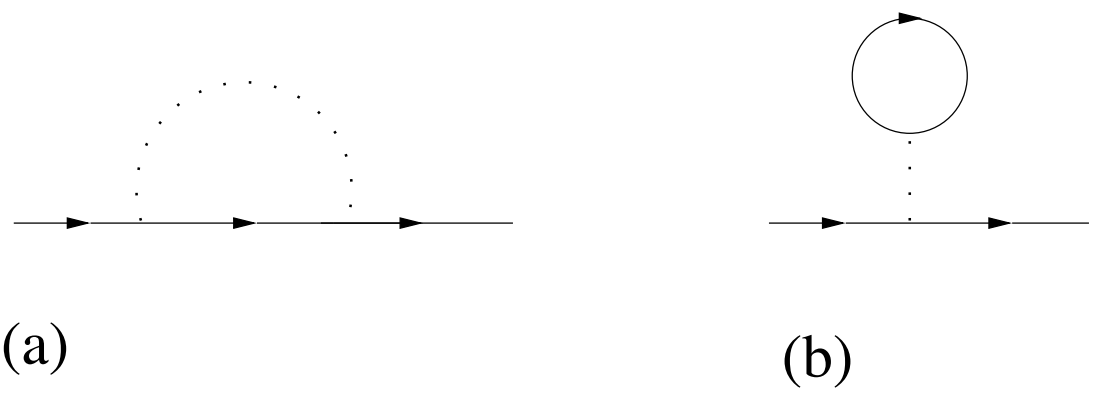


Figure 23.4: (a) An allowed diagram in disorder-averaged perturbation theory. This diagram is the lowest-order contribution to the self-energy. (b) A disallowed diargam in disorder-averaged perturbation theory.

we will just summarize these properties here and describe their qualitative features.

Many of the properties of a dirty metal are qualitatively the same as those of a clean metal: the compressibility is finite and approximately temperature-independent at low-temperatures, as is the magnetic susceptibility. The specific heat is linear in T. The principal new feature in a dirty metal is that the DC conductivity does not diverge at low-temperatures; it approaches a constant value at $T = 0$.

This can be seen with a perturbative calculation. Consider the electron Green function. The lowest-order contribution to the self-energy comes from the diagram depicted in figure 23.4a. We will call the imaginary part of this diagram $(i/2\tau)\,\mathrm{sgn}(\epsilon_n)$, for reasons which will be clear shortly. If the interaction between the electrons and the impurities is

$$H_{\rm imp} = \int \frac{d^3k}{(2\pi)^3}\,\frac{d^3k'}{(2\pi)^3}\,\frac{d\epsilon}{2\pi}\,V(k'-k)\,\psi^\dagger(k,\epsilon)\,\psi(k',\epsilon) \tag{23.46}$$

where, for simplicity, we take

$$\overline{V(\mathbf{q})} = 0\ , \qquad \overline{V(\mathbf{q})\,V(\mathbf{q'})} = n_i v^2\,\delta^{(3)}(\mathbf{q}+\mathbf{q'}) \tag{23.47}$$

The disorder-averaged value of this diagram is equal to

$$\frac{1}{2\tau}\,\mathrm{sgn}(\epsilon_n) \equiv \mathrm{Im}\Sigma(\epsilon,k)$$

$$= \mathrm{Im}\left\{ \int \frac{d^3k'}{(2\pi)^3} \frac{\overline{V(\mathbf{k}-\mathbf{k}')\,V(\mathbf{k}'-\mathbf{k})}}{i\epsilon_n - \xi_k} \right\}$$
$$= 2\pi N_F\, n_i v^2\, \mathrm{sgn}(\epsilon_n) \tag{23.48}$$

The real part merely renormalizes the chemical potential.

If we add this self-energy to the inverse Green function, we have a Matsubara Green function of the form:

$$\mathcal{G}(\mathbf{k},\epsilon_n) = \frac{1}{i\epsilon_n - \xi_k + \frac{i}{2\tau}\mathrm{sgn}(\epsilon_n)} \tag{23.49}$$

The corresponding retarded and advanced Green functions are:

$$G_{\mathrm{ret,adv}}(\mathbf{k},\epsilon) = \frac{1}{\epsilon - \xi_k \pm \frac{i}{2\tau}} \tag{23.50}$$

In the presence of impurities, momentum eignenstates are no longer energy eigenstates. The inverse-lifetime $1/\tau$ is the energy-uncertainty of a momentum eigenstate. Writing $\xi_k = v_F|k-k_F|$, we see that the mean-free-path $\ell = v_F\tau$ is the momentum uncertainty of an energy eigenstate. It is essentially the average distance between scatterings of an electron since the momentum is the inverse of the spatial rate of change of the wavefunction and the latter is scrambled at collisions.

Fourier transforming the fermion Green function at the Fermi energy into real space, we find

$$G_{\mathrm{ret}}(\mathbf{x},\epsilon) \sim e^{-|x|/\ell} \tag{23.51}$$

Thus, the single-particle Green function decays exponentially, unlike in a clean system, where it has power-law behavior.

23.4.3 Conductivity

With this single-particle Green function, we can perturbatively compute the conductivity. Consider the basic 'bubble' diagram, figure 23.5, at $\mathbf{q} = 0$ but with each electron line representing the Green function (23.49) rather than the bare Green function.

$$\langle j(0,\omega_n)\, j(0,-\omega_n)\rangle =$$
$$\frac{1}{3}v_F^2 N_F \int d\xi_k \frac{1}{\beta}\sum_n \frac{1}{i\epsilon_n + i\omega_m - \xi_k + \frac{i}{2\tau_{n+m}}} \frac{1}{i\epsilon_n - \xi_k + \frac{i}{2\tau_n}} \tag{23.52}$$

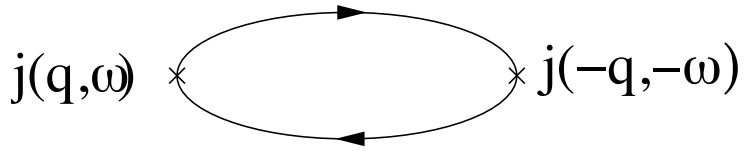


Figure 23.5: The basic conductivity bubble. The electron lines represent lines dressed by the one-loop self-energy correction computed in the previous section.

where we have introduced the shorthand $\tau_n = \tau\,\mathrm{sgn}(\epsilon_n)$. The factor of $1/3$ comes from the angular average over the Fermi surface. As usual, we convert the sum over Matsubara frequencies to an integral

$$\langle j(0,\omega_n)\, j(0,-\omega_n)\rangle = \\ -\frac{v_F^2}{3} N_F \int d\xi_k \int_c \frac{dz}{2\pi i}\, n_F(z) \frac{1}{z+i\omega_n-\xi_k+\frac{i}{2\tau_{z+m}}} \frac{1}{z-\xi_k+\frac{i}{2\tau_z}} \quad (23.53)$$

The integrand is non-analytic if z or $z+i\omega_m$ is real so the integral is equal to the contributions from the contour as is passes just above and just below these lines. Let us assume that $\omega_n > 0$.

$$\langle j(0,\omega_n)\, j(0,-\omega_n)\rangle = \\ -\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i}\, n_F(\epsilon) \frac{1}{\epsilon+i\omega_n-\xi_k+\frac{i}{2\tau}} \left(\frac{1}{\epsilon-\xi_k+\frac{i}{2\tau}} - \frac{1}{\epsilon-\xi_k-\frac{i}{2\tau}} \right) \\ -\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i}\, n_F(\epsilon) \left(\frac{1}{\epsilon-\xi_k+\frac{i}{2\tau}} - \frac{1}{\epsilon-\xi_k-\frac{i}{2\tau}} \right) \frac{1}{\epsilon-i\omega_n-\xi_k-\frac{i}{2\tau}} \quad (23.54)$$

Only the second term in parenthesis in the first line of (23.54) contributes to the imaginary part of the integral and only the first term in parenthesis in the second line of (23.54) contributes to the imaginary part of the integral because the other terms have poles on the same side of the axis. Hence, we

have

$$\begin{aligned}\langle j(0,\omega_n)\, j(0,-\omega_n)\rangle = \\ &\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i}\, n_F(\epsilon) \frac{1}{\epsilon + i\omega_n - \xi_k + \frac{i}{2\tau}} \frac{1}{\epsilon - \xi_k - \frac{i}{2\tau}} \\ -\ &\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi i}\, n_F(\epsilon) \frac{1}{\epsilon - \xi_k + \frac{i}{2\tau}} \frac{1}{\epsilon - i\omega_n - \xi_k - \frac{i}{2\tau}}\end{aligned} \tag{23.55}$$

Taking $i\omega_n \to \omega$ and dividing by ω, we have

$$\begin{aligned}\sigma(\omega) = \\ &\frac{1}{\omega}\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\, n_F(\epsilon) \frac{1}{\epsilon + \omega - \xi_k + \frac{i}{2\tau}} \frac{1}{\epsilon - \xi_k - \frac{i}{2\tau}} \\ -\ &\frac{1}{\omega}\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi}\, n_F(\epsilon) \frac{1}{\epsilon - \xi_k + \frac{i}{2\tau}} \frac{1}{\epsilon - \omega - \xi_k - \frac{i}{2\tau}}\end{aligned} \tag{23.56}$$

Shifting $\epsilon \to \epsilon + \omega$ in the second integral and combining the two terms, we have:

$$\begin{aligned}\sigma(\omega) = \\ &\frac{1}{\omega}\frac{v_F^2}{3} N_F \int d\xi_k \int_{-\infty}^{\infty} \frac{d\epsilon}{2\pi} \left(n_F(\epsilon) - n_F(\epsilon+\omega)\right) \frac{1}{\epsilon + \omega - \xi_k + \frac{i}{2\tau}} \frac{1}{\epsilon - \xi_k - \frac{i}{2\tau}}\end{aligned} \tag{23.57}$$

Taking $\omega \to 0$ and noting that $\left(n_F(\epsilon) - n_F(\epsilon+\omega)\right)/\omega \to \delta(\epsilon)$, we have

$$\begin{aligned}\sigma_{\rm DC} &= \frac{v_F^2}{6\pi} N_F \int d\xi_k \frac{1}{\xi_k^2 + \left(\frac{1}{2\tau}\right)^2} \\ &= \frac{1}{3} v_F^2 \tau\, N_F \\ &\equiv \frac{1}{3\pi^2} k_F^2 \ell\end{aligned} \tag{23.58}$$

As expected, it is a constant determined by the lifetime due to impurity scattering. In the final line, we have written $v_F N_F = k_F^2/\pi^2$. (This is the $3D$ expression. In general, it is proportional to k_F^{d-1}.)

The finite conductivity of electrons reflects the fact that electrons move diffusively rather than ballistically. If we define a diffusion constant $D = \frac{1}{3} v_F^2 \tau$, then $\sigma = N_F D$, which is the Einstein relation. Further insight into this can be gained by computing the density-density correlation function, to which we turn in the next section.

Note that we have not included vertex corrections in our calculation. This is not a problem because there are no vertex corrections to the *current vertex* to this order, as may be seen from the Ward identity, which relates vertex corrections to the derivative of the self-energy.

$$\Gamma^\mu(p,p,0) = \frac{\partial \Sigma(p)}{\partial p_\mu} \tag{23.59}$$

As usual, we have used the notation $p_\mu = (\omega, p_i)$, $i = 1,2,3$. Since the self-energy is momentum-independent, there are no vertex corrections to the current vertices Γ^0. On the other hand, the self-energy is energy-dependent, so there are important vertex corrections to the *density vertex*, Γ^0.

Note that we computed this integral by performing the energy integrals first and then the momentum integrals. When these integrations do not commute, e.g. when the integrands are formally divergent by power-counting, it is safest to introduce a momentum cutoff. When this is done, the order of integrations does not matter. A contour integration over ξ_k cannot be done, however, as a result of the cutoff, so the most natural way to proceed is to perform the energy integral first. The resulting momentum integral is then convergent in most cases. When the integrals are convergent, it doesn't matter in what order they are done. When they do not converge, as in the present case, the order does matter. The order in which we did the integrals is safest since it can be done with a momentum cutoff. If we had ignored this subtlety and done the integrals in the opposite order, and done the momentum integral without a cutoff (e.g. by contour integration) then we would have actually obtained the same result for the *real part* of the current-current correlation function. However, the imaginary part, which includes a term which cancels the diamagnetic term, would be missing.

23.4.4 Diffusion

The finite diffusion constant of electrons, D, which we found indirectly in the DC conductivity above, can be obtained explicitly by computing the density-density correlation function. As we will see, this correlation function has a very different form in a disordered electron system.

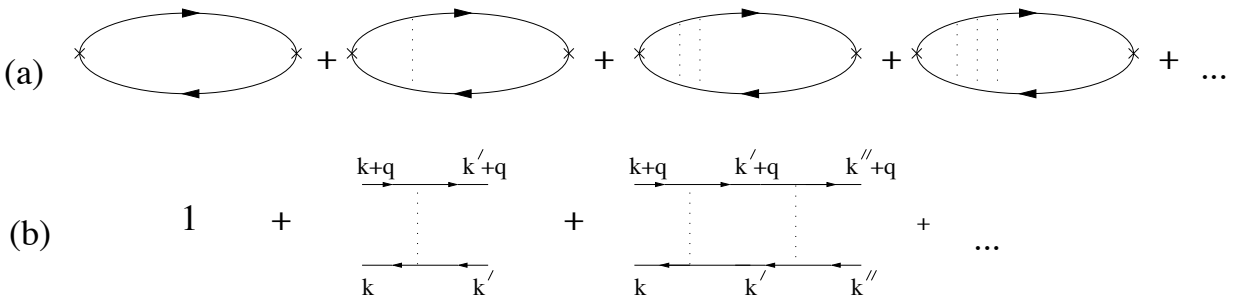


Figure 23.6: (a) A geometric series of diagrams which contribute to the density-density correlation function. (b) The quantity $\Gamma(q, \epsilon_n, \epsilon_n+\omega_m)$ which is the ladder sum which sits inside the density-density bubble.

Consider the set of diagrams in figure (23.6)a (with, again, the electron Green function dressed with the one-loop self-energy, i.e. the lifetime). All of these diagrams contribute to the density-density correlation function. They form a geometric series which we can sum once we have obtained the value of the first. The higher-order diagrams constitute vertex corrections to the first diagram in the series. As noted in the previous subsection, there are no vertex corrections to the conductivity (i.e. the corresponding diagrams for the current-current correlation function do not correct the vertex in the DC limit, which is why we computed only a single diagram). However, such vertex corrections are important for the density-density correlation function. We will show that the resulting correlation function is consistent with the Ward identity.

