P683 HW2

Due Wednesay Feb 3, 2010

Problem 1. Typical Spectral Density One is typically interested in A, but can most easily calculate G. Here we will imagine we did a calculation of G, and want to know what A looks like. Let us take

$$G(\omega) = \frac{1}{\omega - \epsilon - \lambda\sqrt{-\omega}/2},$$

where ϵ and λ are real. We will derive this G in a future class. It is typical of an atomic state. By $\sqrt{x}$ we mean the principle branch of the square root. Thus if $\mathrm{Im}(\omega) > 0$ and $\mathrm{Re}(\omega) > 0$, then $\mathrm{Im}(\sqrt{-\omega}) > 0$.

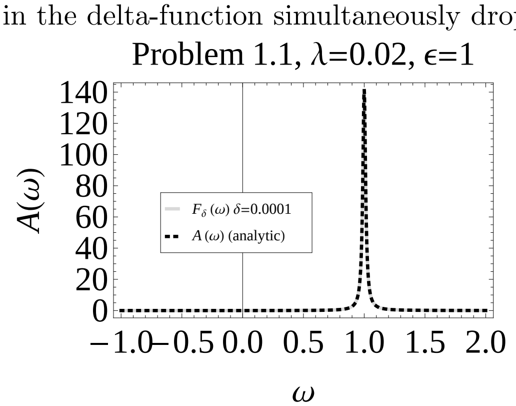
Define $F_\delta(\omega) = i[G(\omega + i\delta) - G(\omega - i\delta)]$, so that $A(\omega) = \lim_{\delta\to 0} F_\delta(\omega)$.

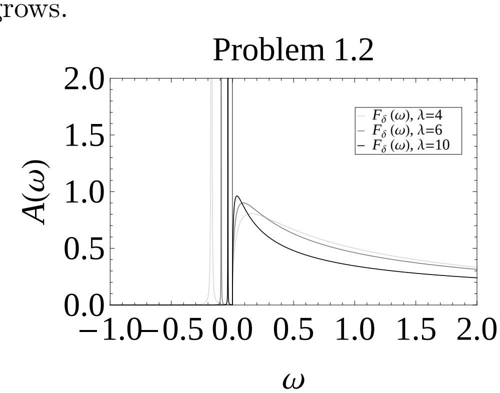
1.1. Plot $F_\delta(\omega)$ for $\delta = 0.0001, \lambda = 0.02, \epsilon = 1$, over the range $-1 < \omega < 2$. Take the vertical axis to go from 0 to 200. Play a bit with the parameters. What happens as one takes $\delta \to 0$?

1.2. Plot $F_\delta(\omega)$ for $\delta = 0.0001, \lambda = 4, \epsilon = -1$, over the range $-1 < \omega < 2$. Take the vertical axis to go from 0 to 2. Play a bit with the parameters. What happens as one takes $\delta \to 0$? What happens as one makes λ bigger? You will need to adjust the vertical scale.

Solution 1.1 (and 1.1)**.** For $\epsilon > 0$, the function F_δ is independent of δ for small δ. There is a single, roughly Lorentzian, peak, centered at $\omega = \epsilon$, whose width is set by λ: bigger λ means wider peak.

For $\epsilon < 0$, there is a sharp peak for $\omega < 0$ and a continuum of states for $\omega > 0$. As $\delta \to 0$ the peak sharpens into a delta function, but the continuum stays largely unchanged. As one makes λ larger, the peak approaches the continuum, which develops a sharper edge and longer tails. Although it might not look like it on this scale, the total spectral weight in the continuum grows as λ becomes larger (go ahead and numerically do the integral and you will see). The total weight in the delta-function simultaneously drops as λ grows.





1.3. (bonus) Analytically calculate A. Its not that hard – but the expression is not as revealing as making the previous plots.

Solution 1.2. We separately consider $\omega > 0$ and $\omega < 0$. When $\omega > 0$ the square roots are purely imaginary, but one uses a different branch for $G(\omega + i\delta)$ and $G(\omega - i\delta)$], which yields

$$A(\omega > 0) = \frac{\lambda\sqrt{\omega}}{(\omega - \epsilon)^2 + \omega\lambda^2/4}.$$

For $\omega < 0$, the only possible singularity of G is a simple pole. The location of the pole is found by setting $G^{-1} = 0$, which gives a quadratic equation for $\sqrt{-\omega}$. We only want solutions of this quadratic where we are on the principle branch of the square root: ie where $\sqrt{-\omega}$ is real and positive. If $\epsilon > 0$ there are no such physical solutions, and $A(\omega < 0) = 0$. If $\epsilon < 0$ there is one such solution (the other is on a separate Rieman sheet with $\sqrt{-\omega} < 0$). The pole is at

$$\omega = \xi = \sqrt{(\lambda/2)^2/4 - \epsilon)} - \lambda/4.$$

The residue of G at $\omega = \xi$ is

$$Z = \frac{1}{1 + \lambda/(4\xi}).$$

Putting this together we have

$$A(\omega) = \theta(\omega)\frac{\lambda\sqrt{\omega}}{(\omega - \epsilon)^2 + (\lambda/2)^2\omega} + \theta(-\omega)2\pi Z\delta(\omega - \xi)$$

where $\theta(\omega)$ is the Heavyside step function.

Problem 2. Analytic Structure of G The greens function is related to the spectral density by

$$G(\omega) = \int \frac{dz}{2\pi}\frac{A(z)}{\omega - z}.$$

2.1. Let $A(z) = 2\pi\delta(z - \epsilon)$, where ϵ is real. What is G? Is it analytic away from the real axis?

[Note, since any A can be written as some limit of delta-functions, this immediately gives us a "physicist proof" of the analyticity of G away from the real axis.]

Solution 2.1.

$$G(\omega) = \frac{1}{\omega - \epsilon}$$

2.2. Suppose

$$A(z) = \frac{\Gamma}{(z - \epsilon)^2 + (\Gamma/2)^2}$$

with real ϵ and Γ. What is $G(\omega)$. Note G is discontinuous across the real ω axis, so one has to separately consider the case $\text{Im}(\omega) > 0$ and $\text{Im}(\omega) < 0$

Solution 2.2. One can either decompose $A(\omega)$ into it's poles and do contour integrals for $\operatorname{Im}\omega > 0, \operatorname{Im}\omega < 0$(doesn't matter whether one closes the contour in the upper or lower half plane) or guess $G(\omega)$ (as I did) the result and check via the relation from Problem 1 that it gives the right $A(\omega)$. The result is

$$G(\omega) = \frac{1}{\omega - \epsilon + i(\Gamma/2)\,\mathrm{sign}(\operatorname{Im}\omega)}$$

Problem 3. By using the definition $A(\omega) = G^{>}(\omega) \mp G^{<}(\omega)$, show that $A(\omega)$ is real [for real ω].

Solution 3.1. The spectral representations for $G_p^{<}(\omega) = Z^{-1}\sum_{i,j} e^{-\beta E_i}|\langle i|\psi_p|j\rangle|^2 2\pi\delta(\omega - (E_i - E_j)), G_p^{>}(\omega) = Z^{-1}\sum_{i,j} e^{-\beta E_j}|\langle i|\psi_p^{\dagger}|j\rangle|^2 2\pi\delta(\omega - (E_i - E_j))$ derived in class are manifestly real for real ω. This implies that $A(\omega)$ is real for real ω.