Lecture 6

Berry's phase in Hall Effects and Topological Insulators

Given the analogs between Berry's phase and vector potentials, it is not surprising that Berry's phase can be important in the Hall effect. In this lecture I will show how it modifies the classical Hall effect. Next I will show that there is a topological invarient in the integer quantum Hall effect which is closely related to the same Berry Phase which was important in the polarization problem. Finally I will introduce topological insulators, which are insulators in which similar topological invariants are non-trivial.

A. Anomalous Velocity

We started thinking about adiabaticity by putting a Bloch electron in an electric field. Lets revisit that problem. We can very formally look at this in terms of adiabatic evolution by using a gauge where the electric field comes from a time dependent vector potential:

$$H = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}/c)^2 + V(\mathbf{r}) \tag{6.1}$$

where V is the potential of the ions. For a constant electric field we take

$$\mathbf{A} = -\mathbf{E}t. \tag{6.2}$$

If this is varying slowly enough in time, we know the state at time t will simply be

$$|\psi(t)\rangle = e^{-i\Phi(t)}|p_0 - eA/c\rangle, \tag{6.3}$$

where p_0 is the momentum at time 0. That is, the momentum just increases with time – as in our previous discussion. There is, however, the phase factor,

$$\Phi(t) = \int^t \epsilon(t)dt + \int^t (\partial_t eA/c)\langle p|\partial_p|p\rangle, \tag{6.4}$$

where

$$p = p_0 - eA/c. \tag{6.5}$$

As before, when discussing the polarization, this extra phase factor gives an extra contribution to the group velocity of a wave packet. Hence

$$\mathbf{v} = \partial_p \epsilon_p - \frac{e}{\hbar}\mathbf{E} \times \mathbf{\Omega}(\mathbf{p}), \tag{6.6}$$

where

$$\mathbf{\Omega}(\mathbf{p}) = i(\nabla_p \langle p|) \times (\nabla_p |p\rangle). \tag{6.7}$$

Thus even in the absence of a magnetic field you can get a Hall effect: ie. you place a voltage across the $\hat{x}$ axis, and get a current flowing along the $\hat{y}$ axis. This is known as the "Anomalous Hall effect".

The natural question is what materials have non-zero $\mathbf{\Omega}$ and hence a significant Anomalous Hall effect?

First of all $\mathbf{\Omega} = \mathbf{0}$ if the system has both time reversal and inversion symmetry. Time reversal takes $v \to -v$, $E \to E$ and $k \to -k$. Consequently, if the system has time reversal symmetry then $\Omega(-k) = -\Omega(k)$. Inversion symmetry takes $v \to -v$, $E \to -E$ and $k \to -k$. Consequently if the system has inversion symmetry then $\Omega(-k) = \Omega(k)$. The only consistent way to have both symmetry is to have $\Omega = 0$.

Time reversal symmetry is broken in ferromagnets and antiferromagnets. In fact, it is often the Hall effect in ferromagnets which is referred to as the "Anomalous Hall Effect". A simple example would be a two-dimensional tight binding model with a Rashba spin-orbit term $\hat{z} \cdot (S \times p))$ and an exchange splitting

$$\begin{aligned} H &= \sum_{i,\sigma,\tau} \left[a^\dagger_{i,\sigma} a_{i-\hat{x},\tau}(-t\delta_{\sigma\tau} + i\alpha(S_y)_{\sigma\tau}) + a^\dagger_{i,\sigma} a_{i-\hat{y},\tau}(-t\delta_{\sigma\tau} - i\alpha(S_x)_{\sigma\tau}) \right] + \mathrm{HC} \\ &\quad + \epsilon \sum_i (a^\dagger_{i\uparrow} a_{i\uparrow} - a^\dagger_{i\downarrow} a_{i\downarrow}). \end{aligned} \tag{6.8}$$

The fact that spin-orbit coupling is important is somehow natural. The idea is that as you adiabatically move through k-space, your spin rotates. This rotation in spin space yields a Berry phase.

Problem 6.1. Calculate the band-structure of Eq. (6.8). Although its not particularly physical, the structure is clearest when $t = 0$, so lets set $t = 0$. Make a plot of Ω as a function of k_x and k_y. Show both the case of $\epsilon \ll \alpha$ and the case of $\epsilon \gg \alpha$.
What would the qualitative difference be in Ω if you took the ϵ term to be a next nearest neighbor (diagonal) hopping, instead of just an on-site term?

A typical model without inversion symmetry would be graphene with an extra superlattice potential.

If you have a conductor with non-zero Ω, the Anomalous Hall conductivity will come from summing up all the velocities from all the occupied states:

$$\sigma_{xy} = \frac{e^2}{\hbar} \int \frac{d^d k}{(2\pi)^d} f(\epsilon_k) \Omega_{k_x,k_y}, \tag{6.9}$$

where f is a step function at the Fermi surface. Using Stoke's theorem (and specializing to 2d), we can write this as

$$\sigma_{xy} = \frac{e^2}{\hbar} \oint dk \cdot A_k. \tag{6.10}$$

Thus the Hall conductivity can be viewed as the Berry phase accumulated in moving around the Fermi surface.

B. Hall conductivity of an insulator

The Hall conductivity of an insulator comes simply from summing Eq. (6.6) over the filled bands. Specializing to two dimensions,

$$\sigma_{xy} = \frac{e^2}{\hbar} \int_{\mathrm{BZ}} \frac{d^2 k}{2\pi}^2 \Omega_{kx,ky}. \tag{6.11}$$

It turns out that this integral must be an integer. The idea is that the Berry phase accumulated in any closed loop in k-space is unique. Using Stoke's theorem, this phase can be written as an integral of the Berry curvature. There are two possible places the integral can be done over. These phases are only the same (modulo 2π) if the integral is a multiple of an integer. This integer is known as the first Chern number.