The sum of these diagrams is

$$\langle \rho(q,\omega_m)\, \rho(-q,-\omega_m)\rangle = \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, \Gamma(q,\epsilon_n,\epsilon_n+\omega_m)\, \mathcal{G}(\mathbf{k},\epsilon_n)\, \mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m) \tag{23.60}$$

where $\Gamma(q, \epsilon_n, \epsilon_n + \omega_m)$ is the infinite series of ladder diagrams in (23.6)b

$$\Gamma(q,\epsilon_n,\epsilon_n+\omega_m) = 1 + n_i v^2 I(q,\epsilon_n,\epsilon_n+\omega_m) + \left(n_i v^2\right)^2 \left(I(q,\epsilon_n,\epsilon_n+\omega_m)\right)^2 + \ldots \tag{23.61}$$

and the integral $I(q, \epsilon_n, \epsilon_n + \omega_m)$ is given by

$$I(q,\epsilon_n,\epsilon_n+\omega_m) = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \mathcal{G}(\mathbf{k},\epsilon_n)\, \mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m) \tag{23.62}$$

Written explicitly,

$$I(q,\epsilon_n,\epsilon_n+\omega_m) = \frac{N_F}{2}\int d\xi_k\, d(\cos\theta)\,\frac{1}{i\epsilon_n + i\omega_m - \xi_k - v_F q\cos\theta + \frac{i}{2\tau_{n+m}}}\,\frac{1}{i\epsilon_n - \xi_k + \frac{i}{2\tau_n}} \tag{23.63}$$

where we have used the shorthand $\tau_n = \tau\mathrm{sgn}(\epsilon_n)$. Integrating ξ_k, we have:

$$I(q,\epsilon_n,\epsilon_n+\omega_m) = i\pi N_F\int d(\cos\theta)\frac{\theta(\epsilon_n)-\theta(\epsilon_n+\omega_m)}{i\omega_m - v_F q\cos\theta + \frac{i}{\tau}\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]} \tag{23.64}$$

Integrating $\cos\theta$,

$$\begin{aligned} I(q,\epsilon_n,\epsilon_n+\omega_m) &= \\ &\frac{i\pi N_F}{v_F q}\left[\theta(\epsilon_n)-\theta(\epsilon_n+\omega_m)\right]\ln\left(\frac{i\omega_m - v_F q + \frac{i}{\tau}\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]}{i\omega_m + v_F q + \frac{i}{\tau}\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]}\right) \\ = &\frac{i\pi N_F}{v_F q}\left[\theta(\epsilon_n)-\theta(\epsilon_n+\omega_m)\right]\ln\left(\frac{1+(\omega_m\tau + iv_F\tau q)\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]}{1+(\omega_m\tau - iv_F\tau q)\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]}\right) \end{aligned} \tag{23.65}$$

Expanding to lowest non-trivial order in ω_m and q, we have

$$\begin{aligned} I(q,\epsilon_n,\epsilon_n+\omega_m) = &-\frac{i\pi N_F}{v_F q}[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)]^2\times \\ &\left(2iv_F\tau q - 2iv_F\tau q\,\omega_m\tau\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right] + \frac{2}{3}(iv_F\tau q)^3\right) \end{aligned} \tag{23.66}$$

This expression vanishes unless ϵ_n, ϵ_{n+m} have opposite signs, in which case:

$$\begin{aligned} I(q,\epsilon_n(\epsilon_n+\omega_m)<0) &= 2\pi N_F\tau\left(1-|\omega_m|\,\tau - \frac{1}{3}v_F^2\tau^2q^2\right) \\ &= 2\pi N_F\tau\left(1-|\omega_m|\,\tau - D\tau q^2\right) \end{aligned} \tag{23.67}$$

where the diffusion constant D is given by $D = v_F^2\tau/3$.

Summing the geometric series (23.61),

$$\Gamma(q,\epsilon_n,\epsilon_n+\omega_m) = \frac{1}{1-n_i v^2 I(q,\epsilon_n,\epsilon_n+\omega_m)}$$

$$= \frac{1}{1-(1-|\omega_m|\,\tau - D\tau q^2)\left[\theta(\epsilon_n+\omega_m)-\theta(\epsilon_n)\right]^2}$$
$$= \frac{1}{\tau}\,\frac{1}{|\omega_m|+Dq^2}\,\theta(-\epsilon_n\epsilon_{n+m}) + \theta(\epsilon_n\epsilon_{n+m}) \qquad (23.68)$$

where we have used $2\pi N_F \tau n_i v^2 = 1$. The same form is obtained in any dimension, with minor differences, such as $D = v_F^2\tau/d$ in d dimensions. This form indicates that the electron density diffuses.

To make this a little more concrete, consider the density-density correlation function (23.60):

$$\langle \rho(q,\omega_m)\,\rho(-q,-\omega_m)\rangle = \frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\theta(\epsilon_n\epsilon_{n+m})\,\mathcal{G}(\mathbf{k},\epsilon_n)\,\mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m)$$
$$+\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\tau}\,\frac{1}{|\omega_m|+Dq^2}\,\theta(-\epsilon_n\epsilon_{n+m})\;\mathcal{G}(\mathbf{k},\epsilon_n)\,\mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m) \qquad (23.69)$$

The first term is essentially the static response of free fermions with $\tau = \infty$ up to corrections of order $1/\tau$, ω_m:

$$\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\theta(\epsilon_n\epsilon_{n+m})\,\mathcal{G}(\mathbf{k},\epsilon_n)\,\mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m) =$$
$$\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{i\epsilon_n - \xi_k - v_F q\cos\theta}\,\frac{1}{i\epsilon_n-\xi_k} + O(1/\tau,\omega_m)$$
$$= -N_F + O(1/\tau,\omega_m) \qquad (23.70)$$

In order to do this integral correctly, we must integrate frequency first or else work with a cutoff in momentum space. Integrating in the opposite order will give us an answer which vanishes in the $\omega_m = 0$ limit, i.e. we will miss the static part. By separating the density-density correlation function into static and dynamic pieces – the two terms of (23.69) – we are left with a dynamic piece which is strongly convergent because the Matusbara sum is over a finite range. Hence, we can do the integrals in the opposite order in evaluating that term, to which we now turn.

The second term can be evaluated by doing a ξ_k contour integral:

$$\begin{aligned}\frac{1}{\beta}\sum_n \int \frac{d^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{\tau}\,\frac{1}{|\omega_m| + Dq^2}\,\theta(-\epsilon_n\epsilon_{n+m})\;\mathcal{G}(\mathbf{k},\epsilon_n)\,\mathcal{G}(\mathbf{k}+\mathbf{q},\epsilon_n+\omega_m) = \\ \frac{2\pi i}{\beta}\,\frac{N_F}{2}\sum_n \frac{1}{\tau}\,\frac{1}{|\omega_m| + Dq^2}\,\frac{[\theta(\epsilon_{n+m}) - \theta(\epsilon_n)]}{i\omega_m - v_F q\cos\theta + \frac{i}{\tau}\,[\theta(\epsilon_{n+m}) - \theta(\epsilon_n)]} \\ \approx \frac{2\pi i}{\beta}\,\frac{N_F}{2}\sum_n \frac{1}{\tau}\,\frac{1}{|\omega_m| + Dq^2}\,\frac{\theta(-\epsilon_n\epsilon_{n+m})}{\frac{i}{\tau}} \\ = N_F\,\frac{|\omega_m|}{|\omega_m| + Dq^2} \qquad (23.71)\end{aligned}$$

Combining the two terms, we have

$$\langle \rho(q,\omega_m)\,\rho(-q,-\omega_m)\rangle = -N_F\,\frac{Dq^2}{|\omega_m| + Dq^2} \qquad (23.72)$$

This correlation function has a pole at $|\omega_m| = -Dq^2$. In order to continue to real time, observe that $f(z) = -iz\,\mathrm{sgn}\,(\mathrm{Im}(z))) = |\omega_m|$ if $z = i\omega_m$. If we continue to $z = \omega + i\delta$, then we have $f(\omega + i\delta) = -i\omega$, so that the retarded density-density correlation function becomes:

$$\langle \rho(q,\omega)\,\rho(-q,-\omega)\rangle = -N_F\left(\frac{Dq^2}{-i\omega + Dq^2}\right) \qquad (23.73)$$

The existence of a pole at $i\omega = -Dq^2$ means that the Fourier transform has the form:

$$\langle [\rho(\mathbf{x},t),\rho(0,0)]\rangle\,\theta(t) \sim \frac{1}{t^{d/2+1}}\,e^{-\frac{x^2}{2Dt}}\,\theta(t) \qquad (23.74)$$

Hence, the electron density moves diffusively since a small, highly localized wavepacket prepared at $t = 0$ will spread out over a region of size $x \sim t^{1/2}$ at time t.

We obtained a diffusive form for the propagation of electron density by summing a particular class of diagrams, but these are clearly not the only possible diagrams. However, since the density-density correlation function which we obtained vanishes at $q = 0$, as required by the Ward identity (see below for a recapitulation of this fact), our neglect of other diagrams is at least consistent with charge conservation. There is a further justification for neglecting other diagrams: they are suppressed by a power of $1/k_F\ell$. (**Exercise: check this.**) However, we will find in the next section that such an argument is too quick.

With the density-density correlation function (23.73) in hand, we can re-visit the conductivity. Charge conservation tells us that

$$\partial_t \rho + \nabla \cdot \mathbf{j} = 0 \tag{23.75}$$

Fourier transforming this equation, we have the following relation between correlation functions:

$$q^2 \langle j(q,\omega)\, j(-q,-\omega)\rangle = \omega^2 \langle \rho(q,\omega)\, \rho(-q,-\omega)\rangle \tag{23.76}$$

(The vanishing of the right-hand-side for $q = 0$ confirms that our expression for the density-density correlation function satisfies the Ward identity.) Hence, from the previous expression for the density-density correlation function, we have

$$\frac{1}{i\omega} \langle j(q,\omega)\, j(-q,-\omega)\rangle = N_F \frac{i\omega\, D}{i\omega + Dq^2} \tag{23.77}$$

Taking $q = 0$, and then $\omega \to 0$, we have the DC conductivity.

The spin-spin correlation function also has a diffusive form. Consider, for instance, the $S_z - S_z$ correlation function. For non-interacting electrons, it is precisely the same as the density-density correlation function because $S_z = \rho_\uparrow - \rho_\downarrow$ implies that $\langle S_z S_z\rangle = \langle \rho_\uparrow \rho_\uparrow\rangle + \langle \rho_\uparrow \rho_\uparrow\rangle = \langle \rho\rho\rangle$ (the cross-terms vanish for non-interacting electrons). By rotational symmetry, the other spin-spin correlation functions must be identical. This can also be easily seen by direct calculation of $\langle S_+ S_-\rangle$: the electron line will have spin $+1/2$ while the hole line will have spin $-1/2$, but the diagrams will be unchanged from the density-density calculation because the fermion propagators are spin-independent:

$$\langle S_+(q,\omega)\, S_-(-q,-\omega)\rangle = -N_F \left(\frac{Dq^2}{-i\omega + Dq^2}\right) \tag{23.78}$$

In the presence of impurities, electrons no longer move ballistically at long length scales. They move diffusively because they undergo many collisions with impurities. We can think of this in the following RG language. Consider the effective Lagrangian for a Fermi liquid in the presence of impurities.

$$\begin{aligned} S = \frac{k_F^{d-1}}{(2\pi)^d} \int dl\, d^d\Omega\, \frac{d\epsilon}{2\pi} \psi^\dagger \left(i\epsilon - v_F l\right) \psi \\ + \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{d\epsilon}{2\pi} V(\mathbf{k}-\mathbf{k}')\, \psi^\dagger(\mathbf{k},\epsilon)\, \psi(\mathbf{k}',\epsilon) \end{aligned} \tag{23.79}$$

To avoid clutter, we suppress the marginal four-Fermi interactions parametrized by Landau parameters which are inessential to this discussion.

For a fixed $V(\mathbf{k})$, this term is precisely marginal. This is what would be expected from a single-impurity. Consider, now, a random $V(\mathbf{k})$. When we integrate over different impurity configurations according to (23.47) we generate the effective action

$$S_{\text{eff}} = \frac{k_F^{d-1}}{(2\pi)^d} \int dl\, d^d\Omega\, \frac{d\epsilon}{2\pi} \psi^\dagger \left(i\epsilon - v_F l\right) \psi$$
$$+ n_i v^2 \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{d^3\mathbf{k}'}{(2\pi)^3} \frac{d\epsilon}{2\pi} \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{d\epsilon'}{2\pi} \psi^\dagger(\mathbf{k},\epsilon)\, \psi(\mathbf{k}',\epsilon) \psi^\dagger(\mathbf{p},\epsilon')\, \psi(\mathbf{p}',\epsilon') \tag{23.80}$$

with the proviso that we should only allow diagrams which would remain connected even if all of the impurity lines (connecting the two pairs of fermions in the second line in equation (23.80)) were cut. (We will introduce a formal way of implementing this in the next chapter.) The resulting impurity-scattering term is strongly-relevant, scaling as s^{-1}. (It is similar to a four-Fermi interaction but with one fewer energy integration.) Thus the clean Fermi liquid is unstable to impurity-scattering. The RG flow is towards the *diffusive Fermi liquid*. At this fixed point, the density-density and spin-spin correlation functions takes the diffusive form (23.73), (23.78) while the single-fermion Green function (23.49) is short-ranged as a result of the i/τ in the denominator. In the next chapter, we will construct an effective field theory for the long-wavelength, low frequency modes of the diffusive fixed point.

23.4.5 Weak Localization

In the previous two subsections, we expended substantial effort to compute the electron Green function, and the density-density and spin-spin correlation functions. However, the result of all of these diagrammatic calculations was no more than simple qualitative considerations and naive Drude theory would have given us: electrons have a finite conductivity which is determined by the lifetime τ. This lifetime also enters the diffusive form of various correlation functions through the diffusion constant $D = v_F^2\tau/d$. One hardly needs sophisticated field-theoretic techniques to discover this, although it is, perhaps, some consolation that we can confirm our intuition by direct calculation.

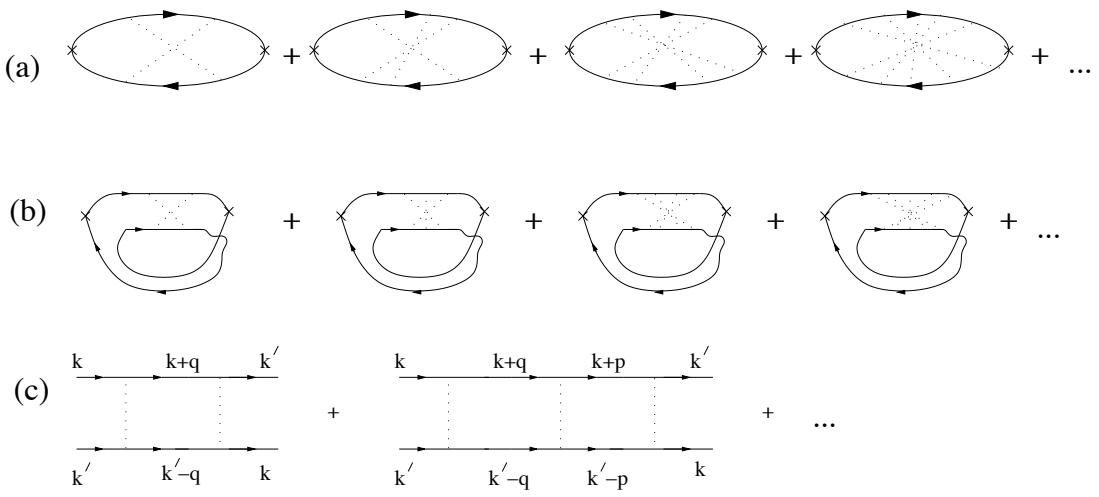


Figure 23.7: (a) Diagrams contributing to the conductivity with impurity lines maximally-crossed. (b) A re-drawing of the maximally-crossed diagrams which emphasizes the role of the particle-particle diagrams shown in (c).

One might naively think that, in the absence of interactions, this is all that there is to the conductivity of a dirty metal. However, the subleading (in $1/k_F\ell$) impurity contributions to the conductivity are also interesting. They are due to quantum interference effects. In two dimensions, the 'subleading' correction is divergent at low-temperatures and, in fact, non-interacting electrons are always insulating in $d = 2$. Field-theoretic techniques are invaluable in obtaining and understanding these corrections.

