Collective Modes

Feb 20, 2008

Learning Goals

* Idea of probing collective modes
  * phonons, plasmons,...
* Equations of motion (collective coordinate) approach
* Virial Theorem
* Sum Rules
* Linear Response

Very specific example

Setup

N atoms in harmonic trap (anisotropic)

$$H = \sum_i \left[\frac{p_i^2}{2m} + \sum_\alpha \frac{1}{2} m \omega_\alpha^2 r_{i\alpha}^2 \right] + \frac{U_0}{2} \sum_{i<j} \delta(\mathbf{r}_i - \mathbf{r}_j)$$

Change omega's and watch response

Measure in an experiment:

$$Q_\alpha = \sum_j r_{j\alpha}^2$$

(breathing modes, quadrupole modes)

Goal today: calculate frequencies

Equation of motion approach

$$\dot{Q}_\alpha = \frac{1}{i\hbar}[Q_\alpha, H] = \frac{1}{m}\sum_i r_{i\alpha}p_{i\alpha} + p_{i\alpha}r_{i\alpha}$$

$$\ddot{Q}_\alpha = \frac{1}{i\hbar}[\dot{Q}_\alpha, H] = \frac{4}{m}\left(T_\alpha - V_\alpha + \frac{1}{2}U\right)$$

$$T_\alpha = \sum_i \frac{p_{i\alpha}^2}{2m} \qquad V_\alpha = \frac{1}{2}m\omega_\alpha^2 Q_\alpha \qquad U = \frac{1}{2}U_0 \sum_i \delta(r_i - r_j)$$

(AKA collective coordinates)

Equation of motion approach

$$\dot{Q}_\alpha = \frac{1}{i\hbar}[Q_\alpha, H] = \frac{1}{m}\sum_i r_{i\alpha}p_{i\alpha} + p_{i\alpha}r_{i\alpha}$$ = "virial"

$$\ddot{Q}_\alpha = \frac{1}{i\hbar}[\dot{Q}_\alpha, H] = \frac{4}{m}\left(T_\alpha - V_\alpha + \frac{1}{2}U\right)$$

$$T_\alpha = \sum_i \frac{p_{i\alpha}^2}{2m} \qquad V_\alpha = \frac{1}{2}m\omega_\alpha^2 Q_\alpha \qquad U = \frac{1}{2}U_0\sum_i \delta(r_i - r_j)$$

(AKA collective coordinates)

Virial Theorem

$$\ddot{Q}_\alpha = \frac{1}{i\hbar}[\dot{Q}_\alpha, H] = \frac{4}{m}\left(T_\alpha - V_\alpha + \frac{1}{2}U\right)$$

In equilibrium Q is time independant

$$\langle T_\alpha \rangle - \langle V_\alpha \rangle + \frac{1}{2}\langle U \rangle = 0$$

Also useful to note

$$H = \sum_\alpha T_\alpha + \sum_\alpha V_\alpha + U$$

Closing the equations

$$\ddot{Q}_\alpha = \frac{1}{i\hbar}[\dot{Q}_\alpha, H] = \frac{4}{m}\left(T_\alpha - V_\alpha + \frac{1}{2}U\right)$$

Case 1: U=0

$$H_\alpha = T_\alpha + V_\alpha$$ is constant of motion

$$\ddot{Q}_\alpha = -4\omega_\alpha^2\left[Q_\alpha - \frac{H_\alpha}{m\omega_\alpha^2}\right]$$

Shift Q: $$\tilde{Q}_\alpha = Q_\alpha - \frac{H_\alpha}{m\omega_\alpha^2}$$

$$\ddot{\tilde{Q}}_\alpha = -4\omega_\alpha^2\tilde{Q}_\alpha$$ Oscillates at twice trap frequency

Closing the equations

$$\ddot{Q}_\alpha = \frac{1}{i\hbar}[\dot{Q}_\alpha, H] = \frac{4}{m}\left(T_\alpha - V_\alpha + \frac{1}{2}U\right)$$

