Problem 1.

1.1. An experiment measures the susceptibility $\chi(T)$ in a magnet for temperatures $T$ slightly above the ferromagnetic transition temperature $T_c$. They find their data is fit well by the form

$$\chi(T) = A(T - T_c)^{-1.25} + B + C(T - T_c) + D(T - T_c)^{1.77}.$$ 

What is the critical exponent $\gamma$?

**Solution 1.1.** As $T \to T_c$ only the most singular part contributes, so $\gamma = 1.25$

1.2. A different experiment, on a different (three dimensional) material, finds that the spin-spin correlation function is

$$C(r, T) = \langle S(x)S(x + r) \rangle = r^{-1.026} g(r(T - T_c)^{0.65}).$$

What is the critical exponent $\nu$? The exponent $\eta$?

**Solution 1.2.** One expects $C(r, t) \sim r^{-(2-d+\eta)} g(r/\xi)$, where $\xi \sim (T - T_c)^{-\nu}$, so $\nu = 0.65$ and $\eta = 0.026$.

Problem 2. A numerical experiment calculates the ratios of the fourth and second moment of the order parameter

$$X = \frac{\langle m^4 \rangle}{\langle m^2 \rangle^2}$$

as a function of temperature for different system sizes. Their data is shown below. Is this a first or second order phase transition? Give your reasoning.
Solution 2.1. This is a first order phase transition. For a second order phase transition all of these curves should cross at one point.

Problem 3. In the following RG flow equations, $K_1 = \beta J_1$ and $K_2 = \beta J_2$ are both externally controlled coupling constants.

$$\ell \frac{\partial K_1}{\partial \ell} = K_1^2 - \frac{K_1}{1 + K_2^2}$$

$$\ell \frac{\partial K_2}{\partial \ell} = K_2(K_1 - 2)$$

3.1. Find the fixed points of this flow for $K_j \geq 0$. [Remember to include the possibility of a fixed-point at infinity.] How many unstable directions do each of them have.

Solution 3.1. There are five fixed points: $(K_1, K_2) = (0, 0), (1, 0), (\infty, 0), (0, \infty)$, and $(\infty, \infty)$. These respectively have 0, 1, 1, 2, 0 unstable directions.

3.2. Sketch the flow diagram in the $K_1 - K_2$ plane for $K_1 > 0$ and $K_2 > 0$. 

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3.3. Which fixed point corresponds to the critical point? Which to the high temperature phase? Which to the low temperature phase?

**Solution 3.3.** The critical point is at \((K_1, K_2) = (1, 0)\), the low temperature phase flows to \((K_1, K_2) = (\infty, \infty)\), and the high temperature phase flows to \((K_1, K_2) = (0, 0)\).

3.4. What is the critical exponent \(\nu\)?

**Solution 3.4.** Linearizing about the critical point one writes \(K_1 = 1 + k_1\), \(K_2 = k_2\), and finds [to linear order]

\[
\ell \frac{dk_1}{d\ell} = k_1 \\
\ell \frac{dk_2}{d\ell} = -k_2,
\]

and \(k_1\) is relevant, while \(k_2\) is irrelevant, with \(k_1(\ell) = \ell k_1(1)\), and \(k_2(\ell) = \ell^{-1}k_2(1)\). Under scaling the coherence length evolves as \(\xi(\ell) = \ell^{-1}\xi(1) \propto k_1(\ell)^{-1}\). Since \(K_1\) is inversely proportional to \(T\), \(k_1\) is proportional to the deviation of \(T\) from \(T_c\), implying that \(\xi \propto (T - T_c)^{-1}\), and that \(\nu = 1\).

3.5. In an experiment where \(J_1/k_B = 3\) Kelvin and \(J_2/k_B = 1\) Kelvin, what is the transition temperature? [Try for at least 2 significant figures]
Solution 3.5. Given these values of $J$, the physical system must lie on the line $K_2 = K_1/3$. One needs to find the intersection of this line with the critical manifold. There are really only two reasonable ways that I know how to do this: numerically or with a series expansion around the critical point.

For the series expansion, one begins by noting that any flow line obeys the differential equation

$$\frac{\partial K_1}{\partial K_2} = \frac{\left( \frac{dK_1}{dt} \right)}{\left( \frac{dK_2}{dt} \right)} = \frac{K_1^2 - \frac{K_1}{1+K_2}}{K_2(K_1 - 2)}$$

As we already saw from linearizing about the critical point, for small $K_2$, the critical manifold is the line $K_1 = 1$, which gives a first estimate: $T_c = J_1 = 3$ Kelvin. To improve this result we substitute our zeroth order form into the differential equation to get

$$\left. \frac{\partial K_1}{\partial K_2} \right|_{K_1=1} \approx -K_2,$$

which is integrated to find the approximation $K_1 \approx 1 - \frac{K_2^2}{2}$. This intersects the line $K_2 = K_1/2$ when $K_1 = 1 - \frac{K_2^2}{18} \approx 17/18$. Which gives $T_c = (18/17)J_1 = 3.2$ Kelvin. If you continue in this direction you can get more accuracy.

The exact answer (from integrating the full differential equation and numerically computing the intersection) is that the intersection occurs at $K_1 = 0.968128$, which gives $T_c = 3.09877$ Kelvin.