Consider the diagrams of figure 23.7a. In these diagrams, the impurity lines are maximally-crossed. This is a particular class of diagrams contributing to the conductivity. In this subsection, we will see why these diagrams give a significant contribution to the condutivity. Observe that they can be re-drawn in the manner shown in figure 23.7b. The diagrams of 23.7b are very similar to those of the diffusion propagator. The main difference is that the arrows are in the same direction, rather than opposite directions, i.e. the middle of the diagram is a particle-particle diagram rather than a particle-hole diagram. However, time-reversal symmetry relates these two. Call the sum of particle-particle diagrams in figure 23.7c $W(\mathbf{k}, \mathbf{k}', \epsilon_n, \epsilon_{n+m})$. By time-reversal symmetry, we can reverse the direction of one of the arrows if we also reverse the sign of the momentum. Because particles and holes

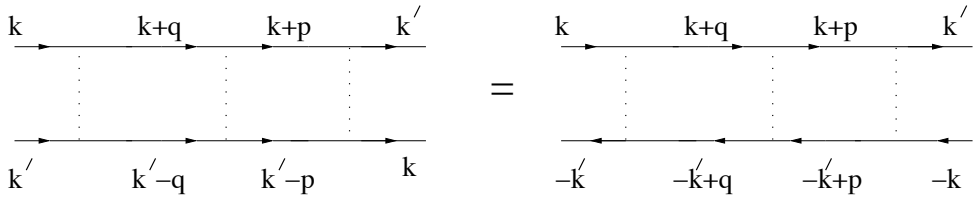


Figure 23.8: A particle-particle diagram is equal to a particle-hole diagram with particle momenta reversed to become hole momenta.

both cost positive energy, but carry opposite momentum, a hole line running through a diagram can be replaced with a particle line at the same energies but opposite momenta without changing the value of the diagram. This is depicted in figure 23.8. From this observation, we see that:

$$\begin{aligned} W(\mathbf{k},\mathbf{k}',\epsilon_n,\epsilon_{n+m}) &= \big(n_i v^2\big)^2 I(\mathbf{k}+\mathbf{k}',\epsilon_n,\epsilon_{n+m}) \\ &\quad + \big(n_i v^2\big)^3 \big(I(\mathbf{k}+\mathbf{k}',\epsilon_n,\epsilon_{n+m})\big)^2 + \dots \\ &= \frac{\big(n_i v^2\big)^2 I(\mathbf{k}+\mathbf{k}',\epsilon_n,\epsilon_{n+m})}{1-(n_i v^2) I(\mathbf{k}+\mathbf{k}',\epsilon_n,\epsilon_{n+m})} \\ &= \frac{n_i v^2}{\tau}\left(\frac{1}{|\omega_m| + D(\mathbf{k}+\mathbf{k}')^2}\right)\theta(-\epsilon_n\epsilon_{n+m}) \end{aligned} \tag{23.81}$$

If we substitute this into a conductivity bubble to compute its contribution to the conductivity, as in figure 23.7b, we obtain

$$\begin{aligned} \delta\sigma(\omega_m) = \frac{1}{\omega_m}\frac{v_F^2}{d}\int \frac{d^d k}{(2\pi)^d}\frac{d^d k'}{(2\pi)^d}\frac{1}{\beta}\sum_n \mathcal{G}(\mathbf{k},\epsilon_n)\,\mathcal{G}(\mathbf{k},\epsilon_n+\omega_m)\times \\ W(\mathbf{k},\mathbf{k}',\epsilon_n,\epsilon_{n+m})\,\mathcal{G}(\mathbf{k}',\epsilon_n)\,\mathcal{G}(\mathbf{k}',\epsilon_n+\omega_m) \end{aligned} \tag{23.82}$$

From (23.81), we see that $W(\mathbf{k},\mathbf{k}',\epsilon_n,\epsilon_{n+m})$ is sharply peaked at $\mathbf{k}\approx-\mathbf{k}'$. Hence, we write $\mathbf{q}=\mathbf{k}+\mathbf{k}'$ and change the variables of integration to $\mathbf{k}$, $\mathbf{q}$. In the second pair of Green functions, we take $\mathbf{k}=-\mathbf{k}'$, neglecting the weak

q dependence of those factors. We now have:

$$\begin{aligned}
\delta\sigma(\omega_m) &= \frac{1}{\omega_m} N_F \frac{v_F^2}{d} \int d\xi_k \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \frac{n_i v^2}{\tau} \left(\frac{\theta(-\epsilon_n \epsilon_{n+m})}{|\omega_m| + Dq^2} \right) \times \\
&\qquad \left(\frac{1}{i\epsilon_n - \xi_k + \frac{i}{2\tau_n}} \right)^2 \left(\frac{1}{i\epsilon_n + i\omega_m - \xi_k + \frac{i}{2\tau_{n+m}}} \right)^2 \\
&= \frac{4\pi i}{\omega_m} N_F \frac{v_F^2}{d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \frac{n_i v^2}{\tau} \left(\frac{\theta(\epsilon_{n+m}) - \theta(\epsilon_n)}{|\omega_m| + Dq^2} \right) \times \\
&\qquad \left(\frac{1}{i\omega_m + \frac{i}{\tau}(\theta(\epsilon_{n+m}) - \theta(\epsilon_n))} \right)^3 \\
&\approx \frac{4\pi i}{\omega_m} N_F \frac{v_F^2}{d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \frac{n_i v^2}{\tau} \left(\frac{\theta(\epsilon_{n+m}) - \theta(\epsilon_n)}{|\omega_m| + Dq^2} \right) \times \\
&\qquad \left(\frac{1}{\frac{i}{\tau}(\theta(\epsilon_{n+m}) - \theta(\epsilon_n))} \right)^3 \\
&= -\frac{4\pi}{\omega_m} N_F \frac{v_F^2}{d} \tau^3 \int \frac{d^d q}{(2\pi)^d} \frac{1}{\beta} \sum_n \frac{n_i v^2}{\tau} \left(\frac{\theta(-\epsilon_{n+m} \epsilon_n)}{|\omega_m| + Dq^2} \right) \\
&= -\frac{1}{\pi} D \int \frac{d^d q}{(2\pi)^d} \frac{1}{|\omega_m| + Dq^2} \\
&= -\frac{1}{\pi} \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})\,(2\pi)^d} \frac{1}{d-2} \left(\frac{1}{\ell^{d-2}} - \left(\frac{|\omega_m|}{D} \right)^{\frac{d-2}{2}} \right)
\end{aligned} \tag{23.83}$$

We have taken the ultraviolet cutoff to be $1/\ell$. For shorter wavelengths, one should use ballistic, rather than diffusive propagators, so this is the appropriate cutoff.

While (23.82) is a small correction in the low frequency limit for $d > 2$, it is divergent in this limit for $d < 2$. In $d = 2$, it is;

$$\begin{aligned}
\delta\sigma(\omega_m) &= -\frac{1}{4\pi^2} \ln\left(\frac{D}{|\omega_m|\ \ell^2} \right) \\
\frac{\delta\sigma(\omega_m)}{\sigma_0} &= -\frac{1}{\pi k_F \ell} \ln\left(\frac{D}{|\omega_m|\ \ell^2} \right)
\end{aligned} \tag{23.84}$$

which is also divergent.

Hence, in $d \leq 2$, the correction (23.82)-(23.84), which is formally smaller than the (semiclassical) Boltzmann result by a power of $k_F \ell \sim D$ is, in

fact, divergent as $\omega \to 0$. Thus, the starting point of the diffusive fixed point is unstable. In the next section, we will discuss the result of this instability in more detail but, for now, we observe that it suppresses the conductivity. Thus, it drives the system towards localization. In $d = 2$, when this correction is still weak (sufficiently high frequencies or, as we will describe below, temperatures) and growing only logarithmically, it is called *weak localization*.

One might wonder whether is is valid to stop with the diagrams which we have calculated. Isn't it possible that if we computed some more diagrams, we would find other divergent corrections to the conductivity which might change our conclusions entirely? The answer is that this will not happen and we can stop here. The divergences which we find are due to the existence of slow modes in the system. The diffusive mode is a slow mode because it is related to the conservation of charge. The particle-particle mode (called a 'Cooperon') is slow because it is related to the diffusion mode by time-reversal symmetry. There is no other mode which is guaranteed to be slow by symmetry, so we do not need to worry about other entirely new divergences, although these same slow modes can lead to subleading divergences. We will discuss this point in the next chapter using an effective field theory for the slow modes of the system.

A simple physical picture for weak-localization is given by the notion of coherent backscattering. The amplitude for an electron to go from $\mathbf{r}$ to $\mathbf{r}'$ is, a complicated function of the trajectory because of the possibility of repeated impurity scatterings along the way. Different trajectories from $\mathbf{r}$ to $\mathbf{r}'$ will therefore interfere with each other – sometimes constructively, sometimes destructively. However, a trajectory from $\mathbf{r}$ back to $\mathbf{r}$ itself will always interfere constructively with the time-reversed trajectory which is scattered by the same impurities, but in reverse order. Thus, there will be an enhanced backscattering amplitude from $\mathbf{r}$ back to $\mathbf{r}$, compared to the amplitude to go to any other $\mathbf{r}'$. This enhanced backscattering is the source of weak localization. Consider the probability for an electron at $\mathbf{r}$ to return to $\mathbf{r}$ within time t_L:

$$P(t_L) = \int_\tau^{t_L} dt\, (Dt)^{-d/2} \tag{23.85}$$

where $t_L = D/L^2$ is the time by which the electron has diffused across the system. The correction to the conductivity is

$$\frac{\delta\sigma}{\sigma} \sim P(t_L) \sim \tau^{1-d/2} - {t_L}^{1-d/2} \tag{23.86}$$

Any effect which inhibits constructive interference, such as inelastic scat-

tering or a magnetic field will, therefore, spoil weak localization. Thus far, we have considered non-interacting electrons and neglected phonons, etc. If such sources of *inelastic scattering* are considered, they cut off the above divergences. At length scales longer than the Thouless length

$$\ell_{\rm Thouless} = \sqrt{D\tau_{\rm in}} \tag{23.87}$$

inelastic processes will thwart localization and cause ohmic conduction. For example, the $2D$ weak-localization correction is:

$$\delta\sigma(\omega_m) = -\frac{1}{2\pi^2}\,\ln(\ell_{\rm Thouless}/\ell) \tag{23.88}$$

If $\tau_{\rm in} \sim T^{-p}$, then the frequency is replaced by $\sim T^p$, *not* T, as one might naively assume.

A weak magnetic field will also cut off localization at length scales longer than the magnetic length $\ell_H = 1\sqrt{H}$. (Restoring the fundamental constants, it is $\ell_H = \sqrt{\hbar c/eH}$.) At these length scales, the time-reversed trajectory differs in phase by $\sim \pi$ as a result of the magnetic flux which it encloses and no longer interferes constructively. Thus, in the weak localization regime, where

$$\delta\sigma(\omega_m) = -\frac{1}{2\pi^2}\,\ln(\ell_H/\ell) \tag{23.89}$$

the conductivity *increases* as the magnetic field is increased, i.e. the magnetoresistance is *negative*.

23.4.6 Weak Magnetic Fields and Spin-Orbit Interactions: the Unitary and Symplectic Ensembles

23.4.7 Electron-Electron Interactions in the Diffusive Fermi Liquid

How are the conclusions of the previous subsections altered by the influence of electron-electron interactions? How does the diffusive motion of electrons affect the relevance or irrelevance of interactions at the diffusive fixed point, as compared to their marginality at the clean Fermi liquid fixed point? As we will see in this subsection, diffusive motion is so slow that it enables electron-electron interactions to cause divergent corrections at the diffusive fixed point.

The simplest example of such a divergence is in the single-electron Green function, from which we can obtain the single-electron density of states:

$$N(E) = \int \frac{d^d k}{(2\pi)^d} G(E, \mathbf{k}) \tag{23.90}$$

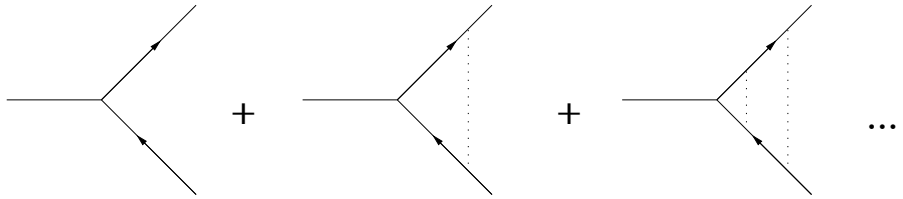


Figure 23.9: The renormalization of the density vertex by repeated impurity scatterings. The unarrowed solid line on the left is the interaction V.

Consider the self-energy diagrams of figure 23.10. Let us assume that the electrons interact through some effective density-density interaction $V(q)$. Then the main effect of disorder is to modify the nature of the interaction vertex. Physically, since the electrons move more slowly when the diffuse, they spend more time in close proximity and, therefore, interact more strongly. The basic density vertex is renormalized by repeated impurity scatterings according to (23.68). Hence, the interaction vertex $V(q)$ is renormalized in a corresponding way, as depicted in figure 23.9:

$$V(q) \to V(q)(\Gamma(q, \epsilon_n, \epsilon_n + \omega_n))^2 \tag{23.91}$$

where $\Gamma(q, \epsilon_n, \epsilon_n+\omega_n)$ is given in (23.68). There are two factors of $\Gamma(q, \epsilon_n, \epsilon_n+\omega_n)$, one for each particle-hole pair which is coupled by V.

Consider, then, the first diagram (exchange) in figure 23.10. Its leading contribution to the self-energy is:

$$\Sigma_{\rm ex}(\epsilon_n, k) = \frac{1}{\beta}\sum_m \int \frac{d^d q}{(2\pi)^d}\, V(q) \left(\frac{1}{\tau}\, \frac{1}{|\omega_m| + Dq^2}\, \theta(-\epsilon_n \epsilon_{n+m})\right)^2 \times \frac{1}{i\epsilon_n + i\omega_m - \xi_{k+q} + \frac{i}{2\tau_{n+m}}} \tag{23.92}$$

Clearly, the integral is dominated by $\omega_m \approx 0$, $q \approx 0$ so we write

$$\begin{aligned}\Sigma_{\rm ex}(\epsilon_n, k) &\approx \frac{V(0)}{i\epsilon_n - \xi_k + \frac{i}{2\tau_n}} \frac{1}{\beta}\sum_m \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{\tau}\, \frac{\theta(-\epsilon_n \epsilon_{n+m})}{|\omega_m| + Dq^2}\right)^2 \\ &\approx -2iV(0)\tau\, \mathrm{sgn}(n) \frac{1}{\beta}\sum_m \int \frac{d^d q}{(2\pi)^d} \left(\frac{1}{\tau}\, \frac{\theta(-\epsilon_n \epsilon_{n+m})}{|\omega_m| + Dq^2}\right)^2 \end{aligned} \tag{23.93}$$

Consider the case $d = 2$.

$$\Sigma_{\rm ex}(\epsilon_n, k) \approx \frac{-iV(0)}{2\pi\tau}\, \mathrm{sgn}(n) \frac{1}{\beta}\sum_m \int dq^2 \left(\frac{\theta(-\epsilon_n \epsilon_{n+m})}{|\omega_m| + Dq^2}\right)^2$$

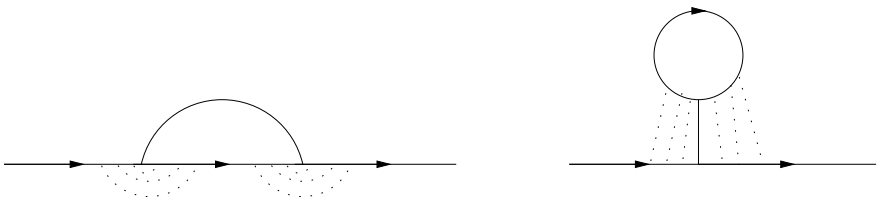


Figure 23.10: The exchange (Fock) and Hartree diagrams contributing to the electron self-energy. Where particle and hole lines are connected by dotted lines, the vertices are dressed by repeated impurity scatterings so that they take a diffusive form. The unarrowed solid line on the left of the vertex is the interaction V.

$$\approx \frac{-iV(0)}{(2\pi)^2 D\tau}\,\mathrm{sgn}(n)\,\frac{2\pi}{\beta}\sum_m \frac{\theta(-\epsilon_n\epsilon_{n+m})}{|\omega_m|} \tag{23.94}$$

Let us suppose, without loss of generality, that $\epsilon_n < 0$. Then the Matsubara frequency sum runs over $\omega_m > -\epsilon_n$. The ultraviolet cutoff is $1/\tau$, since for higher frequencies the diffusive form is not correct. For low T, the sum can be approximated by an integral, and we find:

$$\Sigma_{\rm ex}(\epsilon_n, k) \approx -i\frac{V(0)}{(2\pi)^2 D\tau}\,\mathrm{sgn}(n)\,\ln(|\epsilon_n|\tau) \tag{23.95}$$

The correction to the density-of-states is

$$\begin{aligned}\delta N_{\rm ex}(\epsilon) &= \int \frac{d^2k}{(2\pi)^2}\;\mathrm{Im}\big((G_0(k,\epsilon))^2\,\Sigma(\epsilon,k)\big) \\ &= \frac{V(0)N_F}{(2\pi)^2 D}\;\ln(|\epsilon|\tau) \\ &= \frac{V(0)N_F^2}{2\pi\epsilon_F\tau}\;\ln(|\epsilon|\tau)\end{aligned} \tag{23.96}$$

Thus, the correction is singular.