Case 2: T=0 (good approximation for a BEC)

$$U = H - \sum_\beta V_\beta$$

$$\frac{d^2}{dt^2}\left(\begin{array}{c} Q_x \\ Q_y \\ Q_z \end{array}\right) = \frac{2}{m}H - \left(\begin{array}{ccc} 3\omega_x^2 & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & 3\omega_y^2 & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & 3\omega_z^2 \end{array}\right)\left(\begin{array}{c} Q_x \\ Q_y \\ Q_z \end{array}\right)$$

constant (shift away)

eigenvalues give oscillation frequencies

Q: if all omegas are equal,
what are oscillation frequencies?

$$\frac{d^2}{dt^2}\begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix} = \frac{2}{m}H - \begin{pmatrix} 3\omega_x^2 & \omega_x\omega_y & \omega_x\omega_z \\ \omega_x\omega_y & 3\omega_y^2 & \omega_y\omega_z \\ \omega_x\omega_z & \omega_y\omega_z & 3\omega_z^2 \end{pmatrix}\begin{pmatrix} Q_x \\ Q_y \\ Q_z \end{pmatrix}$$

constant
(shift away)

eigenvalues give oscillation frequencies

Closing the equations

Generically the equations do not close so easily

-- introduce more formal tools for dealing with it

Changing trap constants for short time:

$$H_{\mathrm{pert}} = \lambda_\alpha(t) Q_\alpha$$

Response

$$\begin{aligned}\langle Q_\beta(t)\rangle &= \langle T e^{i\int^t d\tau (H_0 + H_{\mathrm{pert}})} Q_\beta e^{-i\int^t d\tau (H_0 + H_{\mathrm{pert}})}\rangle \\ &= \langle Q_\beta\rangle_0 + \int dt' \chi^R_{\alpha\beta}(t-t')\lambda_\alpha(t') + \cdots\end{aligned}$$

$$\chi^R_{\alpha\beta} = \frac{\theta(t)}{i}\langle [Q_\beta(t), Q_\alpha(0)]\rangle_0$$

Interaction picture

(only if requested)

$$U = Te^{-i\int^t d\tau (H_0 + H')}$$

$$\bar{U} = e^{iH_0 t} U$$

$$\begin{aligned} i\partial_t \bar{U} &= e^{iH_0 t} H' U \\ &= H'(t)\bar{U} \end{aligned}$$

so

$$\bar{U}(t) \approx 1 - i\int^t d\tau H'(\tau)$$

Equations of motion

$$\chi^R_{\alpha\beta} = \frac{\theta(t)}{i}\langle[Q_\beta(t), Q_\alpha(0)]\rangle_0$$

$$\partial_t^2\chi_{\alpha\beta} = \frac{\delta'(t)}{i}\cancelto{0}{\langle[Q_\beta(0), Q_\alpha(0)]\rangle_0} + \frac{\delta(t)}{i}\langle[\dot{Q}_\beta(0), Q_\alpha(0)]\rangle + \frac{\theta(t)}{i}\langle[\ddot{Q}_\beta(t), Q_\alpha(0)]\rangle$$

Substitute in EOM for Q

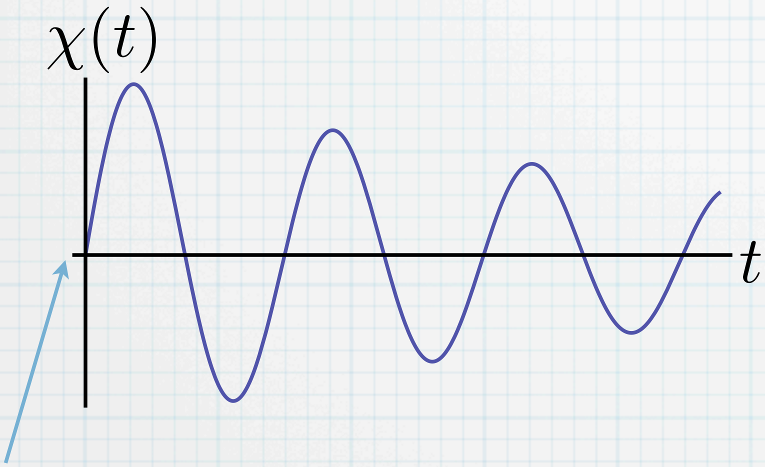
When EOM for Q close -- so do EOM for chi

