Problem 1. A gas of zero temperature bosonic $^{23}$Na atoms are trapped in a “box”. The box consists of a hard horizontal “wall” at which the wavefunction $\phi$ must vanish, and gravity which acts perpendicular to the wall. The areal density of the gas is $n_{2D} = 10^{10} \text{cm}^{-2}$ and the scattering length is $a_s = 5 \cdot 10^{-9} \text{m}$.

1.1. Sketch the vertical density profile of the gas as a function of the distance from the wall.

Solution. A rough picture should have the density rising from zero to a maximum in a distance given by the coherence length, and then falling nearly linearly.

1.2. Approximately how high above the wall does the density of particles become negligible?

Solution. We use the Thomas-Fermi approximation

$$n = \frac{\mu_0 - V}{\lambda},$$

where $\lambda = 4\pi \hbar^2 a_s / m$, and $V(z) = mgz$. Taking $z_0$ to be the place where this vanishes, we can write

$$n(z) = \frac{mg}{\lambda}(z_0 - z).$$

We use this expression to relate $n_{2D}$ to $z_0$:

$$n_{2D} = \int_0^{z_0} dz \, n(z) = \frac{mg z_0^2}{2\lambda}.$$ 

We therefore find

$$z_0 = \sqrt{\frac{2\lambda n_{2D}}{mg}} = 2.9 \mu\text{m}.$$
**Problem 2.** In class we discussed the Thomas-Fermi approximation to the density profile of a cloud of $N$ noninteracting Fermions in a 3D harmonic oscillator potential. Here we consider fermions in 1D.

2.1. What is the density of a zero temperature 1D fermi gas in free space with chemical potential $\mu$?

**Solution.**

$$n = \int_{-k_f}^{k_f} \frac{dk}{2\pi} = \frac{k_f}{\pi} = \frac{\sqrt{2m\mu}}{\pi}$$

2.2. Using the result from (??) and the Thomas-Fermi approximation, $\mu(x) = \mu_0 - V(x)$, calculate the density profile of a trapped 1D gas with chemical potential $\mu_0$.

**Solution.**

$$n(x) = \frac{\sqrt{2m}}{\pi} \sqrt{\mu_0 - \frac{1}{2}m\omega^2 x^2}$$

2.3. Given that $N = \int dx n(x)$, relate $\mu_0$ to $N$.

**Solution.**

$$N = \int dx n(x) = \mu_0/\omega$$

2.4. The exact density profile for $N$ fermions in a 1-D harmonic trap is

$$n(x) = \sum_{j=0}^{N} |\phi_j(x)|^2$$

$$\phi_j(x) = \frac{1}{(\pi d)^{1/4}} \frac{1}{\sqrt{2^n n!}} H_n(x/d) e^{-x^2/(2d^2)},$$

where $d = \sqrt{\hbar/m\omega}$ and $H_n(y)$ is the $n$’th Hermite polynomial.

Numerically graph the exact 5-particle density and compare it with the Thomas-Fermi prediction for $N = 5$.

**Solution.** The Thomas-Fermi solution (dashed line) is remarkably close to the exact result (solid line).
**Problem 3.** A soliton is a wave packet which propagates without changing shape. This requires that the dispersion is exactly balanced by interactions. The time dependant Gross-Pittaevskii equation supports solitons.

3.1. Show that for any \( v \), the following function is a solution of the one dimensional Gross-Pittaevskii equation,

\[
\psi(x,t) = \sqrt{n} e^{-i\Phi t/\hbar} \left[ i\frac{v}{c} + \sqrt{1 - \frac{v^2}{c^2}} \tanh \left( \frac{x - vt}{\sqrt{2}\xi} \sqrt{1 - \frac{v^2}{c^2}} \right) \right],
\]

where \( c = \sqrt{gn/m} \) is the speed of sound, and \( \xi = \sqrt{\hbar^2 / 2mg} \) is the healing length.

**Solution.** This is a simple algebra exercise using the Gross-Pittaevskii equation:

\[
i\partial_t \psi = -\frac{\partial^2 \psi}{2m} + g|\psi|^2 \psi.
\]

3.2. The phase difference \( \Phi \) across the soliton is defined by

\[
e^{i\Phi} = \frac{\psi(x - vt \to \infty)}{\psi(x - vt \to -\infty)}.
\]

What is \( \Phi \)?

**Solution.**

\( \Phi = 2 \arccos(v/c) \)

3.3. The depth of the soliton is defined by

\[
D = 1 - \frac{n_{\text{min}}}{n},
\]

where \( n_{\text{min}} \) is the minimal value of the density \( n(x) = |\psi(x)|^2 \) and \( n \) is the density at \( \infty \). What is \( D \)?