The second diagram (Hartree) in figure 23.10 has opposite sign as a result of the closed fermion loop and an extra factor of two from the spin sum in

the loop. It has leading contribution:

$$\Sigma_{\rm Hartree}(\epsilon_n,k) = -2\,\frac{1}{\beta}\sum_m \int \frac{d^2p}{(2\pi)^2}\,\frac{d^2p'}{(2\pi)^2}\,\frac{d^2q}{(2\pi)^2}\,V(\mathbf{p}-\mathbf{p}')\left(\frac{1}{\tau}\,\frac{\theta(-\epsilon_n\epsilon_{n+m})}{|\omega_m|+Dq^2}\right)^2 \times \frac{1}{i\epsilon_n - \xi_p + \frac{i}{2\tau_n}}\,\frac{1}{i\epsilon_{n+m} - \xi_{p+q} + \frac{i}{2\tau_{n+m}}} \times \frac{1}{i\epsilon_n - \xi_{p'} + \frac{i}{2\tau_n}}\,\frac{1}{i\epsilon_{n+m} - \xi_{p'+q} + \frac{i}{2\tau_{n+m}}} \quad (23.97)$$

As before, we set $\omega_m = q = 0$ in the electron propagators; notice that the interaction is not at zero momentum, however.

$$\Sigma_{\rm Hartree}(\epsilon_n,k) \approx -\frac{2}{\beta}\sum_m \int \frac{d^2p}{(2\pi)^2}\,\frac{d^2p'}{(2\pi)^2}\,\frac{d^2q}{(2\pi)^2}\,V(\mathbf{p}-\mathbf{p}')\left(\frac{1}{\tau}\,\frac{\theta(-\epsilon_n\epsilon_{n+m})}{|\omega_m|+Dq^2}\right)^2 \times \frac{1}{i\epsilon_n - \xi_p + \frac{i}{2\tau_n}}\,\frac{1}{i\epsilon_n - \xi_p + \frac{i}{2\tau_n}} \times \frac{1}{i\epsilon_n - \xi_{p'} + \frac{i}{2\tau_n}}\,\frac{1}{i\epsilon_n - \xi_{p'} + \frac{i}{2\tau_n}} \quad (23.98)$$

The electron Green functions are all peaked around the Fermi surface, so we have

$$\begin{aligned}{\rm Im}\Sigma_{\rm Hartree}(\epsilon_n,k) \approx \\ 2\tau^2\frac{1}{\beta}\sum_m \int \frac{d^2q}{(2\pi)^2}\left(\frac{1}{\tau}\,\frac{\theta(-\epsilon_n\epsilon_{n+m})}{|\omega_m|+Dq^2}\right)^2 \int \frac{d\theta}{2\pi}\,V(2k_F\cos\theta) \\ = \frac{2FV(0)}{(2\pi)^2 D\tau}\,{\rm sgn}(n)\ln(|\epsilon_n|\tau) \quad (23.99)\end{aligned}$$

where

$$F = \frac{1}{V(0)}\int \frac{d\theta}{2\pi}\,V(2p_F\cos\theta) \quad (23.100)$$

is an average of the interaction over all momenta connecting two points on the Fermi surface.

Hence, adding the exchange and Hartree terms we have

$$\frac{\delta N(\epsilon)}{N_F} = \frac{V(0)N_F}{2\pi\epsilon_F\tau}\,(1-2F)\,\ln(|\epsilon|\tau) \quad (23.101)$$

Since $F < 1$ (because $V(q)$ decreases with increasing q), the exchange terms wins, and the net effect of interactions at lowest-order is to suppress the

conductivity. When this correction becomes large, $|\epsilon|\tau \ll 1$, it is no longer valid to use lowest-order perturbation theory; but one can speculate that this suppression eventually leads to a vanishing of the density-of-states at the Fermi energy, analogous to the Coulomb gap discussed in the context of insulators. A similar correction is expected for the specific heat.

The basic physics behind this correction is that the interaction between two electrons is renormalized by processes in which they diffuse back to each other and interact again:

$$V(0) \to V(0) \left(1 + \int_\tau^{1/\epsilon} dt\, P(t) \right) \tag{23.102}$$

where $P(t) \sim 1/(Dt)^{d/2}$. In $d = 2$, this is:

$$V(0) \to V(0)\,(1 + c \ln(|\epsilon|\tau)) \tag{23.103}$$

The effects of interactions bear the signature of this correction.

Thus, interactions lead to singular corrections to the density of states in $d = 2$ (and worse singularities in $d = 1$ (**exercise**)). This is in contrast to a non-interacting disordered system, in which the density-of-states is perfectly smooth. On the other hand, the conductivity receives singular corrections even in the non-interacting system. How do interactions affect this? Similar, but more complicated diagrams than those of fig. 23.10 determine the lowest-order interaction correction to the conductivity of a diffusive metal. They give a contribution

$$\delta\sigma(\omega) = \frac{V(0) N_F}{2\pi^2} (1 - F) \ln(|\omega|\tau) \tag{23.104}$$

If the electrons interact through the screened Coulomb interaction,

$$V(q) = \frac{\frac{4\pi e^2}{q^2}}{1 + \frac{4\pi e^2}{q^2} \Pi(q, \omega)} \tag{23.105}$$

and $\Pi(q, \omega)$ is the density-density correlation function. Hence, $V(q \to 0, \omega = 0) = (\Pi(q \to 0, 0))^{-1} = N_F^{-1}$. This implies, for instance, that the conductivity correction is of the form

$$\delta\sigma_{\text{scr. Coul.}}(\omega) = \frac{1}{2\pi^2} (1 - F) \ln(|\omega|\tau) \tag{23.106}$$

which is nearly identical in form and magnitude as the weak-localization correction. It is difficult to disentangle the two effects, but it can be done

by measuring the magnetoresistance since the weak-localization and interaction corrections are affected very differently by a magnetic field (the latter, through the Zeeman coupling; while the former is through the destruction of interference).

The diagrams which we have computed in this subsection are the lowest-order diagrams which have diffusively-dressed interaction vertices such that the particle-hole lines which are difffusing have low momentum/energy. One could imagine dressing the diagrams of figure 23.10 with impurity lines in other ways, but the diffusing particle-hole pair would not be near $\omega = q = 0$. Thus, while it may, at first glance, seem as if we have chosen an arbitrary set of diagrams, in fact, we have chosen precisely the ones which have the most singular diffusive corrections. If this seems a little unsatisfying, then in the next chapter we will introduce an effective field theory for the diffusive degrees of freedom, in which these corrections come from the only diagrams present at lowest order, so they can be obtained by simply turning the crank.

23.5 The Metal-Insulator Transition

23.5.1 Percolation

Consider a disordered solid which is insulating as a result of impurities. Although it is insulating, there will be some isolated regions which are 'conducting'. In a system of non-interacting fermions, such regions would have single-particle states at the Fermi energy which extend across them but do not leak substantially into the surrounding insulating regions. One way to think about this is to consider the potential $V(x)$, whose spatial variation is random as a result of impurities. The regions in which $V(x) < \epsilon_F$ are classically-allowed for electrons at the Fermi energy. Let us consider these to be the 'conducting' regions. The regions in which $V(x) > \epsilon_F$ are classically-forbidden for electrons at (or below) the Fermi energy. Let us call these 'insulating' regions.

Suppose we decrease the impurity concentration. Then the conducting regions will grow in size. As we continue to decrease the impurity concentration, the conducting regions will grow larger and larger and will begin to merge. Eventually, it will become possible to go from one end of the system to the other within one long conducting region which spans the system. If electrical conduction were purely classical and ohmic, this impurity concentration – which is called the *percolation threshold* – would be the transition point between metallic and insulating behavior.

However, electrical conduction is not classical. Even below the percolation threshold, electrons can tunnel quantum mechanically from one conducting region to another. Conversely, an electron can take two different paths from point A to point B, and the quantum mechanical amplitudes for these processes might interfere destructively. Consequently, a system might be insulating even above the percolation threshold. Nevertheless, at temperatures which are higher than the characteristic phase coherence temperatures of the system – so that coherent quantum tunneling cannot occur – but not so high that the conductivity of the 'insulating' regions (insulating and metallic are precise distinctions only at zero temperature; at finite temperatures, all systems have finite conductivity) is comparable to that of the conducting ones, we expect the conductivity and other physical properties to show behavior characteristic of *percolation* processes. Since percolation is, in a sense, the classical limit of the metal-insulator transition, we will discuss it briefly.

There are two basic lattice models of percolation, *site percolation* and *bond percolation*. In site percolation, one randomly paints black some of the sites of a lattice and considers the properties of the remaining sites, which are colored red. We imagine that an electron can only be on a red site, never on a black one. It can hop directly from a red site to its nearest neighbors if they are also red, but not to other sites. Thus, if there is an isolated red site which is surrounded by black sites, then an electron on that site is stuck there. Sites which are red can be grouped into clusters of sites which are continguous with each other. One site in a cluster can be reached from any other site in the cluster by a sequence of nearest-neighbor hops entirely within the cluster. In *bond percolation*, on the other hand, one randomly removes bonds from a lattice. An electron can hop from one lattice site to a nearest-neighbor site only if the bond between them is present. Sites can again be grouped into clusters which are connected by bonds so that any site in a cluster can be reached from any other site in the cluster.

For the sake of concreteness, let us consider *bond percolation*. When the average bond density, which we will call p, is low, clusters tend to be small. As the bond density is increased, clusters grow until the percolation threshold, p_c, is reached. At this point, one cluster extends across the length of the system; it is called the *percolating cluster*. For $p < p_c$, the average distance between two sites on the same cluster, ξ, scales as

$$\xi \sim (p_c - p)^{-\nu} \tag{23.107}$$

For $p > p_c$, the fraction, P, of the system which belongs to the percolating

cluster is

$$P(p) \sim (p - p_c)^{\beta} \tag{23.108}$$

(For $p < p_c$, $P = 0$ because the largest cluster is finite even in a system of infinite size.) Notice the analogy between these exponents and the correlation length and order-parameter exponents associated with critical phenomena.

The precise value of p_c depends on the particular lattice and whether it is bond or site percolation. However, the exponents ν, β, etc. are universal, and depend only on the dimension.

One might imagine that the conductivity (assuming that bonds which are present are considered to be conducting and those which are absent are insulating) would be proportional to the fractions of the system belonging to the largest cluster. However, this is not the case because many of the bonds in the cluster lead to dead-ends which do not contribute to the conductivity. Instead, the conductivity is associated with a different exponent, μ, (sometimes called t) which describes the 'backbone' of the percolating cluster – i.e. the cluster with dead-ends removed (dead-ends are those bonds which do not have two completely independent routes to the two ends of the cluster).

$$\sigma(p) \sim (p - p_c)^{\mu} \tag{23.109}$$

At the percolation point itself, the system will have power-law correlations. Many of the properties of the percolation point in two dimensions have been revealed using conformal field theory and, more recently with the use of the stochastic Loewner equation. For instance, $\nu = 4/3$ in two dimensions.

However, as we pointed out earlier, percolation does not correctly describe metal-insulator transitions because it misses quantum-mechanical effects. In the remainder of this section, we discuss alternate approaches which take these effects into account.

23.5.2 Mobility Edge, Minimum Metallic Conductivity

Let us now consider a metal-insulator transition in a system of non-interacting fermions. Such a transition can occur as a result of varying either the chemical potential or the disorder strength. If the system is weakly-disordered, then the energy spectrum will still approximately break up into bands, although there may not be a true gap between the bands because there will be exponentially-small tails in the density of states. Near the bottom of a band, the states will all be localized because the kinetic energies of these states will be low. Near the top of the band, the same will be true. It is useful

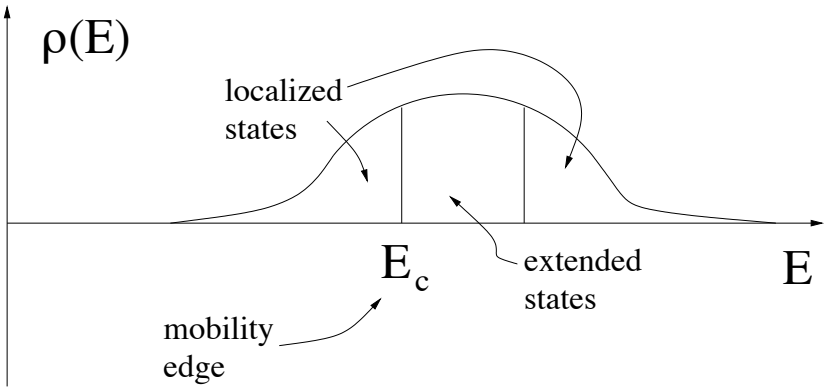


Figure 23.11: The density-of-states in a disordered system, with extended and localized states and mobility edges shown.

to think of a band with chemical potential near the top as a nearly empty hole band; in such a case, the hole kinetic energies near the top of the band are low. Near the middle of the band, on the other hand, the states will be extended if the disorder strength is sufficiently small because the kinetic energies will be large in comparison. (If the disorder strength is too large, then even at the middle of the band the kinetic energy will not be large enough to overcome it.) What happens as the Fermi energy is swept from the bottom of the band to the middle? To answer this question, one should first note that there are no energies at which there are both extended and localized states because localization is not stable in the presence of extended states at the same energy. A small change in the particular realization of disorder (e.g. the locations of the impurities) would cause mixing between the localized states and the extended ones. When a localized state mixes with extended states, it becomes extended.

Hence, there will be sharp energies which separate localized states from extended ones. Such an energy is called a *mobility edge*. As depicted in figure 23.11, the mobility edge separates the strongly-localized states with low kinetic energy in the band tails from the extended states with high kinetic energy in the center of the band. (One can imagine there being more than a single pair of such energies, but this would not be generic and would require some special circumstances.)