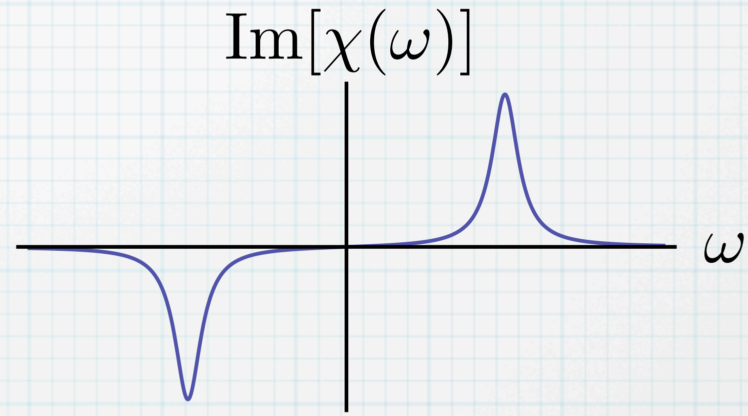
$$\langle[\dot{Q}_\beta(0), Q_\alpha(0)]\rangle = \delta_{\alpha\beta}\frac{4\langle Q_\alpha\rangle}{m}$$

$$\dot{Q}_\alpha = \frac{1}{i\hbar}[Q_\alpha, H] = \frac{1}{m}\sum_i r_{i\alpha}p_{i\alpha} + p_{i\alpha}r_{i\alpha}$$

Expected Structure:

Damped Harmonic oscillator





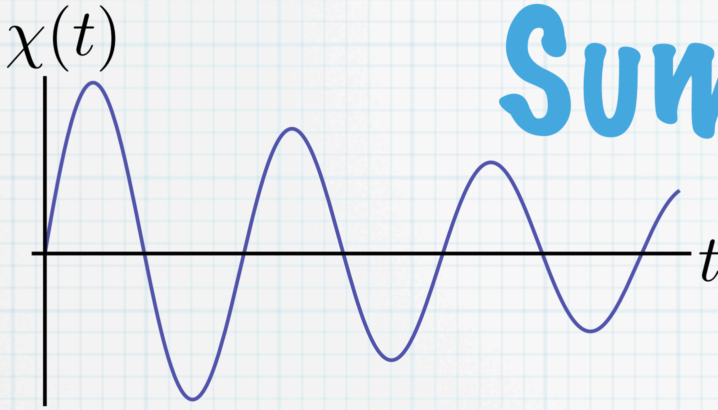
Probe and response commute
at t=0

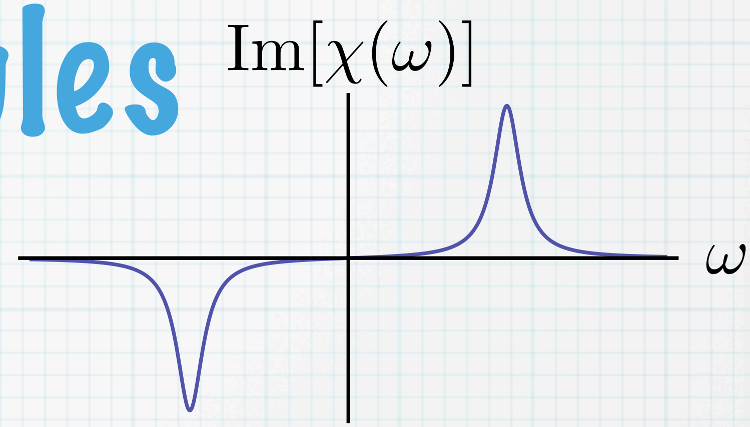
Sum rules:

Make Ansatz for chi(t)

Fit parameters from t=0
chi and derivatives

Sum Rules





f-sum rule: (always set by kinematics)

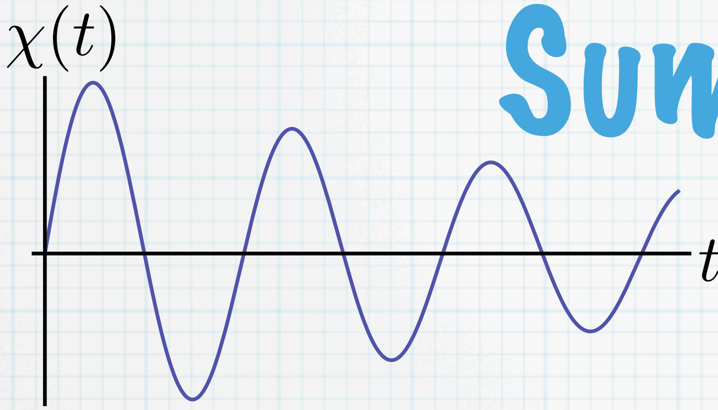
$$M_1 = \int \frac{d\omega}{2\pi}(i\omega)\chi(\omega) = -i\langle[Q'(0), Q(0)]\rangle$$

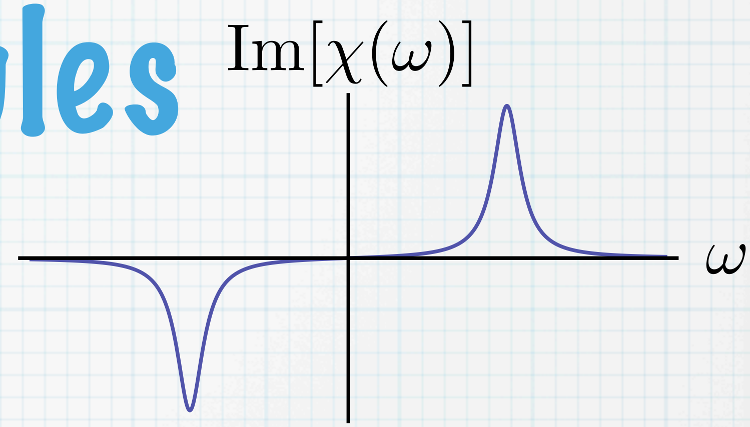
$$= \frac{4}{m}\langle Q_\alpha\rangle\delta_{\alpha\beta}$$

$$M_3 = \int \frac{d\omega}{2\pi}(i\omega)^3\chi(\omega) = -i\langle[Q''(0), Q'(0)]\rangle$$

$$= \frac{8}{m^2}\left[\delta_{\alpha\beta}(2\langle T_\alpha\rangle + 2\langle V_\alpha\rangle) + \frac{1}{2}\langle U\rangle\right]$$