B.1. Integer Quantum Hall Effect as a Chern Number

We wish to understand the single particle problem defined by the Hamiltonian

$$H = \frac{(p - eA/c)^2}{2m} \tag{6.12}$$

where

$$B = \nabla \times A \tag{6.13}$$

is constant. This is a standard textbook problem, and if you have never done it before, open up Landau and Lifshitz and read the relevant chapter. Here we will approach it from a slightly more formal direction, which will make contact with Berry's phase.

To understand the structure, we introduce magnetic translation operators

$$T_\delta = e^{i\mathbf{p}\cdot\delta - i(e/c)(\boldsymbol{\Delta}_\delta \mathbf{A})\cdot\mathbf{r}} \tag{6.14}$$

$$(\boldsymbol{\Delta}_\delta \mathbf{A}) = A(r+\delta) - A(r). \tag{6.15}$$

The quantity $\boldsymbol{\Delta}_\delta \mathbf{A}$ is a constant, independent of $\mathbf{r}$.

Group Activity: What is

$$[T_\delta, H] \tag{6.16}$$

These magnetic translations commute with the Hamiltonian. Hence if I have one eigenfunction of the Hamiltonian $H\psi = E\psi$, I can generate a new one $T_\delta\psi$ with the same eigenvalue. Thus the eigenstates will be degenerate.

Group Activity: What is wrong with this argument when $B = 0$.

Typically what we would next do is try to find the mutual eigenstates of the T's and H. In the magnetic field-free case these are plane waves. The magnetic translations do not, however, commute with one-another. In fact

$$T_{\delta_1} T_{\delta_2} = e^{i2\pi\phi/\phi_0} T_{\delta_2} T_{\delta_1}, \tag{6.17}$$

where $\phi = \mathbf{B}\cdot\delta_1 \times \delta_2$ in the flux through the closed path formed by moving along the path $\delta_1, \delta_2, -\delta_1, -\delta_2$, and $\phi_0 = hc/e$ is the quantum of flux. Thus we cannot be simultaneously an eigenstate of all of the T's. Hence no plane waves.

On the other hand, if ϕ is an integer multiple of ϕ_0 they do commute. Thus we can find a subset of the T's which commute with one-another and with H. This is the same structure as Bloch waves. The connection is more than just formal. Lets take our commuting set of T's to be

$$T_{nm} = T_{n\ell\hat{\mathbf{x}}} T_{m\ell\hat{\mathbf{y}}} \tag{6.18}$$

where $\ell^2 = (hc)/(eB)$ is the "magnetic length". These operators are unitary, so their eigenvalues must have modulus 1. Thus we define the state $|q_x, q_y\rangle$ with $-\pi < q < \pi$ and satisfying

$$T_{nm}|q_x, q_y\rangle = e^{2\pi i(q_x n + q_y m)}|q_x, q_y\rangle. \tag{6.19}$$

As you can see, there is an arbitrary phase hanging around – which once again will lead us to a geometric phase. An equivalent definition is

$$[p_x - (e/c)(\partial_x A_y)y]|q_x, q_y\rangle = \frac{2\pi}{\ell} q_x |q_x, q_y\rangle, \tag{6.20}$$

up to a possible integer multiple of 2π.

To find the degeneracy of the "Landau levels" we put the system in a $L \times L$ box with "magnetoperiodic" boundary conditions – meaning $T_{L\hat{\mathbf{x}}}\psi = T_{L\hat{\mathbf{y}}}\psi = \psi$ – and we take L to be an integer multiple of ℓ. We see there are exactly $(\ell/L)^2$ ways to choose q_x and q_y – resulting in that many degenerate states.

Lets consider an integer quantum Hall state. This is the "insulator" formed when the entire Landau level is filled. I'll spare you the details, but it probably does not come as much of a surprise that the Hall conductivity is the same as what we found for an ordinary insulator:

$$\sigma_{xy} = -\frac{e^2}{\hbar} \int_{\mathrm{BZ}} \frac{dk}{(2\pi)^2} \Omega_{kx,ky}, \tag{6.21}$$

where

$$\Omega = i\left[(\partial_x \nabla\langle q|)(\partial_y|q\rangle) - (\partial_y \nabla\langle q|)(\partial_x|q\rangle)\right]. \tag{6.22}$$

By our now standard Stokes theorem arguments, the Hall conductivity is quantized.

C. Other topological invariants in insulators

Any insulator which has a topological invariant is a "topological insulator". So far we have seen one invariant, the first Chern number. There are interesting varients of this invarient: for example, there is at least one model for which the up-spins have a Chern number of +1, and the down spins -1. This gives rise to a "Spin Hall Effect".

The most famous other topological invariant was introduced by Kane and Mele for time reversal invariant models. These generalize the Spin Hall Effect to cases where the spin projection is not conserved. I am afraid if I do not have a clear enough understanding of it to really say something useful.

D. Edge excitation in topological insulators

As we have seen, topological insulators, including those seen in the quantum Hall effect, have non-trivial topological invariants. A consequence of this is that in any boundary between a topological insulator and vacuum (or between two topological insulators with different topological numbers) there must exist gapless "edge states". The idea is that the topological numbers are only quantized if the state is an insulator. The only way for the number to change is if the gap closes – hence edge states.