As the chemical potential is moved through the mobility edge, E_c, the system undergoes a metal-insulator transition. For $E_F < E_c$, the zero-

temperature DC conductivity σ vanishes. For $E_F > E_c$, it is finite. What happens at $E_F = E_c$? Is it equal to some minimum non-zero value $\sigma_{\rm min}$, the *minimum metallic conductivity*? According to the percolation model described in the preceding section, the conductivity vanishes precisely at the transition, $\sigma \sim |E_F - E_c|^\mu$. ($E_F - E_c$ or any other control parameter will act as a proxy for $p - p_c$ near the transition.) On the other hand, the following heuristic argument suggests that it is finite. The conductivity is given in Boltzmann transport theory by

$$\sigma = A_d \frac{e^2}{\hbar} k_F^{d-1} \ell \tag{23.110}$$

where ℓ is the mean-free-path and A_d is a dimension-dependent constant. (Here, we have restored the factors of e, $\hbar$ which were set to one earlier.) If scattering is due entirely to impurities, the mean-free path cannot be shorter than the inter-atomic spacing a in the solid. Since $k_F \sim \pi/a$ and $\ell > a$, $k_F \ell \gtrsim \pi$. Hence, we conclude that

$$\sigma_{\rm min} = A_d \frac{e^2}{\hbar} k_F^{d-2} \tag{23.111}$$

In particular, in $d = 2$, the minimum metallic conductivity can be written entirely in terms of fundamental constants, $\sigma_{\rm min} = A_d e^2/\hbar$.

However, one should be cautious in applying the Boltzmann transport formula (23.110) – which is semiclassical and valid for large $k_F \ell$ – in the regime $k_F \ell \sim 1$. Indeed, as we will see in the next subsection, the preceding logic fails in the case of non-interacting electrons.

23.5.3 Scaling Theory for Non-Interacting Electrons

Thouless had the important insight that one should study the scaling behavior of the conductance. Consider a system of non-interacting electrons of size L. Thouless wrote the conductance of the system in the following form, which we encountered earlier (23.6):

$$g(L) = \frac{\overline{\delta E}}{\overline{\Delta E}} \tag{23.112}$$

δE is the disorder-avergaged energy change of eigenstates at E_F when the boundary conditions are changed from periodic to antiperiodic. ΔE is the mean level spacing of eigenstates. (In order to derive such a relation analytically from the Kubo formula, one must consider disorder-averaged quantities; otherwise, one could probably consider such a relation for a fixed realization of disorder.)

To understand this formula, suppose that $D(L)$ is the diffusion constant for an electron at the Fermi energy. The time required for an electron to diffuse across the system is the Thouless time:

$$t_L = \frac{L^2}{D(L)} \tag{23.113}$$

According to the uncertainty principle,

$$\overline{\delta E} \sim \frac{1}{t_L} \sim \frac{D(L)}{L^2} \tag{23.114}$$

Meanwhile,

$$\overline{\Delta E} = \frac{1}{N_F\, L^d} \tag{23.115}$$

Hence,

$$\frac{\overline{\delta E}}{\overline{\Delta E}} = N_F\, D(L)\, L^{d-2} = \sigma\, L^{d-2} = g(L) \tag{23.116}$$

The second equality follows from the Einstein relation between the diffusion constant and the conductivity.

We now study how $g(L)$ scales as the system size is increased. We could increase the system size from L to bL by stacking b subsystems of size L. The key assumption of Abrahams, Anderson, Licciardello, and Ramakrishnan ('the Gang of Four') is that:

$$g(bL) = f\left(b, g(L)\right) \tag{23.117}$$

In particular, L does not appear explicitly on the right-hand-side. In a macroscopic Ohmic system, such a relation is obvious, $g(bL) = b^{d-2} g(L)$. However, the Ohmic scaling form is not exactly right microscopically. While $\overline{\Delta E(bL)} = b^{-d}\overline{\Delta E(L)}$ should hold generally, we need to find a relation for $\overline{\delta E(bL)}$. This relation might be complicated since we have to match the eigenfunctions from each size L block at their boudaries in order to construct the eigenfunctions of the size bL system. However, the sensitivity of the eigenfunctions to their boundary conditions is precisely what is encapsulated in $g(L)$. Thus, it is very reasonable to assume that $\overline{\delta E(bL)}$ depends only on b and $g(L)$, and, therefore, that $g(bL)$ does. As noted above, this assumption holds in the ohmic regime. It also holds in the localized regime, where $g(L) = g_0\, e^{-L/\xi}$: $g(bL) = g_0\, (g(L)/g_0)^b$. The scaling assumption of the gang of four was that the form (23.117) holds throughout the entire region between these limits.

If (23.117) holds, then we can define

$$\frac{d\ln g}{d\ln L} = \frac{1}{g}\left(\frac{df}{db}\right)_{b=1} = \beta(g) \tag{23.118}$$

In strongly-disordered systems, g is small, and all states are localized. In this limit, we expect $\beta(g) = \ln(g/g_0)$. In the clean, or large g, limit, we expect Ohmic scaling, and $\beta(g) = d-2$.

How do we connect the two limits? For g large, we can compute $\beta(g)$ in powers of $1/g$ or, equivalently, $1/k_F\ell$. In the previous section, we have already computed the first correction in $1/k_F\ell$ to the conductivity. Suppose we reconsider this calculation in a system of finite-size L. Then $1/L$ is an infrared momentum cutoff, and we can take the other cutoffs used in the previous section (frequency $1/\sqrt{\omega/D}$, Thouless length $\sqrt{D\tau_{in}}$, magnetic length,...) to infinity. We thus obtain $\sigma(L)$ in $d=2$ for large $k_F\ell$. For $d = 2+\epsilon$, we can compute in powers of ϵ. Since $g = \sigma$ in $d=2$,

$$g(L) = g_0\left(1 - \frac{1}{2\pi^2 g_0}\ln\left(\frac{L}{\ell}\right)\right) \tag{23.119}$$

For $d = 2+\epsilon$, corrections to this formula for σ are $O(\epsilon)$, which we drop. Differentiating, we obtain $\beta(g)$.

$$\beta(g) = \epsilon - \frac{1}{2\pi^2 g} + \ldots \tag{23.120}$$

where $\ldots$ represents $O\left(1/g^2\right)$ terms and higher-order in ϵ terms have been dropped. This is interpolated with the small g limit in figure 23.12.

There are several consequences which follow immediately. (1) There is no metallic state of non-interacting fermions in $d \leq 2$ (and, therefore, no metal-insulator transition, of course). (2) There is a metal-insulator transition with no minimum metallic conductivity in $d > 2$. (3) For $d = 2+\epsilon$, the metal-insulator transition occurs at $g \sim 1/\epsilon$, which is in the perturbative regime. The first statement is clear from figure 23.12: since $\beta(g) < 0$ for all g, the conductance always flows to the localized regime of small g. The second statement follows from the positivity of $\beta(g)$ for large g. In order to connect to the negative $\beta(g)$ localized regime at small g, $\beta(g)$ must pass through zero at some g_c. For $g > g_c$, $\beta(g) > 0$ and the conductance flows to large values, where it is Ohmic; this is the metallic state. For $g < g_c$, $\beta(g) < 0$ and the conductance flows to small values; this is the insulating state. For $g = g_c$, the conductance remains constant; this is the transition point. However, if the conductance remains constant as the system size L goes to infinity, then

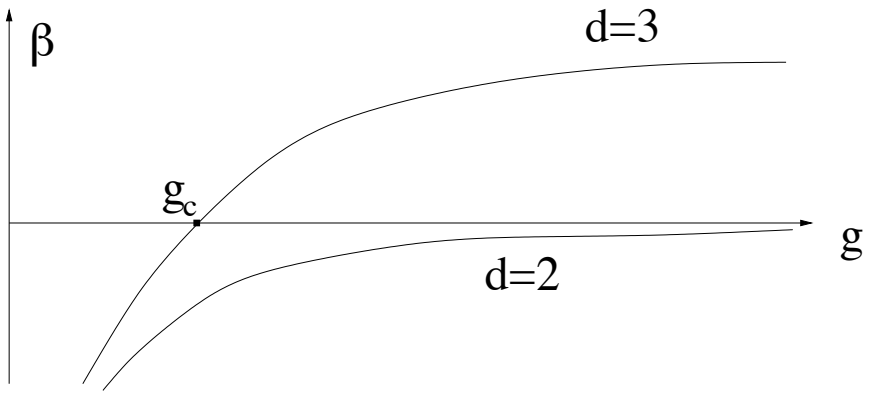


Figure 23.12: The β-function of Abrahams, *et al.* which is obtained by smoothly connecting the Ohmic limit with the localized one.

the conductivity $\sigma_c = g_c L^{2-d} \to 0$. Hence, there is no minimum metallic conductivity.

The third statement follows immediately from (23.120). We can say more about the critical region by expanding the β-function about g_c:

$$\beta(g) = \frac{1}{\nu}\left(\frac{g - g_c}{g_c}\right) \tag{23.121}$$

From our β-function (23.120), we see that $1/\nu = \epsilon$ and $g_c = 1/4\pi^2\epsilon$, where $\epsilon = d - 2$. Since the ... in (23.120) can be neglected only for g large, we can trust our $g_c = 1/4\pi^2\epsilon$ conclusion only for ϵ small, i.e. only near two dimensions.

Suppose $|g - g_c|$ is small at the microscopic scale ℓ. The linearized approximation to the β-function is applicable. Let us integrate until $|g - g_c|/g_c \sim 1$. We define the length scale at which this is reached as ξ:

$$\xi = \ell \left|\frac{g - g_c}{g_c}\right|^{-\nu} \tag{23.122}$$

This is an important length scale because the system will cross over from critical behavior at short-distances to Ohmic metallic ($g > g_c$) or localized insulating ($g < g_c$) behavior at longer length scales. On the insulating side, ξ is the localization length.

On the conducting side as well, ξ is a quantum-classical crossover length. For length scales longer than ξ, electrical conduction is Ohmic and, therefore,

essentially classical. For length scales shorter than ξ, quantum interference effects can occur and classical ideas are inappropriate. Since the conductivity is independent of length in the classical Ohmic regime, it is equal to

$$\sigma = \frac{g_c}{\xi^{d-2}} \tag{23.123}$$

The vanishing of the minimum metallic conductivity can be understood as the divergence of ξ at the metal-insulator transition.

Thus, the scaling hypothesis of Abrahams, *et. al* is equivalent to the assumption that there is a single length scale, ξ, which becomes large near the metal-insulator transition. This is completely analogous to the case of classical thermal phase transitions studied in chapter .. or quantum phase transitions studied in chapter .. However, there is an important difference: *a metal-insulator transition of non-interacting fermions is not a thermodynamic phase transition.* While the conductivity changes sharply at the transition, all thermodynamic properties, such as the ground state energy, are perfectly smooth across the transition. The latter follows from the smoothness of the density-of-states across the transition (which, in turn, was proved by Thouless). A simple way of seeing why it is smooth is to consider the disorder-averaged single-particle Green function. It is short-ranged on both sides of the transition and does not show any qualitative difference between the metallic and insulating states, so it (and the density-of-states which is derived from it) varies smoothly across the transition. For instance, weak-localization effects, which suppress the conductivity, do not affect the single-particle Green function, as one may see by drawing diagrams which might renormalize the single-particle Green function. Within the effective field theory approach of the next chapter, this will be a little more obvious.

In an interacting system, however, a metal-insulator transition might be a true thermodynamic phase transition. Because interacting disordered systems are technically challenging, we will postpone further discussion of this issue to the next chapter, after we have developed the appropriate effective field theories.

23.6 The Integer Quantum Hall Plateau Transition

CHAPTER 24

Non-Linear σ-Models for Diffusing Electrons and Anderson Localization

24.1 Introduction

In this chapter, we will present a formalism which will elucidate the results of the previous chapter and cast them in the form of a field theory which brings out analogies between the metal-insulator transition and other critical phenomena. The basic physical observation is that the only low-energy, long-wavelength correlation functions (after disorder-averaging) are density-density, spin-spin, etc. correlation functions which are diffusive in form as a result of a conservation law. All other disorder-averaged correlation functions decay exponentially. Hence, we construct an effective field theory for the low-energy, long-wavelength correlation functions at the diffusive Fermi gas fixed point of figure 24.1. (However, it is worth remembering that there is still a constant, non-zero density of states, which reflects the fact that there is still a Fermi surface's worth of single-particle states near the Fermi energy. The 'mass term' in the Green function, $G^{-1} = \epsilon - \xi_k + i/2\tau$ is *imaginary*. It's not that there are no low-energy states, as would be the case with a real mass term. Rather, it is that there are low-energy states, but they are not momentum eigenstates. These states show up in, for instance, the specific heat. Upon disorder-averaging, however, only the diffusion of conserved currents remains at long-wavelengths. Thus, we would like an effective field theory which directly gives us this diffusion physics without

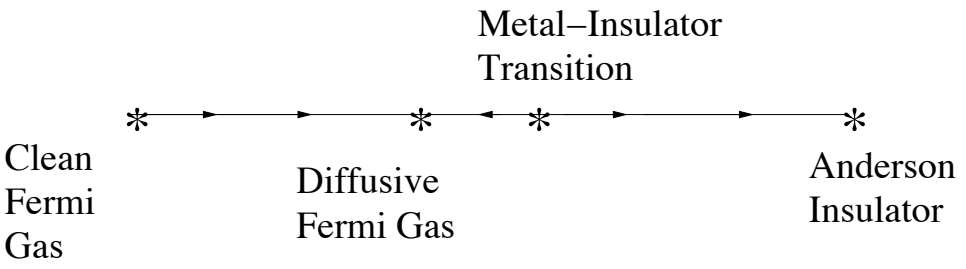


Figure 24.1: Renormalization group flow diagram for a Fermi gas in the presence of disorder. The clean fixed point is highly unstable to disorder and flows into the diffusive metal. When the disorder strength becomes sufficiently strong, a metal-insulator transition occurs, beyond which the flows are into the Anderson insulating fixed point.

the encumbrance of the low-energy single-particle states.)

One might be tempted, at first, to construct an effective theory built on the density operator, since it has long-ranged correlations. Let's imagine trying to do this. The first problem which one must confront is that the density satisfies the diffusion equation, so we would expect it to have an action like

$$S[\rho] = \int d^d x dt \, \rho \left(\partial_t + \nabla^2 \right) \rho \tag{24.1}$$

However, the time-derivative term is a total derivative, so we can drop it from the action!

Consider, also, the problem of interaction terms. The density should have self-interactions – perhaps of the form ρ^4 – since these must generate the quantum interference effects which we computed in the previous chapter. Also, there should be some coupling of the density to the electron operator, presumably of the form $\rho\psi^\dagger\psi$. However, there are problems with such terms: for instance, the latter would induce singular corrections to the electron Green function, which shouldn't be present in the absence of electron-electron interactions.

Thus, we see that a simple-minded approach to an effective field theory won't work, but, as we will see later, the general philosophy is correct. In order to endow our theory with the necessary properties, we will need some more formalism, to which we turn in the next section.

24.2 The Replica Method

Let us return now to the problem of performing the disorder average of the free energy.

$$\begin{aligned}\overline{F} &= \int \mathcal{D}V\, F[V]\, e^{-\int d^x V^2(x)/2n_i v^2} \\ &= \int \mathcal{D}V\, \ln Z[V]\, e^{-\int d^d x V^2(x)/2n_i v^2}\end{aligned} \tag{24.2}$$

While $Z[V]$ has a functional integral representation, $\ln Z[V]$ does not. Hence, we cannot write, say, an effective action for the disorder-averaged system. However, we now observe that

$$\ln Z[V] = \lim_{n\to 0} \frac{1}{n}\left(Z^n - 1\right) \tag{24.3}$$

This is a useful expression because Z^n has a simple functional integral representation if n is an integer: we just imagine that we have n independent copies of the system. Thus, apart from a constant $1/n$ term, the free energy is

$$\begin{aligned}\overline{F} &= \lim_{n\to 0} \frac{1}{n} \int \mathcal{D}V \prod_{a=1}^{n} \mathcal{D}\psi_a \mathcal{D}\psi_a^\dagger e^{-\sum_{a=1}^n S[\psi_a,\psi_a^\dagger,V]}\, e^{-\int d^d x V^2(x)/2n_i v^2} \\ &= \int \prod_{a=1}^{n} \mathcal{D}\psi_a \mathcal{D}\psi_a^\dagger \left(\int \mathcal{D}V e^{-\sum_{a=1}^n S[\psi_a,\psi_a^\dagger,V]}\, e^{-\int d^d x V^2(x)/2n_i v^2} \right)\end{aligned} \tag{24.4}$$

The replica index $a = 1, 2, \ldots, n$ labels the n identical copies of the system. Hence, we have a disorder-averaged action:

$$e^{-S_{\rm avg}[\psi_a,\psi_a^\dagger]} = \int \mathcal{D}V e^{-\sum_{a=1}^n S[\psi_a,\psi_a^\dagger,V]}\, e^{-\int d^d x V^2(x)/2n_i v^2} \tag{24.5}$$

After V has been integrated out, the n independent ψ_as become coupled together.

