Problem 1. Susceptibility of O(n) model

Consider a system of spins which sit in d-dimensional space and which can point in n dimensions, with Landau Free energy

\[ F = \int d^d r (c/2)(\nabla_\mu S_i(r))(\nabla_\mu S_i(r)) + (a/2)S_i(r)S_i(r) + (b/4)(S_i(r)S_i(r))(S_j(r)S_j(r)) - h_i(r)S_i(r), \]

where Einstein summation is assumed with \( \mu \) running from 1 to \( d \) and \( i, j \) running from 1 to \( n \). We will flip between using integers and letters to denote the directions. For example if I say \( h = h^\hat{x} \), that is equivalent to saying \( h_j = \delta_{j1} h \).

In class we considered the case where \( h = 0 \), here we will consider \( h \neq 0 \).

1.1. First consider the case where \( h(r) = h^\hat{x} \) is uniform and points in the \( \hat{x} \) direction [ie \( h_i = h \delta_{j1} \)]. Minimize \( F \), and show that \( S = S^\hat{x} \), and \( S \) satisfied the cubic equation

\[ aS + bS^3 - h = 0. \]

1.2. Fix \( b > 0 \), and find the boundary in the \( h - a \) plane between where this equation has one and three solutions. For \( h \neq 0 \) this defines the spinodal.

1.3. Let \( h = h^\hat{x} + \delta h^\parallel e^{ik\cdot r}\hat{x} \), and let \( S = S^\hat{x} + \delta S e^{i\cdot r}\hat{x} \) minimize the free energy. Calculate \( \delta S \) to linear order in \( \delta h^\parallel \). Your expression may contain \( S \).

The longitudinal susceptbility is defined as

\[ \chi^\parallel = \frac{\delta S}{\delta h^\parallel} \]

Verify that when \( h \rightarrow 0 \) you recover the expression from class

\[ \chi^\parallel \bigg|_{h=0} = \frac{1}{ck^2 + 2|a|}. \]

1.4. What happens to the longitudinal susceptbility in the metastable state at the spinodal? [ie. at the spinodal, one of the minima disapears. Evaluate the susceptability of that metastable state.]

1.5. Show that even in the presence of nonzero \( \delta h^\parallel \) that \( S_j = 0 \) for all \( j \neq 1 \), and hence that

\[ \chi^{yx} = \frac{\delta S_y}{\delta h_x} = 0, \]

where \( h_x = h^\parallel \).
1.6. Now let’s consider a transverse perturbation. Let \( h = h x + \delta h_x e^{ikr \hat{y}} \), and let \( S = S_x + \delta S_y e^{ir \hat{y}} \).

minimize the free energy. To linear order in \( \delta h_x \), calculate \( \delta S_y \). Show that as \( h \to 0 \) one recovers the result from class that

\[
\chi_{\perp}|_{h=0} = \frac{1}{ck^2},
\]

Problem 2. Continuum limit of x-y model Consider a microscopic x-y model on a square lattice in two dimensions,

\[
H = -J \sum_{\langle ij \rangle} S_i \cdot S_j = -JS^2 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j),
\]

where \( S \) is the length of the spins, and \( \theta \) defines their directions. We will derive a continuum version of this model, and evaluate the energy of some important quantities.

2.1. Suppose that \( \theta_i \) varies slowly from one site to the next. Let \( \theta(r) \) be a smooth function for which \( \theta(r_i) = \theta_i \). Show that

\[
H \approx \int d^2 r - \frac{JS^2}{2} |\nabla \theta(r)|^2,
\]

independent of the lattice spacing.

2.2. There can be spin configurations which are not smooth. An example is a vortex: \( \theta(r) = \arctan(y/x) = \text{Im} \log(x + Iy) \).

This configuration is smooth except for a region near \( r = 0 \). Let \( \xi \sim a \) be a length for which \( \theta(r) \) is smooth when \( r > \xi \). If the size of the system is \( L \), estimate the contribution to the energy of a vortex configuration from all spins at \( r > \xi \). This is described as the region "outside the vortex core".

You should find that this energy diverges as \( L \to \infty \).

Hint 1: The continuum approximation works in this region.

Hint 2: Take the sample to be circular in shape.

2.3. Estimate the energy contributions from outside the vortex cores of a vortex-antivortex pair separated by a distance \( d(\gg \xi) \): \( \theta(r) = \text{Im} [\log(x - d/2 + Iy) - \log(x + d/2 + Iy)] \).

Hint 1: Take the limit of an infinite system, this energy is finite in that limit.

Hint 2: Use Stoke’s Theorem (ie integrate by parts):

\[
\int_{\Omega} d^2 r |\nabla \theta|^2 = \int_{\partial \Omega} d\ell \cdot \theta \nabla \theta - \int_{\Omega} d^2 r \theta \nabla^2 \theta.
\]

Note that \( \nabla^2 \theta = 0 \). Look out for branch cuts.
Problem 3. Correlation functions in harmonic crystal

As a simple model of a crystal, consider a system of particles that want to form a square lattice in $d$ dimensions, with lattice constant $a$. If we only consider the interaction between neighboring atoms one can approximate the Hamiltonian as

$$H = \sum_{\langle ij \rangle} V(r_i - r_j) + \sum_i \frac{p_i^2}{2m},$$

where $r_i$ is the position of the $i$'th particle and $p_i$ is the momentum of that particle. We now assume that each particle stays near its equilibrium position, $r_i^{(0)}$, in which case $r_i = r_i^{(0)} + \delta_i$. Presumably $V$ has a minimum at this point, so we can expand and get to an Einstein model,

$$H = \sum_{\langle ij \rangle} \frac{m\omega_0^2}{2} (\delta_i - \delta_j)^2 + \sum_i \frac{p_i^2}{2m}. $$

3.1. Find the normal modes and their frequencies. What is the energy cost of exciting each of these modes with some given amplitude.

This is a system with a spontaneously broken continuous symmetry. How is Goldstone’s theorem manifested in these modes?

3.2. The equipartition theorem says that at finite temperature each degree of freedom should have an energy $kT/2$. Use the equipartition theorem and the normal modes to estimate $\langle |\delta_i|^2 \rangle$ as the system size becomes large.

What happens for $d = 1, 2$?

Hint 1: This result is independant of $i$, so you might as well take $r_i^0 = 0$.

Hint 2: Turn the sum into an integral. The integral is dominated by the modes of lowest energy. Approximate $\cos(x) \approx 1 - x^2/2$.

3.3. Use the same method to write down an integral for $g_{ij} = \langle \delta_i \delta_j \rangle$ as a function of the distance $r_i^{(0)} - r_j^{(0)}$. How does this integral behave in the infinite system as $r_i^{(0)} - r_j^{(0)} \to \infty$.

How is this related to the Mermin-Wagner theorem?