**Solution.**

\( D = 1 - \frac{v^2}{c^2} \)

3.4. How many particles are “removed” to form the soliton:

\[
N = \int dx \ (n - |\psi|^2).
\]

**Solution.** This is straightforward to calculate using

\[
N = n \int dx \ \left( 1 - \frac{v^2}{c^2} \right) \left( 1 - \tanh^2(\theta) \right),
\]

where

\[
\theta = \left( \frac{x - vt}{\sqrt{2}\xi} \sqrt{1 - \frac{v^2}{c^2}} \right).
\]

One notes that \( 1 - \tanh^2 \theta = (d/d\theta) \tanh \theta \). Integrating by parts gives

\[
N = n \left( 1 - \frac{v^2}{c^2} \right) \frac{dx}{d\theta} \tanh(\theta) \bigg|_{x=\infty}^{x=-\infty}
\]

\[
= 2\sqrt{2}n\xi \sqrt{1 - \frac{v^2}{c^2}}.
\]
3.5. This “hole” moves with velocity $v$. It should therefore carry momentum $p = Nmv$, where $m$ is the mass of the particles. Compare this value with a direct calculation of the momentum:

$$p = \frac{m\hbar}{2i} \int dx \left( \psi^* \frac{d\psi}{dx} - \frac{d\psi^*}{dx} \psi \right).$$

**Solution.** The direct calculation gives

$$p = \text{Im} \int dx \psi^* \frac{d\psi}{dx}$$

$$= -n \frac{v}{c} \sqrt{1 - \frac{v^2}{c^2}} \int dx \frac{\partial_x \tanh \theta}{}$$

$$= -2n \frac{v}{c} \sqrt{1 - \frac{v^2}{c^2}}.$$

Aside from a ‘$-$’ sign, this agrees with the intuitive approach. The ‘$-$’ sign is correct, as this is a ‘hole’, which carries momentum in the direction opposite to its velocity.
Problem 4. The speed of sound in a fluid is completely determined by thermodynamic variables.

4.1. Suppose a fluid satisfies Euler’s equations:

\[
\frac{dn}{dt} + \nabla \cdot j = 0
\]

\[
mn \frac{dv}{dt} + \nabla P = 0,
\]

which are a consequence of conservation of mass and Newton’s second law. The density is \( n \), pressure \( P \), velocity \( v \), particle mass \( m \) and particle current \( j = nv \). By linearizing about the solution \( n = n_0, v = 0 \), calculate the speed of sound in terms of \( \partial P/\partial n \).

Solution. The linearized equations read

\[
\partial_t \delta n + n_0 \nabla \cdot v = 0 \quad (1)
\]

\[
m n_0 \partial_t v + \frac{\partial P}{\partial n} \nabla \delta n = 0. \quad (2)
\]

Taking the derivative with respect to time of (1) and substituting in (2) gives

\[
\partial_t^2 \delta n + \frac{1}{m} \frac{\partial P}{\partial n} \nabla^2 \delta n = 0.
\]

This is the wave equation where the speed of sound is

\[
c^2 = \frac{1}{m} \frac{\partial P}{\partial n}
\]

4.2. Relate \( \partial \mu/\partial n \) to \( \partial P/\partial n \), and therefore derive a formula for the speed of sound in terms of \( \partial \mu/\partial n \).

Solution. The free energy is related to the number of particles and the volume by

\[
dE = \mu dN - pdV.
\]

By the equality of mixed partial derivatives, we have

\[
\left( \frac{\partial \mu}{\partial V} \right)_N = - \left( \frac{\partial P}{\partial N} \right)_V.
\]

This is readily converted into the form we want by applying the chain rule

\[
\left( \frac{\partial P}{\partial N} \right)_V = \frac{1}{V} \left( \frac{\partial P}{\partial V} \right)_V
\]

\[
\left( \frac{\partial \mu}{\partial V} \right)_N = \left( \frac{\partial \mu}{\partial n} \right)_N \left( \frac{\partial n}{\partial V} \right)_N = - \frac{n}{V} \left( \frac{\partial \mu}{\partial n} \right)_N.
\]

Thus we find that

\[
\frac{\partial P}{\partial n} = n \frac{\partial \mu}{\partial n},
\]

and that the speed of sound is

\[
c = \sqrt{\frac{n \frac{\partial \mu}{\partial n}}{m \frac{\partial \mu}{\partial n}}}.
\]
4.3. Use the Hartree-Fock expression for \( \mu(N) \) for three dimensional zero-T bosons with scattering length \( a \) to calculate the speed of sound.

**Solution.** For bosons, \( \mu = gn \), so \( \partial \mu / \partial n = g \) and the speed of sound is

\[
c = \sqrt{gn/m},
\]

in agreement with our previous Bogoliubov calculation.

4.4. Use the Hartree-Fock expression for \( \mu(N) \) for three dimensional zero-T noninteracting fermions to calculate the speed of sound.

**Solution.** For three dimensional fermions

\[
\mu = \frac{\hbar^2}{2m} (3\pi^2 n)^{2/3}
\]

Substituting this into our expression for the speed of sound gives

\[
c = \frac{\hbar}{m} (3\pi^2 n)^{1/3},
\]

which is equal to the Fermi velocity.