This method for averaging the free energy is called the *replica method*, and the limit $n \to 0$ is called the replica method. There is some sleight of hand involved here since we can do the calculation for integer n but we then take the limit $n \to 0$. However, within perturbation theory, there is no difficulty in taking this limit; it is simply a bookkeeping device which discards unwanted diagrams with closed electron loops. Beyond perturbation theory, little is known about the validity of taking the replica limit; one usually proceeds without answering this question and hopes for the best.

Consider, for example, a system of non-interacting electrons moving in a random potential:

$$S[\psi,\psi^\dagger] = \int d\tau\, d^dx\, \psi^\dagger \left(\partial_\tau + \frac{1}{2m}\nabla^2 + \mu\right)\psi + \int d\tau\, d^dx\, V(x)\psi_a^\dagger(x,\tau)\psi_a(x,\tau) \quad (24.6)$$

Following the replica procedure, we find that

$$\overline{F} = \frac{1}{n}\int \prod_{a=1}^{n} \mathcal{D}\psi_a \mathcal{D}\psi_a^\dagger\, e^{-S_{\rm avg}[\psi_a,\psi_a^\dagger]} \quad (24.7)$$

where

$$S_{\rm avg}[\psi_a,\psi_a^\dagger] = \int d\tau\, d^dx\, \psi_a^\dagger \left(\partial_\tau + \frac{1}{2m}\nabla^2 + \mu\right)\psi_a + n_i v^2 \int d\tau\, d\tau'\, d^dx\, \psi_a^\dagger(x,\tau)\psi_a(x,\tau)\,\psi_b^\dagger(x,\tau')\psi_b(x,\tau') \quad (24.8)$$

The second term, which results from the disorder average, is non-local in time (reflecting the time-independence of the disorder) and couples different replicas. This action has both figures 23.4a,b but with a free summation over a replica index in 23.4b so that this diagram is proportional to n. In the replica limit, this diagram vanishes.

The RG perspective on the diffusive Fermi liquid which we introduced in the previous chapter applies directly to this replicated action.

Mention Supersymmetry, Schwinger-Keldysh

24.3 Non-Interacting Electrons

24.3.1 Derivation of the σ-model

We will focus on interacting electrons or, at least, on methods which can be applied to both non-interacting and interacting electrons. There are tow reasons for this: (1) there is a great deal of formalism which has evolved to describe non-interacting electrons, which is beyond the scope of this book and is well-described in books and review articles, and (2) in the real world, electrons interact, and these interactions are evidently important in two-dimensions, as we saw when we studied perturbative corrections in the previous chapter. Hence, we focus on interacting electrons in the presence of

disorder. However, the problem of non-interacting electrons in under much better control, and the theory is on much safer footing. Some of the analogies with other critical phenomena can be made more precise. Hence, we introduce this theory briefly and comment on its physical interpretation, before moving on to the interacting case.

In order to compute the conductivity or density-density correlation function, we need to compute the product of an advanced and a retarded electron Green function at the chemical potential. Such products of an advanced and a retarded Green function have their phase cancelled, so that impurity averaging does not make them short-ranged, unlike the electron Green function. For instance, the advanced Green function is the complex conjugate of a retarded Green function at the same energy, so

$$\left|\left\langle \psi^\dagger(E+i\delta,x)\psi(E+i\delta,x)\right\rangle\right|^2 = \left\langle \psi^\dagger(E+i\delta,x)\psi(E+i\delta,x)\right\rangle \left\langle \psi^\dagger(E-i\delta,x)\psi(E-i\delta,x)\right\rangle \quad (24.9)$$

These Green functions can be obtained from the following functional integral:

$$\left\langle \psi^\dagger(E+i\delta,x)\psi(E+i\delta,x)\right\rangle \left\langle \psi^\dagger(E-i\delta,x)\psi(E-i\delta,x)\right\rangle = \int \mathcal{D}\psi_R(\mathbf{x})\,\mathcal{D}\psi_R^\dagger(\mathbf{x})\,\mathcal{D}\psi_A(\mathbf{x})\,\mathcal{D}\psi_A^\dagger(\mathbf{x})\,\left(\psi_R^\dagger(\mathbf{x})\psi_R(\mathbf{x})\,\psi_A^\dagger(\mathbf{x})\psi_A(\mathbf{x})\right)\, e^{-S_{E,E}} \quad (24.10)$$

where

$$S_{E,E'} = \int d^dx\, \left(\psi_R^\dagger\left(E+i\delta+\frac{1}{2m}\nabla^2\right)\psi_R + V(x)\psi_R^\dagger(x)\psi_R(x) + \psi_A^\dagger\left(E'-i\delta+\frac{1}{2m}\nabla^2\right)\psi_A + V(x)\psi_A^\dagger(x)\psi_A(x)\right) \quad (24.11)$$

In order to perform the disorder average, we use the replica method. We introduce n copies of the functional integral

$$Z^n[V] = \int \prod_{a=1}^{n} \mathcal{D}\psi_{aR}(\mathbf{x})\,\mathcal{D}\psi_{aR}^\dagger(\mathbf{x})\,\mathcal{D}\psi_{aA}(\mathbf{x})\,\mathcal{D}\psi_{aA}^\dagger(\mathbf{x})\, e^{-S_{n\,E,E'}} \quad (24.12)$$

where

$$S_{n\,E,E'} = \int d^dx \left(\psi^\dagger_{aR} \left(E + i\delta + \frac{1}{2m}\nabla^2 \right) \psi_{aR} + V(x)\psi^\dagger_{aR}(x)\psi_{aR}(x) + \psi^\dagger_{aA} \left(E' - i\delta + \frac{1}{2m}\nabla^2 \right) \psi_{aA} + V(x)\psi^\dagger_{aA}(x)\psi_{aA}(x) \right) \quad (24.13)$$

We now average over disorder:

$$\begin{aligned} \overline{Z^n} &= \int [V e^{-\int d^dx V^2(x)/2n_i v^2}\, Z^n[V] \\ &= \int \prod_{a=1}^{n} \mathcal{D}\psi_{aR}(\mathbf{x})\, \mathcal{D}\psi^\dagger_{aR}(\mathbf{x})\, \mathcal{D}\psi_{aA}(\mathbf{x})\, \mathcal{D}\psi^\dagger_{aA}(\mathbf{x})\, e^{-S^{\rm rep}_{E,E'}} \end{aligned} \quad (24.14)$$

with

$$\begin{aligned} S^{\rm rep}_{E,E'}[\psi^\dagger_{aR}, \psi_{aR}, \psi^\dagger_{aA}, \psi_{aA}] = \int d^dx \Big[& \psi^\dagger_{aR} \left(E + i\delta + \frac{1}{2m}\nabla^2 \right) \psi_{aR} \\ & + \psi^\dagger_{aA} \left(E' - i\delta + \frac{1}{2m}\nabla^2 \right) \psi_{aA} \\ & + n_i v^2 \left(\psi^\dagger_{aR}\psi_{aR} + \psi^\dagger_{aA}\psi_{aA} \right) \left(\psi^\dagger_{bR}\psi_{bR} + \psi^\dagger_{bA}\psi_{bA} \right) \Big] \end{aligned} \quad (24.15)$$

We introduce a capital Latin index $B = R, A$, so that this can be re-written

$$\begin{aligned} & S^{\rm rep}_{E+\omega/2,E-\omega/2}[\psi^\dagger_{aB}, \psi_{aB}] = \\ & \int d^dx \left[\psi^\dagger_{aB} \left(\delta_{BC} \left(E + \frac{1}{2m}\nabla^2 \right) + \Lambda_{BC} \left(\frac{\omega}{2} + i\delta \right) \right) \psi_{aC} \right. \\ & \left. + n_i v^2\, \psi^\dagger_{bB}\psi_{bB}\, \psi^\dagger_{cC}\psi_{cC} \right] \end{aligned} \quad (24.16)$$

where $\Lambda_{BC} = \mathrm{diag}(1, -1)$.

If we were to compute perturbatively with this effective action, we would recover the results of the previous chapter. However, as we noted earlier, the final term, is a strongly-relevant perturbation. It is no-local in time or, equivalently, it is a function of only two, not three energies. Thus, we should not expect perturbation theory to be valid for this term. Indeed, we did not us low-order perturbation theory in the previous chapter, but had to sum an infinite set of diagrams in order to obtain physically-reasonable results. Here, we will deal with this relevant perturbation by decoupling it with a Hubbard-Stratonovich transformation.

There are two choices of Hubbard-Stratonovich decomposition. (We will make the simplification of ignoring spin in this section.) We can either introduce a Hubbard-Stratonovich field which is essentially equal to $\psi^\dagger_{bB}\psi_{bB}$ or one which is equal to $\psi^\dagger_{bB}\psi_{cC}$. The former is just the density of electrons at energy $E+\omega/2$ plus the density of electrons at energy $E-\omega/2$. If it develops an expectation value, it is just a shift of the chemical potential. The latter, however, is non-local in time (since it is a function of two energies, $E\pm\omega/2$) and can, in principle, have non-trivial structure in replica and advanced/retarded indices. Thus, we decouple the quartic term in this channel. In so doing, we re-write the final term in (24.16) as:

$$\psi^\dagger_{bB}\psi_{bB}\,\psi^\dagger_{cC}\psi_{cC} = -\psi^\dagger_{bB}\psi_{cC}\,\psi^\dagger_{cC}\psi_{bB} \tag{24.17}$$

We introduce a Hubbard-Statonovich field $Q_{bc;BC}$ by rewriting the partition function as:

$$\overline{Z^n} = \int \prod_{a=1}^{n} \mathcal{D}\psi_{aR}(\mathbf{x})\,\mathcal{D}\psi^\dagger_{aR}(\mathbf{x})\mathcal{D}Q_{bc;BC}\; e^{-S_{\rm HS}[Q,\psi^\dagger,\psi]} \tag{24.18}$$

where

$$\begin{aligned} S_{\rm HS}[Q_{bc;BC},\psi^\dagger_{aB},\psi_{aB}] = \\ \int d^dx\,\Big[\psi^\dagger_{aB}\left(\delta_{BC}\left(E+\frac{1}{2m}\nabla^2\right)+\Lambda_{BC}\left(\frac{\omega}{2}+i\delta\right)\right)\psi_{aC} \\ + \frac{i}{2\tau}Q_{bc;BC}\,\psi^\dagger_{bB}\psi_{cC} + \frac{\pi N_F}{4\tau}Q_{bc;BC}Q_{cb;CB}\Big] \end{aligned} \tag{24.19}$$

Note that the minus sign in equation (24.17) resulted in the i in this equation.

To proceed, we will integrate out the electrons and look for a saddle-point for Q. Integrating out the electrons, we find:

$$Z = \int \mathcal{D}Q\, e^{-S_{\rm eff}[Q,]} \tag{24.20}$$

where $S_{\rm eff}[Q,]$ is given by

$$\begin{aligned} S_{\rm eff}[Q] = \\ \int d^dx\,\Big[-{\rm tr}\,\ln\!\left(\delta_{BC}\left(E+\frac{1}{2m}\nabla^2\right)+\Lambda_{BC}\left(\frac{\omega}{2}+i\delta\right)+\frac{i}{2\tau}Q_{BC}\right) \\ + \frac{\pi N_F}{4\tau}{\rm tr}\left(Q^2\right)\Big] \end{aligned} \tag{24.21}$$

If we assume a translationally-invariant saddle-point, the saddle-point equation is:

$$\pi N_F \hat{Q} = i \int \frac{d^2p}{(2\pi)^2} \frac{1}{i\hat{\epsilon_n} + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}\hat{Q}} \tag{24.22}$$

It has solution

$$Q_{bc;BC} = \Lambda_{BC}\, \delta_{bc} \tag{24.23}$$

At this saddle-point, the electron Green functions are of the expected form:

$$\begin{aligned} G_{B=R,A}(\mathbf{k}, E) &= \frac{1}{E - \frac{k^2}{2m} + \frac{i}{2\tau} Q_{BB}} \\ &= \frac{1}{E - \frac{k^2}{2m} \pm \frac{i}{2\tau}} \end{aligned} \tag{24.24}$$

Thus, the saddle-point has self-consistently computed corrections to the propagator and summed them to obtain the self-energy correction.

We can now obtain the spectrum of fluctuations about the saddle-point by expanding (24.21) about $Q = \Lambda$:

$$\begin{aligned} &S_{\text{eff}}[Q] = \\ &\int d^dx \left[-\text{tr } \ln\left(\delta_{BC} \left(E + \frac{1}{2m}\nabla^2 \right) + \Lambda_{BC} \left(\frac{\omega}{2} + \frac{i}{2\tau} \right) + \frac{i}{2\tau} \delta Q_{BC} \right) \right. \\ &\qquad \left. + \frac{\pi N_F}{2\tau} \text{tr}\left(\delta Q \Lambda\right) + \frac{\pi N_F}{4\tau} \text{tr}\left(\delta Q^2\right) \right] \\ &\quad = \int d^dx \left[-\text{tr } \ln\left({G_{BC}}^{-1} + + \frac{i}{2\tau} \delta Q_{BC} \right) \right. \\ &\qquad \left. + \frac{\pi N_F}{2\tau} \text{tr}\left(\delta Q \Lambda\right) + \frac{\pi N_F}{4\tau} \text{tr}\left(\delta Q^2\right) \right] \end{aligned} \tag{24.25}$$

where $G_{RR} = G_R\left(E + \frac{\omega}{2}\right)$, $G_{AA} = G_A\left(E - \frac{\omega}{2}\right)$, and $G_{RA} = G_{AR} = 0$. Expanding the logarithm to second order, we have:

$$S_{\text{eff}}[Q] = \int d^dx \left[-\frac{1}{8\tau^2} \text{tr}\left(G \delta Q\, G \delta Q\right) + \frac{\pi N_F}{4\tau} \text{tr}\left(\delta Q^2\right) \right] \tag{24.26}$$

The linear terms in δQ vanish by the saddle-point condition. Consider the

first term in equation (24.26):

$$\begin{aligned}\mathrm{tr}\,(G\delta Q\, G\delta Q) = \int \frac{d^dp}{(2\pi)^d}\,\frac{d^dq}{(2\pi)^d}\,(&G_R(p+q)\delta Q_{RA}(q)G_A(p)\delta Q_{AR}(-q)\\ &+ G_A(p+q)\delta Q_{AR}(q)G_R(p)\delta Q_{RA}(-q)\\ &+ G_R(p+q)\delta Q_{RR}(q)G_R(p)\delta Q_{RR}(-q)\\ &+ G_A(p+q)\delta Q_{AA}(q)G_A(p)\delta Q_{AA}(-q))\end{aligned} \tag{24.27}$$

In the previous chapter, we computed the p integrals in imaginary time. Continuing to real-time, we have

$$\begin{aligned}\int \frac{d^dp}{(2\pi)^d}\,G_R(p+q,E+\frac{\omega}{2})G_A(p,E-\frac{\omega}{2}) &= 2\pi N_F\tau\left(1-i\omega\tau-D\tau q^2\right)\\ \int \frac{d^dp}{(2\pi)^d}\,G_R(p+q,E+\frac{\omega}{2})G_R(p,E+\frac{\omega}{2}) &= 0\end{aligned} \tag{24.28}$$

and similarly for the other two integrals. Thus, we have to second order in δQ:

$$\begin{aligned}S_{\text{eff}}[Q] = \frac{\pi N_F}{4}\int d^dx\,\Big(&D\mathrm{tr}(\nabla\delta Q_{RA})^2 + D\mathrm{tr}(\nabla\delta Q_{AR})^2\\ &+ i\omega\,\mathrm{tr}\big(\delta Q_{RA}^2+\delta Q_{AR}^2\big) + \frac{1}{\tau}\,\mathrm{tr}\big(\delta Q_{RR}^2+\delta Q_{AA}^2\big)\Big)\end{aligned} \tag{24.29}$$

To this order, Q_{RR} and Q_{AA} are massive fields, while Q_{RA}, Q_{AR} are massless for $\omega = 0$. Since Q_{RR}, Q_{AA} are non-zero at the saddle-point, one might say that the *longitudinal* fields are massive, as in a non-linear σ-model, while the transverse fields, Q_{RA}, Q_{AR} are massless. As we will see momentarily, this interpretation is correct.

