P683 HW3

Due Friday Feb 5, 2010

Problem 1. harmonic oscillator

The canonical example to learn about path integrals is the harmonic oscillator, $H = \hat{a}^\dagger \hat{a}$. [We set the oscillator frequency to one, and neglect the $+1/2$ that one usually includes.] We will introduce "coherent states" defined by $\hat{a}|a\rangle = a|a\rangle$ which obey $\langle a|b\rangle = \exp(a^* b - (|a|^2 + |b|)/2)$ and for which the resolution of the identity is $\mathbf{1} = \int \frac{da^*\, da}{2\pi i} |a\rangle < a|$. [Here $da^*\, da/(2\pi i) = dx\, dy/\pi$, where $a = x + iy$.]

We are going to make an approximation to the partition function Z=Tr$e^{-\beta H} = \int \frac{da^*\, da}{2\pi i} e^{-|a|^2} \langle a|e^{-\beta \hat{H}}|a\rangle$.

1.1. Primitive Approximation As the simplest approximation (good at high temperature) we write

$$\langle a|e^{-\beta\hat{H}}|a\rangle \quad \approx \quad \langle a|(1-\beta\hat{H})|a\rangle \approx \exp(-\beta\langle a|H|a\rangle).$$

Calculate Z and $\langle \hat{a}^\dagger \hat{a}\rangle$ in this classical "primitive approximation".

1.2. Two time-slice Approximation The next level of sophistication involves breaking up the exponential into two pieces:

$$Z = \int \frac{da^*\, da}{2\pi i} \frac{db^*\, db}{2\pi i} \langle a|e^{-\beta\hat{H}/2}|b\rangle\langle b|e^{-\beta\hat{H}/2}|a\rangle.$$

Here the primitive approximation becomes

$$\langle a|e^{-\beta\hat{H}/2}|b\rangle \quad \approx \quad \langle a|(1-\beta\hat{H}/2)|b\rangle \approx \exp[-\beta\langle a|H|b\rangle/2 - (\beta/2)(a^*(b-a) + (a^*-b^*)b)].$$

Calculate Z, $\langle \hat{a}^\dagger(0)\hat{a}(0)\rangle$, $\langle \hat{a}^\dagger(-i\beta/2)\hat{a}(0)\rangle$, and $\langle \hat{a}^\dagger(0)\hat{a}(-i\beta/2)\rangle$ within the two time-slice approximation.

1.3. The exact partition function is $Z = 1/(1-e^{-\beta})$, and the exact mean occupation number is $< \hat{a}^\dagger \hat{a} >= 1/(e^\beta - 1)$. Expand these to second order in β. How do they compare with the results of the primitive and two time-slice approximations?

1.4. *(bonus)* Calculate $\langle a^\dagger a\rangle$ in the n-time-slice approximation. In the limit $n \to \infty$ this is the path integral we discussed in class, and is exact.