Response Function Exercise

This is not a traditional teaser, but it is a good exercise.

Consider a harmonic oscillator $H = \frac{p^2}{2} + \frac{x^2}{2}$ with $[x, p] = i$. Assume at $t = 0$ the system is in thermal equilibrium with temperature $T$.

(a) What are $\langle x(0)^2 \rangle$, $\langle p(0)^2 \rangle$, $\langle x(0)p(0) \rangle$ and $\langle p(0)x(0) \rangle$?

(b) Write the operator $x(t)$ as a function of $x(0)$ and $p(0)$.

(c) Calculate $\chi^R(t) = \theta(t) \langle x(t)x(0) - x(0)x(t) \rangle$. Does this depend on temperature?

(d) Calculate $\chi^>(t) = \langle x(t)x(0) \rangle$. Does this depend on temperature?

(e) Calculate $\chi(t) = \frac{\theta(t)}{i} \langle x(t)x(0) \rangle + \frac{\theta(-t)}{i} \langle x(0)x(t) \rangle$.

(f) Sketch $\chi^>(\omega)$ and $\text{Im} \chi^R(\omega)$.