Suppose we would like to compute $S_{\text{eff}}[Q]$ to higher orders. We could continue to expand in powers of δQ, but this is tedious. A much simpler way is to identify the symmetries of the action; these will constrain the form of $S_{\text{eff}}[Q]$ completely. Inspecting (24.16) for ω, we see that any transformation of the form $\psi_{bB} \to U_{bc;BC}\psi_{cC}$ will leave (24.16) invariant so long as $U^\dagger_{cb;CB}U_{bd;BD} = \delta_{cd}\,\delta_{CD}$. Under such a transformation, $Q_{bc;BC} \to U^\dagger_{bd;BD}Q_{de;DE}U_{ec;EC}$. Since such a transformation must leave the action invariant, (24.23) is not the only saddle-point of the action. In fact, the entire manifold $U^\dagger_{bd;BD}\Lambda_{DE}U_{dc;EC}$ extremizes the action. Thus, the effective action $S_{\text{eff}}[Q]$ must vanish for any Q of this form. Configurations in which Q varies slowly within this manifold must be gapless, since they

have only gradient energy. Small but non-zero $\omega + i\delta$ gives these excitations a small gap because it favors $Q = \Lambda$. Meanwhile, configurations in which Q moves out of this manifold should cost energy; this is why we found that δQ_{RR}, δQ_{AA} are massive while δQ_{RA}, δQ_{AR} are massless.

The above paragraph is almost correct. One gap in these arguments is that the unitary transformations $U_{bd;BD}$ are not the only symmetries of the action (24.16). As a result of time-reversal symmetry, the equations satisfied by ψ are the same as those satisfied by $\psi^\dagger$. In other words, we should not only allow transformations which mix the different ψ_{bB}s, but also those which mix ψ_{bB} with $\psi^\dagger_{cC}$ so long as they leave $\psi^\dagger_{cC}\psi_{cC}$ invariant (and, therefore, leave (24.16) invariant for $\omega + i\delta = 0$). These transformations can be parametrized by defining

$$\begin{pmatrix} \chi_{bB1} \\ \chi_{bB2} \end{pmatrix} = \begin{pmatrix} \psi_{bB} \\ \psi^\dagger_{bB} \end{pmatrix} \tag{24.30}$$

Then we can write

$$\psi^\dagger_{bB}\psi_{bB} = \frac{1}{2}\,\chi_{bBi}J_{ij}\chi_{bBj} \tag{24.31}$$

where $J_{ij} = \epsilon_{ij}$. Hence, all transformations

$$\chi_{bBi} \to M_{bc;BC;ij}\chi_{cCj} \tag{24.32}$$

are symmetries of (24.16) for $\omega + i\delta = 0$ so long as

$$M^T_{bc;BC;ij}\, J_{jk}\, M_{cd;CD;kl} = J_{il}\delta_{bd}\,\delta_{BD} \tag{24.33}$$

or, simply,

$$M^T J M = J \tag{24.34}$$

where $J \equiv J_{il}\delta_{bd}\,\delta_{BD}$. Equation (24.34) is the defining equation of the symplectic group $Sp(2n)$.

$Sp(2n)$ does not act in a simple way on $Q_{bc;BC}$ as we defined it above. However, we can rewrite

$$\psi^\dagger_{bB}\psi_{bB}\,\psi^\dagger_{cC}\psi_{cC} = \frac{1}{4}\,\chi_{bBi}J_{ij}\chi_{bBj}\,\chi_{cCk}J_{kl}\chi_{cCl} = -\frac{1}{4}\,\chi_{bBi}J_{ij}\chi_{cCl}\,\chi_{cCk}J_{kl}\chi_{bBj} \tag{24.35}$$

and decouple this term with Q so that it becomes

$$\frac{i}{2\tau}Q^{jl}_{bc;BC}\,\chi_{bBi}J_{ij}\chi_{cCl} + \frac{\pi N_F}{4\tau}Q^{ij}_{bc;BC}Q^{ji}_{cb;CB} = \frac{i}{2\tau}\,\chi^T J Q \chi + \frac{\pi N_F}{4\tau}\mathrm{tr}\big(Q^2\big) \tag{24.36}$$

With this definition, Q must transform so that these terms remain invariant. In particular, the first term in (24.36) transforms as

$$\chi^T J Q \chi \to \chi^T M^T J Q' M \chi = \chi^T J M^{-1} Q' M \chi \tag{24.37}$$

In the first equality, we have used (24.34). This will remain invariant if

$$Q' = M Q M^{-1} \tag{24.38}$$

which also leaves the second term in (24.36) invariant.

Hence, the saddle-point manifold is given by

$$Q = M \, \Lambda \, M^{-1} \tag{24.39}$$

This manifold is parametrized by $Sp(2n)$ matrices M. However, not every different M leads to a different Q. Any M which doesn't mix retarded fields with advanced ones commutes with Λ. Therefore the corresponding subset of transformations $Sp(n) \times Sp(n) \subset Sp(2n)$ don't lead to new configurations. In other words, $Q \in Sp(2n)/Sp(n) \times Sp(n)$. For such Q, the action is simply:

$$S_{\text{eff}}[Q] = \frac{\pi N_F}{4} \int d^d x \, \left(D \, \text{tr}(\nabla Q)^2 + (i\omega - \delta) \, \text{tr}(\Lambda Q) + \ldots \right) \tag{24.40}$$

where the $\ldots$ refers to higher-derivative terms. The second term has been written in the above manner because non-zero $\omega + i\delta$ favors $Q = \Lambda$, which is equivalent to the condition $Q_{RA} = Q_{AR} = 0$ which we had in the second-order expansion of $S_{\text{eff}}[Q]$.

The effective action $S_{\text{eff}}[Q]$ describes a non-linear σ-model. The main complication is that instead of a field which takes values on the sphere $S^2 = O(3)/O(2)$ (as in an antiferromagnet), we have a field which takes values in a different coset, namely $Sp(2n)/Sp(n) \times Sp(n)$.

If we consider a system of electrons in a weak magnetic field, then transformations which mix ψ and $\psi^\dagger$ are no longer symmetries of the action. The symmetries of the action

$$\begin{aligned} S^{\text{rep}}_{E+\omega/2, E-\omega/2}[\psi^\dagger_{aB}, \psi_{aB}] = \\ \int d^d x \, \left[\psi^\dagger_{aB} \left(\delta_{BC} \left(E + \frac{1}{2m} (\nabla - i\mathbf{A})^2 \right) + \Lambda_{BC} \left(\frac{\omega}{2} + i\delta \right) \right) \psi_{aC} \right. \\ \left. + n_i v^2 \, \psi^\dagger_{bB} \psi_{bB} \, \psi^\dagger_{cC} \psi_{cC} \right] \end{aligned} \tag{24.41}$$

are $U(2n)$ symmetries $\psi_{cC} \to U_{cd;CD} \psi_{dD}$ because the first derivative term $\mathbf{A} \cdot \nabla + \nabla \cdot \mathbf{A}$ is not invariant under the the rest of the $Sp(2n)$ group. Hence,

a system of electrons in a weak magnetic field ('weak' means too weak for the quantum Hall effect to occur) will also be described by a NLσM of the form (24.40), but with Q taking values in $U(2n)/U(n) \times U(n)$. Systems in this universality class – characterized by the absence of time-reversal symmetry – are said to be in the 'unitary ensemble' because of the unitary symmetry group $U(2n)$. Meanwhile, systems with time-reversal invariance, as we discussed above, have symplectic symmetry group $Sp(2n)$ and said to be in the 'orthogonal ensemble'. Finally, systems with spin-orbit scattering have symmetry group $O(2n)$ (see ref.) and are said to be in the 'symplectic ensemble'. The reason for this confusing terminology is that the method which we have used in this section is called 'fermionic replicas' because we could have used bosonic fields to construct the functional integral (24.10). For non-interacting electrons, thee two representations are equivalent. However, the symmetries of the bosonic action are different, and we would have found NLσMs for the cosets $O(n,n)/O(n) \times O(n)$, $U(n,n)/U(n) \times U(n)$, $Sp(n,n)/Sp(n) \times Sp(n)$ for the orthogonal, unitary, and symplectic cases, respectively. However the β-function for $O(n,n)/O(n) \times O(n)$ is the same as the β-function for $Sp(2n)/Sp(n) \times Sp(n)$, etc. so the same physics is obtained.

24.3.2 Interpretation of the σ-model; Analogies with Classical Critical Phenomena

Superficially, our NLσM

$$S_{\text{eff}}[Q] = \frac{\pi N_F}{4} \int d^d x \, \Big(D \text{tr} (\nabla Q)^2 + (i\omega - \delta) \, \text{tr} (\Lambda Q) \Big) \tag{24.42}$$

looks very similar to the NLσM which describes an classical ferromagnet or antiferromagnet near its lower critical dimension, $d = 2$.

$$S_{\text{eff}}[\mathbf{n}] = \frac{\pi N_F}{4} \int d^d x \, \Big(\frac{\rho_s}{2} \text{tr} (\nabla \mathbf{n})^2 + \mathbf{H} \cdot \mathbf{n} \Big) \tag{24.43}$$

In both cases, there is a Goldstone phase. In the former case, this is the metallic phase, with diffusion propagators $(-i\omega + Dq^2)^{-1}$; in the latter case, this is the magnetically-ordered phase, with spin-wave propagators $(H + \rho_s q^2)^{-1}$. In both cases, there is a critical point in $d = 2 + \epsilon$ which is near the Goldstone phase's stable fixed point. In the former case, it is the metal-insulator transition; in the latter case, it is the Curie or Néel point. However, the similarity ends here. In the latter case (24.43), there is a disordered phase in which the symmetry is restored and the order parameter vanishes. While

the same is true for any finite n in (24.42), this does not quite happen in the replica limit, $n \to 0$.

The basic symmetry (replicated and enmeshed with particle-hole symmetry) is between advanced and retarded Green functions. If Q were to vanish and we were to take $i\delta = 0$, these Green functions would be precisely the same.

$$G_{B=R,A}(\mathbf{k}, E) = \frac{1}{E - \frac{k^2}{2m} + \frac{i}{2\tau} Q_{BB}} \tag{24.44}$$

However, if they were precisely equal, the spectral function and, therefore, the density-of-states would vanish. Hence, the density-of-states plays the role of an order parameter. Diffusion modes are slowly-varying oscillations of this order parameter which remain within the low-energy manifold $Q = M\Lambda M^{-1}$.

However, the density-of-states is non-zero in the insulating state as well and varies smoothly through the transition, in contrast to the magnetization at a magnetic transition. As we will show in the next subsection, the critical exponent for the order parameter vanishes, $\beta = 0$, so

$$\rho(E) \sim \frac{1}{n} \operatorname{tr}(\Lambda Q)_{\delta \to 0^+} \sim |E - E_C|^{\beta} \sim \text{const.} \tag{24.45}$$

Thus, *the symmetry is always broken.*

McKane and Stone, who pointed this out, explained this as a special feature of disordered systems. If E is in the regime of localized states, then

$$|G(x, y; E + i\delta)|^2 \to \frac{\pi}{\delta} \sum_l \delta(E - E_l) \; |\psi_l(x)|^2 \, |\psi_l(y)|^2 \tag{24.46}$$

for $\delta \to 0$. Where the E_l are the energies of localized states. This follows from

$$G(x, y; E + i\delta) = \sum_n \frac{\psi_n^*(x)\psi_n(y)}{E_n - E - i\delta} \tag{24.47}$$

and the fact that the cuts due to a continuum of extended states can be deformed away from the integration contour in (24.46) while the isolated poles due to localized states cannot be avoided. Therefore,

$$\langle Q_{RA}(x) Q_{AR}(y) \rangle = |G(x, y; E + i\delta)|^2 \sim \frac{1}{\delta} \, e^{-2|x-y|/\xi} \tag{24.48}$$

where L is the system size. This factor comes from the spatial density of localized states l in (24.46). Hence, the correlation function is divergent

for arbitrary separation as $\delta \to 0$. Fourier transforming, we see that it is divergent for all p:

$$\langle Q_{RA}(p)Q_{AR}(-p)\rangle \sim \frac{1}{\delta}\, g(p) \tag{24.49}$$

where $g(p)$ is a smooth function of p which is finite at $p = 0$. On the other hand, if E is in the regime of extended states, then we have diffusion:

$$\langle Q_{RA}(p)Q_{AR}(-p)\rangle \sim \frac{1}{Dp^2 + 2\delta} \tag{24.50}$$

Comparing these two expressions, the localized phase has $D \to 0$, but *not* the restoration of symmetry or the dynamical generation of a mass. Compare this to the $O(3)$ NLσM which we studied in the context of antiferromagnetism. There, the correlation function in the disordered state is of the form $\langle \mathbf{n}(p)\mathbf{n}(-p)\rangle \sim 1/(Dp^2 + m^2)$; a mass term is generated and the spin stiffness D (the analog of the diffusion constant) is irrelevant compared to it.

Therefore, the $\langle Q_{RA}(p)Q_{AR}(-p)\rangle$ correlation function is divergent in both phases. However, it is divergent only at $p = 0$ in the metallic phase, but divergent at all p in the insulating phase. In the latter case, there are no long-ranged correlations or long-wavelength excitations, but neither is there a gap – the density of states at the chemical potential is finite. This situation should be viewed as a generalization of Goldstone's theorem. When a symmetry is broken, there can be no mass gap because the order parameter is susceptible to infinitesimal perturbations. There are two ways in which this can happen: (a) the conventional way – the gapless states are Goldstone modes – or (b) they can be localized excitations. Thus, the basic symmetry between advanced and retarded Green functions – which differ only by the $i\delta$ in their definition – is broken, so that infinitesimal δ leads to a finite difference between then. The symmetry is broken in both the metallic and insulating phases; the difference is that (a) occurs in the former while (b) occurs in the latter.

24.3.3 RG Equations for the NLσM

We will now compute the one-loop RG equations for the NLσMs introduced in this section, thereby recovering the Gang of Four's scaling equation. In order to do so, we need a convenient parametrization for Q. One simple one is:

$$Q = \begin{pmatrix} \left(1 - VV^\dagger\right)^{1/2} & V \\ V^\dagger & -\left(1 - V^\dagger V\right)^{1/2} \end{pmatrix} \tag{24.51}$$

where V is an unconstrained $n \times n$ matrix. For $Sp(2n)/Sp(n) \times Sp(n)$, the elements of V are quaternions and $\dagger$ means take the transpose and perform the quaternion conjugate. For $U(2n)/U(n) \times U(n)$, the elements of V are complex numbers and $\dagger$ means the adjoint. For $O(2n)/O(n) \times O(n)$, the elements of V are real numbers and $\dagger$ means the transpose.

