P683 HW5

Due Wednesday Feb 17, 2010

Problem 1 (Viscocity)**.** Viscocity is most simply defined in terms of an experiment where two plates are separated by a distance L. The top plate is moved in the $\hat{x}$ direction with speed $v/2$ and the bottom plate is moved in the opposite direction with the same speed. The force F exerted by the fluid on each plate is related to the shear viscosity by $F = \eta A v/L$, where A is the area of each plates.

A completely equivalent definition is to consider a fluid where the x-component of velocity is a function of the z-position: $\mathbf{v} = v(z)\hat{\mathbf{x}}$. Imagine putting an imaginary surface at height z, parallel to the x-y plane. The momentum transfer per unit time per unit area through this surface is defined to be is $\eta v'(z)$.

1.1. Show that these two definitions are equivalent.

Hint: take the former setup and break up the fluid into small pieces. Apply Newton's laws to each piece (which is clearly not accelerating).

Using the second definition, the shear viscosity can be written as

$$\eta v' = \int \frac{d^3p}{(2\pi)^3} \frac{p_z}{m}(p_x - mv) f(\mathbf{p}) \tag{1}$$

where v' is the derivatve of v with respect to z.

We will take the distribution function to be of the form $f = f_0 + \delta f$, with

$$f_0 = \exp\left(-\beta\left(\frac{|\mathbf{p} - mv(z)\hat{\mathbf{x}}|^2}{2m} - \mu\right)\right)$$

being the local equilibrium form, where the velocity $\mathbf{v} = v(z)\hat{\mathbf{x}}$ is a function of z, but points in the $\hat{\mathbf{x}}$ direction. We will assume that δf is small, and solve the Boltzman equation:

$$\frac{\mathbf{p} \cdot \nabla_{\mathbf{r}}}{m} f = I_p(f).$$

Given that $I_p(f_0) = 0$, we will can in principle linearize the right hand side, so it is just a linear function of δf (ie. the convolution of δf and some kernel). We could then ignore δf on the left hand side. Solving the resulting integral equation would give δf.

1.2. Suppose we make a relaxation time approximation $I_p(f) = -\delta f/\tau$. Solve for δf.

1.3. Perform the integral in Eq. (1) to relate η to the relaxation time τ. Express your answer in terms of the density $n = \int \frac{d^3p}{(2\pi)^3} f_0$.

Hint: the equipartition theorem gives

$$\int \frac{d^3p}{(2\pi)^3} \frac{(p_z)^2}{2m} \frac{(p_x - mv)^2}{2m} f_0(\mathbf{p}) = n(k_B T/2)^2$$

1.4. The standard approximation for the relaxation time is $\tau = 1/(n\bar{\sigma}\bar{v})$ where $m\bar{v}^2/2 = kT$ and $\bar{\sigma}$ is the total scattering cross-section. What do you then get for η in terms of $\bar{\sigma}$?

We will now go a little deeper into the expression for τ.

It turns out that the δf in problem 1.2 is not the exact solution to the integral equation, but is still a reasonable approximation. A nice trick for estimating τ is to start with $I_p(f) \approx \delta f/\tau$ and then take moments. The integral over p of both sides vanishes, but the following integrals do not:

$$\int \frac{d^3p}{(2\pi)^3} p_z(p_x - mv) I_p(f) \approx \int \frac{d^3p}{(2\pi)^3} p_z(p_x - mv) \frac{\delta f}{\tau}.$$

Thus a good approximation to τ is

$$\tau^{-1} \approx \frac{\int \frac{d^3p}{(2\pi)^3} p_z(p_x - mv) I_p(f)}{\int \frac{d^3p}{(2\pi)^3} p_z(p_x - mv) \delta f} \tag{2}$$

Note that this is the viscosity relaxation time, and one will want slightly different expressions for looking at the relaxation of different δf's.

For simplicity we will take the scattering to be isotropic so that to linear order the collision integral is

$$\begin{aligned} I_p(f) &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{\sigma}{m} 2\pi\delta\left(\frac{k^2}{2m} + \frac{p^2}{2m} - \frac{(k+q)^2}{2m} - \frac{(p-q)^2}{2m}\right) \\ &\quad \times [f_0(k+q)\delta f_{p-q} + f_0(p-q)\delta f_{k+q} - f_0(k)\delta f_p - f_0(p)\delta f_k] \end{aligned}$$

Inserting δf from 1.2, one can use symmetry arguments to write

$$I_p(f) = -\delta f_p \int \frac{d^3k}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{\sigma}{m} 2\pi\delta\left(\frac{k^2}{2m} + \frac{p^2}{2m} - \frac{(k+q)^2}{2m} - \frac{(p-q)^2}{2m}\right) f_0(k).$$

1.5. Explain the symmetry arguments.

Clearly if the integral right before question 1.5 was independent of p, then the relaxation time approximation would be exact. It is not independent of p, so we must resort to Eq. 2, finding

$$\frac{1}{\tau} = \frac{\int \frac{d^3k}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\frac{d^3p}{(2\pi)^3}\frac{\sigma}{m}2\pi\delta\left(\frac{k^2}{2m}+\frac{p^2}{2m}-\frac{(k+q)^2}{2m}-\frac{(p-q)^2}{2m}\right)f_0(k)f_0(p)p_z^2(p_x-mv)^{2^2}}{\int \frac{d^3p}{(2\pi)^3}f_0(p)p_z^2(p_x-mv)}$$

This is clearly independent of v, so we may set $v=0$. By making the substitution $\bar{m}=k/\sqrt{2mT}$ (and similar substitutions for p and q), one can adimensionalize the integrals, writing

$$\frac{1}{\tau} = n\sigma\sqrt{2mT}I,$$

where I is just some number which can be expressed as a quotient of integrals.

1.6. What is the expression for I?

1.7. (bonus) Calculate I. (I believe it is doable analytically, but I couldn't be bothered – it seemed like more trouble than it is worth. If you do want to do this, some of the tricks are given in Reif Section 14.8. In fact the whole calculation appears there, but since Reif uses slightly different notation I couldn't just pull the answer out. Reif also shows that the approximation in Eq. 2 can be derived from a variational principle.)