Sum Rules





Estimate frequency

$$-\omega^{-2} \approx \text{eigenvalues}(M_1^{-1} M_3)$$

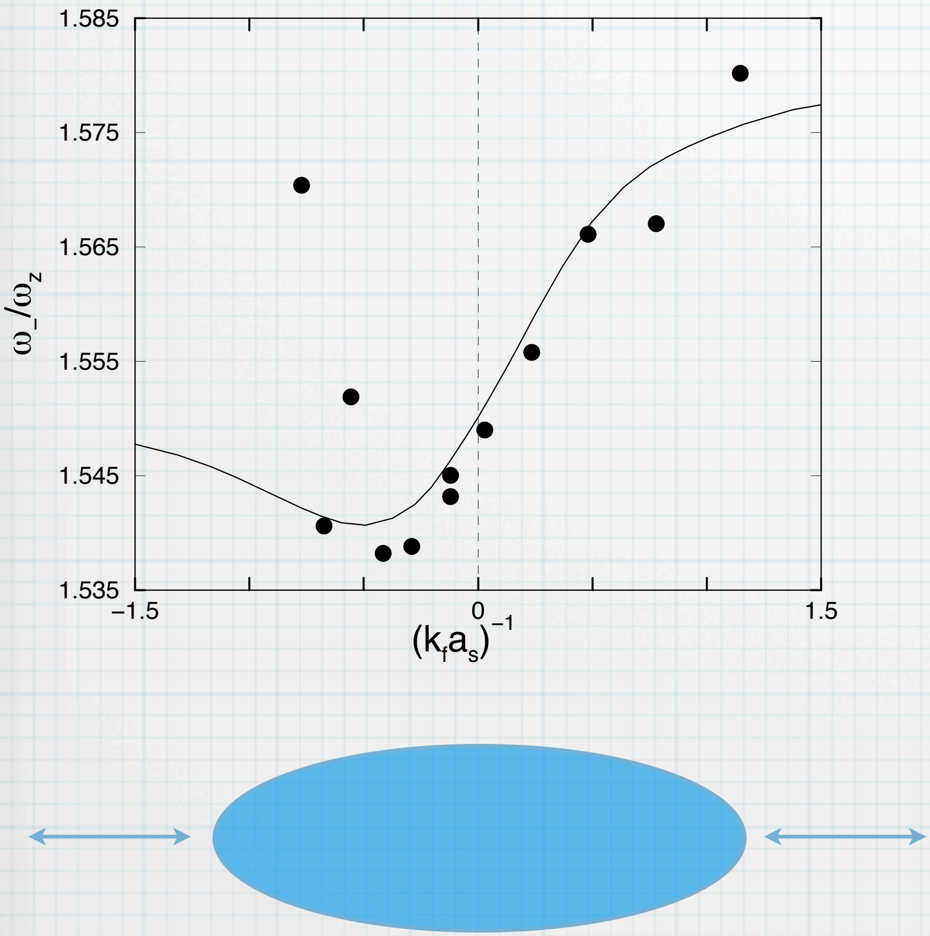
Other tools: **Compressibility sum rule**