In order to compute the one-loop β-function, it is sufficient to expand this action to quartic order and, therefore, sufficient to expand Q to quadratic order in V. The result is simple to state. Let $t = 1/D$ and let G be the symmetry group of the NLσM. Then

$$\beta(t) = \epsilon t - Ct^2 + \dots \tag{24.52}$$

where C is the Casimir (in the adjoint representation) of the symmetry group G. In particular

$$\begin{aligned} \beta_{Sp(2n)}(t) &= \epsilon t - (2n+1)t^2 + \dots \\ \beta_{U(n)}(t) &= \epsilon t - nt^2 + \dots \\ \beta_{O(n)}(t) &= \epsilon t - (n-2)t^2 + \dots \end{aligned} \tag{24.53}$$

Note that the β-function does not depend on the subgroup which is being modded out. Instead of $Sp(2n)/Sp(n) \times Sp(n)$, we could just as easily take $Sp(2n)/Sp(p) \times Sp(2n-p)$, and we would obtain the same result.

For $n \to 0$, this give us

$$\beta(t) = \epsilon t - t^2 + \dots \tag{24.54}$$

for electrons in zero field, which is equivalent to the β-function which we found in the previous chapter. In the presence of spin-orbit coupling, the β-function has sign reversed, while in the presence of a weak magnetic field, the one-loop β-function vanishes, and we must go to two loops.

24.4 Interacting Electrons

We now turn to the problem of interacting electrons. We can no longer consider individual frequencies in isolation. As a result, there is no longer an exact symmetry of the theory which we can exploit. Hence, there is no completely controlled route to an effective theory for diffusion modes. Such a theory must be some form of NLσM in limit that the interaction strength vanishes. In the absence of a clear symmetry justification, we must rely on physical arguments: clearly the low-energy degrees of freedom should be

diffusion modes (at least for weak interactions), so we should write down an effective field theory which captures their physics. It turns out that the most straightforward way to do this is to construct a field theory with infinitely-many terms in its Lagrangian, all of whose coefficients are related. Unlike in the non-interacting case, we can only hope that the relationship between these terms is preserved. At one-loop order this holds. Furthermore, the theory contains terms which are completely non-local in time – such as the non-interacting electron NLσM (which is at a fixed energy and, therefore, is completely non-local in time) – and also terms which are local in time. In principle, terms which are only somewhat non-local in time could be generated. At one-loop, this, too, does not occur.

We begin with a system of interacting electrons moving in a quenched random potential $V(x)$ in $2D$. For simplicity, we consider the unitary ensemble. The imaginary-time action is

$$S = S_0 + S_{\rm disorder} + S_{\rm int} \tag{24.55}$$

where

$$S_0 + S_{\rm disorder} = T\sum_n \int d^2x\, \psi^\dagger_{n,\alpha}(x)\left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu + V(x)\right)\psi_{n,\alpha}(x) \tag{24.56}$$

and

$$S_{\rm int} = T\sum_{n,m,l}\int [\Gamma_1\, \psi^\dagger_{n,\alpha}(\mathbf{k}+\mathbf{q})\psi_{m,\alpha}(\mathbf{k})\psi^\dagger_{l,\beta}(\mathbf{p}-\mathbf{q})\psi_{l+m-n,\beta}(\mathbf{p}) + \Gamma_2\, \psi^\dagger_{n,\alpha}(\mathbf{k}+\mathbf{q})\psi_{m,\alpha}(\mathbf{p})\psi^\dagger_{l,\beta}(\mathbf{p}-\mathbf{q})\psi_{l+m-n,\beta}(\mathbf{k})] \tag{24.57}$$

For later convenience, we have written the second term in momentum space and have split the interaction into nearly forward scattering, in which electrons with momenta and spins $(\mathbf{k},\alpha)$, $(\mathbf{p},\beta)$ are scattered to $(\mathbf{k}',\alpha)$, $(\mathbf{p}',\beta)$, where $\mathbf{k}' \approx \mathbf{k}$, $\mathbf{p}' \approx \mathbf{p}$ and exchange scattering, in which $\mathbf{k}' \approx \mathbf{p}$, $\mathbf{p}' \approx \mathbf{k}$. If the microscopic electron-electron interaction is $V(\mathbf{q})$, then the associated bare couping constants are $\Gamma_1 = V(0)$, $\Gamma_2 = \int (d\theta/2\pi)\, V(2k_F \cos\theta/2)$. As Finkelstein noted, these are subject to finite Fermi liquid renormalization in the ballistic regime, even before they begin to flow in the diffusive regime. We assume that the system is in a weak magnetic field, so that we can ignore Cooper scattering, for which $\mathbf{k}_1 \approx -\mathbf{k}_2$, $\mathbf{k}_3 \approx -\mathbf{k}_4$.

In order to perform the average over the quenched random potential, we replicate the theory and then average the replicated functional integral over the probability distribution $\overline{V(\mathbf{x})V(\mathbf{x}')} = \frac{1}{2\pi N_F \tau}\,\delta^{(d)}(\mathbf{x}-\mathbf{x}')$. Now, we have

$$\begin{aligned} S^{\rm rep} = T\sum_{n;a}\int d^2x\,[\psi^\dagger_{n,\alpha}(x)\left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu\right)\psi_{n,\alpha}(x) + \frac{1}{4\pi N_F\tau}\,\left(\psi^{a\,\dagger}_{n,\alpha}\psi^a_{n,\alpha}\right)^2] \\ T\sum_{n,m,l;a}\int [\Gamma_1\,\psi^{a\,\dagger}_{n,\alpha}(\mathbf{k}+\mathbf{q})\psi^a_{m,\alpha}(\mathbf{k})\psi^{a\,\dagger}_{l,\beta}(\mathbf{p}-\mathbf{q})\psi^a_{l+m-n,\beta}(\mathbf{p}) \\ + \Gamma_2\,\psi^{a\,\dagger}_{n,\alpha}(\mathbf{k}+\mathbf{q})\psi^a_{m,\alpha}(\mathbf{p})\psi^{a\,\dagger}_{l,\beta}(\mathbf{p}-\mathbf{q})\psi^a_{l+m-n,\beta}(\mathbf{k})] \end{aligned} \tag{24.58}$$

where $a = 1, 2, \ldots, N$ is a replica index; eventually, we must take the replica limit, $N \to 0$. N_F is the single-particle density-of-states at the Fermi energy. The resulting effective action now contains four quartic terms: one disorder term and two interaction terms. Note that the disorder term is non-local in time since it only depends on two Matsubara frequencies, and it couples different replicas. The interaction terms depend on three Matsubara frequencies, so they are local in time, and they do not couple different replicas.

We decouple these quartic terms with three Hubbard-Stratonovich fields, $Q^{ab}_{nm,\alpha\beta}$, $Y^a_{nm} = Y^a(n-m)$, $X^a_{nm,\alpha\beta} = X^a_{\alpha\beta}(n-m)$. It will be useful to think of X, Y both as matrices with indices n, m and also as functions of a single frequency ω_{n-m}; they have only a single replica index, however, because interactions do not mix different replicas. Note that $X_{\alpha\beta}\,\vec{\sigma}_{\beta\alpha}$ and Y are essentially the spin and charge density. Integrating out the fermions, we can write the partition function as

$$Z = \int \mathcal{D}Q\,\mathcal{D}X\,\mathcal{D}Y\; e^{-S_{\rm eff}[Q,X,Y]} \tag{24.59}$$

where $S_{\rm eff}[Q,X,Y]$ is given by

$$\begin{aligned} S_{\rm eff}[Q,X,Y] = \sum\int [-{\rm tr}\,\ln\left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}Q + i\Gamma_1\,Y + \Gamma_2\,X\right) \\ + \frac{\pi N_F}{4\tau}{\rm tr}\left(Q^2\right) + \frac{1}{2}\,\Gamma_1\,{\rm tr}\left(Y^2\right) + \frac{1}{2}\,\Gamma_2\,{\rm tr}\left(X^2\right)] \end{aligned} \tag{24.60}$$

In this expression, we use 'tr' to mean the trace over all of the indices which are not explicitly written. Matsubara indices are treated as ordinary matrix indices except that their summations come with factors of T.

The saddle-point equations are

$$\pi N_F \hat{Q} = i \int \frac{d^2p}{(2\pi)^2} \frac{1}{i\hat{\epsilon_n} + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}\hat{Q} + i\Gamma_1 Y + \Gamma_2 X} \tag{24.61}$$

$$X(m) = T\sum_n \int \frac{d^2p}{(2\pi)^2} \frac{1}{i\epsilon_n \delta_{n+m,n} + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}\hat{Q} + i\Gamma_1 \hat{Y} + \Gamma_2 \hat{X}} \tag{24.62}$$

$$Y(m) = iT\sum_n \int \frac{d^2p}{(2\pi)^2} \frac{1}{i\epsilon_n \delta_{n+m,n} + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}\hat{Q}_{n+m,m} + i\Gamma_1 \hat{Y} + \Gamma_2 \hat{X}} \tag{24.63}$$

The momentum integrals on the right-hand-sides of (24.62) and (24.63) are simply $i\pi N_F Q_{n+m,n}$, according to (24.61). Hence, they have the solution

$$\begin{aligned} Q^{ab}_{mn,\alpha\beta} &= \mathrm{sgn}\,(\epsilon_n)\ \delta_{mn}\ \delta_{\alpha\beta}\ \delta^{ab} \\ X^a_{\alpha\beta}(m) &= 0 \\ Y^a(m) &= 0 \end{aligned} \tag{24.64}$$

Note that the right-hand sides of the second and third equations of (24.62) and (24.63) vanish for $m \neq 0$ because the saddle-point Q is diagonal in Matusubara frequency while they vanish for $m = 0$ because of cancellation between positive and negative frequencies. As we will see later, for Γ_2 sufficiently large, $X = 0$ becomes unstable and there is an $X \neq 0$ solution of (24.62). This is the Stoner instability.

The low-energy fluctuations about this saddle-point are transverse fluctuations of Q, which can be parametrized by $V^{ab}_{nm,\alpha\beta}$:

$$Q = \begin{pmatrix} \left(1 - VV^\dagger\right)^{1/2} & V \\ V^\dagger & -\left(1 - V^\dagger V\right)^{1/2} \end{pmatrix} \tag{24.65}$$

The diagonal blocks of Q_{nm} correspond to Matsubara indices $n > 0, m > 0$ and $n < 0, m < 0$ while the upper right block corresponds to $n < 0, m > 0$; the lower left block, to $n > 0, m < 0$. These fluctuations correspond to diffusion of charge and spin. The longitudinal fluctuations of Q as well as fluctuations of X, Y are gapped at the classical level (i.e. tree-level).

In order to derive an effective field theory for these diffusion modes, we

shift $Q \to Q - 2\tau\Gamma_1 Y + 2\tau i\Gamma_2 X$ in (24.60)

$$\begin{aligned} S_{\rm eff}[Q,X,Y] = \sum \int [-{\rm tr}\,\ln\left(i\epsilon_n + \frac{1}{2m}\nabla^2 + \mu + \frac{i}{2\tau}Q\right) \\ + \frac{\pi N_F}{4\tau}{\rm tr}\left(Q^2\right) - \pi N_F\Gamma_1 {\rm tr}\left(QY\right) + i\pi N_F\Gamma_2 {\rm tr}\left(QX\right) \\ + \frac{1}{2}\left(1 + 2\pi\left(\tau T\sum_m\right)N_F\Gamma_1\right)\Gamma_1\,{\rm tr}\left(Y^2\right) \\ + \frac{1}{2}\left(1 - 2\pi\left(\tau T\sum_m\right)N_F\Gamma_2\right)\Gamma_2\,{\rm tr}\left(X^2\right)] \end{aligned} \quad (24.66)$$

Formally, the factors of $\tau\left(T\sum_m\right)$ in the final two terms are infinite as a result of the unrestricted Matsubara sum. However, upon integrating out the massive longitudinal modes of Q, we see that they are the first terms in the series $\tau T\sum_m - 2\pi\left(\tau T\sum_m\right)^2 + \ldots = \tau T\sum_m/\left(1 + 2\pi\tau T\sum_m\right)$, which is simply $1/2\pi$. Thus, we replace the factors of $\tau\left(T\sum_m\right)$ by $1/2\pi$ in (24.66). Alternatively, we could expand the tr $\ln(\ldots)$ about the diffusive saddle-point and keep only terms up to second order in X and Y. We would then obtain the same expressions, namely (24.66) with the factors of $\tau\left(T\sum_m\right)$ replaced by $1/2\pi$. The coefficient of X^2 is now $1 - N_F\Gamma_2$, which becomes negative for $\Gamma_2 > 1/N_F$: the Stoner instability. Finkelstein actually expanded the tr $\ln(\ldots)$ only to linear order in X and Y. For small Γ_1, Γ_2, there is no difference, but the Stoner instabilty is missed.

We now expand the tr $\ln(\ldots)$ about the diffusive saddle-point. Thus, we require that Q is a constrained field of the form given in (24.65).

$$\begin{aligned} S_{\rm eff}[Q] = \pi N_F \int d^dx\,\{D\,{\rm tr}(\nabla Q)^2 - 4iZ{\rm tr}\left(\hat{\epsilon}Q\right) \\ - \Gamma_1 {\rm tr}\left(QY\right) + i\Gamma_2 {\rm tr}\left(QX\right) \\ + \frac{1}{2}\left(1 + N_F\Gamma_1\right)\frac{\Gamma_1}{\pi N_F}\,{\rm tr}\left(Y^2\right) \\ + \frac{1}{2}\left(1 - N_F\Gamma_2\right)\frac{\Gamma_2}{\pi N_F}\,{\rm tr}\left(X^2\right)\} \end{aligned} \quad (24.67)$$

The diffusion constant D is given by $D = v_F^2\tau/2$.

Following Finkelstein, we integrate out X, Y and obtain an effective ac-

tion for the diffusion modes.

$$S_{\rm eff}[Q] = \pi N_F \int d^dx \, \{D \, {\rm tr}(\nabla Q)^2 - 4i{\rm tr}\,(\hat{\epsilon}Q) \\ - \pi N_F \tilde{\Gamma}_1 Q^{aa}_{n_1 n_2,\alpha\alpha} Q^{aa}_{n_3 n_4,\beta\beta} \delta_{n_1 - n_2 + n_3 - n_4} \\ + \pi N_F \tilde{\Gamma}_2 Q^{aa}_{n_1 n_2 \alpha\beta} Q^{aa}_{n_3 n_4 \beta\alpha} \delta_{n_1 + n_2 - n_3 - n_4} \} \quad (24.68)$$

Note that ordinary matrix multiplication rules reflect the non-locality in time of the first term (the 'disorder term') in this action. The $\tilde{\Gamma}_{1,2}$ interaction terms do not involve matrix multiplication and are, consequently, local in time. In this expression, $\tilde{\Gamma}_{1,2} = \Gamma_{1,2}/(1 \pm N_F \Gamma_{1,2})$. These corrections to $\Gamma_{1,2}$ follow from our retention of the X^2 and Y^2 terms which Finkelstein drops. For $\Gamma_{1,2}$ small, $\tilde{\Gamma}_{1,2} = \Gamma_{1,2}$, and Finkelstein's effective action is recovered.

Initially, the coefficient of the ${\rm tr}\,(\hat{\epsilon}Q)$ term is 1, as in (24.67), but quantum corrections cause it to flow, so we have followed Finkelstein in introducing the coupling Z.

24.5 The Metal-Insulator Transition

24.6 Mesoscopic fluctuations***