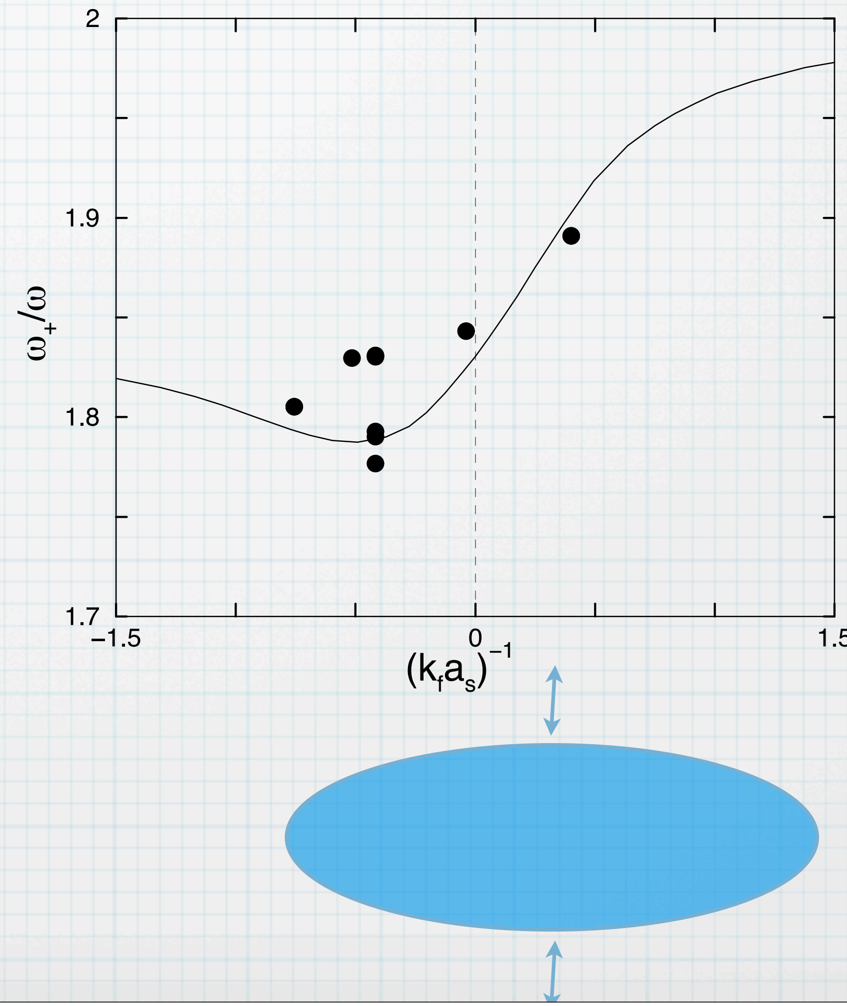
$$M_{-1} = \lim_{t\to 0} \int \frac{d\omega}{2\pi} e^{i\omega t} \frac{\chi(\omega)}{i\omega} = \lim_{t\to 0} \int_{\infty}^{t} \chi(t')dt'$$

$$= \chi(\omega = 0)$$ **Static Response**

(or use Kramers-Kronig)

2-component Fermi Gas





So What

Measurement ⟷ Calculation

Agreement

Aside from learning that our sum rules work -- what do we learn?

A: Modes depend on equation of state -- learn about it

Simpler Case

Dipole Mode (Kohn Mode)

$$X = \sum_i x_i \qquad\qquad P = \sum_i p_{xi}$$

Shift Trap center: $H_{\text{pert}} = \lambda(t) X$

$$\chi = \frac{\theta(t)}{i}\langle X(t)X(0) - X(0)X(t)\rangle$$

Equations of motion:

$$\partial_t X = P/m \qquad\qquad \partial_t P = -m\omega_x^2 X$$

Solution

$$X(t) = \cos(\omega t)X(0) + (m/\omega)\sin(\omega t)P(0)$$

Response function

$$\chi = \frac{\theta(t)}{i}\langle X(t)X(0) - X(0)X(t)\rangle$$

$$\chi = \frac{m}{\omega}\theta(t)\sin(\omega t)$$

Useful to introduce "structure factors"

(we will see these next when we discuss Scattering)

$$\chi^{>} = \langle X(t)X(0)\rangle$$

$$= \langle X^2\rangle\cos(\omega t) + (m/\omega)\langle XP\rangle\sin(\omega t)$$

$$= \frac{m}{2\omega}\left(\coth(\beta\omega/2)\cos(\omega t) + i\sin(\omega t)\right)$$

$$\chi^{<} = \langle X(0)X(t)\rangle$$

know about T

$$= \frac{m}{2\omega}\left(\coth(\beta\omega/2)\cos(\omega t) - i\sin(\omega t)\right)$$

Detailed Balance

$$\chi^{>}(t) = \mathrm{Tr} e^{-\beta H} e^{iHt} X e^{-iHt} X$$

$$= \mathrm{Tr} e^{-\beta H} X e^{-\beta H} e^{iHt} X e^{-iHt} e^{\beta H}$$

$$= \chi^{<}(t + i\beta)$$

Fourier Transform:

$$\chi^{>}(\nu) = \chi^{<}(-\nu) \quad = e^{\beta\nu} \chi^{>}(-\nu)$$