P7654 HW3

Due Wednesday Feb 6, 2013

Problem 1. Elementary Calculation of linear response of harmonic oscillator Here we will use undergraduate level physics to calculate the linear response of a simple Harmonic Oscillator. As always, if this is trivial for you, then skip it.

As discussed in class, we consider a system described by Hamiltonian

$$H = \frac{p^2}{2} + \frac{1}{2}\omega_0^2 x^2 - xF(t).$$

1.1. Use the Heisenberg equations of motion, $i\partial_t A = [A, H]$, derive a second order inhomogeneous equation of motion for the operator x.

Solution 1.1.

$$\partial_t^2 x + \omega_0^2 x = F(t)$$

We will use Greens functions to solve this equation. Suppose $\chi(t - t_0)$ is a solution to this differential equation with $F(t) = \delta(t - t_0)$. A solution to the generic inhomogeneous equation is then

$$x(t) = -\int dt \chi(t - t_0) F(t_0). \tag{1}$$

As is indicated by this careful wording, the Greens function is not unique. The "-" that I just added gives the convention we used in class where we calculate the response to a potential rather than a response to a force.

1.2. Explain in a couple sentences why it is not unique. Given one particular χ, how do you generate the most general Greens function? Hint: what is the most general thing we could write in Eq. 1?

Solution 1.2. We can always add the homogenous solution to the Green's function to generate another one,

$$\chi^{AB}(t - t_0) = \chi(t - t_0) + A\sin(\omega_0 (t - t_0)) + B\cos(\omega_0 (t - t_0)), \tag{2}$$

as $\left(\partial_t^2 + \omega_0^2\right)\chi^{AB}(t - t_0) = \left(\partial_t^2 + \omega_0^2\right)\chi(t - t_0) = \delta(t - t_0)$.

The physically most useful Greens function is the "retarded" one χ^R, which is characterized by $\chi^R(t) = 0$ for $t < 0$.

1.3. Use your differential equation to show that

$$\chi^R(t) = \theta(t)\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i\omega_0}$$

is the retarded Greens function for this equation.

Solution 1.3. Obviously this function has $\chi^R(t<0)=0$. It's left to verify that

$$\left(\partial_t^2+\omega_0^2\right)\chi^R(t-t_0)=\delta(t-t_0)\,. \tag{3}$$

The least confusing approach is to check

$$\begin{aligned}
&\int \mathrm{dt}_0\;\left(\partial_t^2+\omega_0^2\right)\chi^R(t-t_0)\,f(t_0)\\
&=\int \mathrm{dt}_0\;\left(\partial_t^2+\omega_0^2\right)\theta(t-t_0)\,\frac{\sin(\omega_0(t-t_0))}{\omega_0}f(t_0)\\
&=\int \mathrm{dt}_0\;\left[\delta'(t-t_0)\,\frac{\sin(\omega_0(t-t_0))}{\omega_0}+2\delta(t-t_0)\cos(\omega_0(t-t_0))\right]f(t_0)\\
&=\int \mathrm{dt}_0\;\left[\delta(t-t_0)\cos(\omega_0(t-t_0))\,f(t_0)-\delta(t-t_0)\,\frac{\sin(\omega_0(t-t_0))}{\omega_0}f'(t_0)\right]\\
&=f(t)\,,
\end{aligned} \tag{4}$$

having used integration by parts in the next-to last line. [The original factor of 2 error in the question came from the fact I erroneously threw away the δ' term.]

1.4. Fourier transform this expression to get

$$\chi^R(\omega)$$

.

Solution 1.4. Anywhere above the real line, $\chi^R(\omega)$ is well defined by

$$\begin{aligned}
\chi^R(\omega)&=\int \mathrm{dt}\,e^{i\omega t}\chi^R(t)=\int_0^\infty \mathrm{dt}\,\frac{e^{i(\omega+\omega_0)t}-e^{i(\omega-\omega_0)t}}{2i\omega_0}\\
&=\frac{1}{2\omega_0}\left[\frac{1}{\omega+\omega_0}-\frac{1}{\omega-\omega_0}\right]=-\frac{1}{\omega^2-\omega_0^2}.
\end{aligned} \tag{5}$$

As a final connection to your undergraduate physics, consider the case $F(t)=F\cos(\omega t)$,and use the fact that

$$x(t)=x_0(t)+\int dt_0\chi(t-t_0)F(t_0),$$

where $x_0(t)$ is the solution to the homogeneous equation. I'd like to know the RMS value of the position is in steady state:

$$\sigma=\sqrt{\frac{1}{T}\int_0^T\langle x(t)^2\rangle dt}.$$

1.5. What is the natural value of T to use for our integration window?

Solution 1.5. At equilibrium we expect the system to oscillate at the frequency of the driving force, ω, and so the natural period to integrate over is $T=2\pi/\omega$.

1.6. Assuming $\omega \neq \omega_0$, find σ. The result should be familliar.

Assume that before the probe was applied the system was in thermal equilibrium, so

$$\begin{aligned} \langle (1/2)\omega_0^2 x_0(t)^2 \rangle &= \frac{\hbar\omega}{4}\coth\left(\frac{\beta\omega}{2}\right). \\ \langle x_0(t) \rangle &= 0 \end{aligned}$$

Note: you do not need to prove these (but they are not hard). We will give a derivation in Prob. 3

Solution 1.6. Let

$$x^F(t) = \int \mathrm{d}t_0\, \chi(t - t_0)\, F(t_0) = \int_{-\infty}^{t} \mathrm{d}t_0\, \frac{\sin(\omega_0 (t - t_0))}{\omega_0} F \cos(\omega t_0) = \frac{\cos(\omega t)}{(\omega_0^2 - \omega^2)} F \tag{6}$$

taking the oscillations at $t0 \to -\infty$ to cancel out.
We have, given that the homogenous and inhomogenous solutions have different periods,

$$\begin{aligned} \left\langle x(t)^2 \right\rangle &= \left\langle x_0(t)^2 \right\rangle + 2\left\langle x(t)\, x^F(t) \right\rangle + \left\langle x^F(t)^2 \right\rangle \\ &= \left\langle x_0(t)^2 \right\rangle + \left\langle x^F(t)^2 \right\rangle, \end{aligned} \tag{7}$$

Taking x^F to be classical, we have

$$\overline{x^F(t)^2} = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \mathrm{d}t \left(\frac{\cos(\omega t)}{(\omega_0^2 - \omega^2)} F \right)^2 = \frac{F^2}{2\left(\omega_0^2 - \omega^2\right)^2} \tag{8}$$

so

$$\sigma^2 = \frac{\hbar}{2\omega_0}\coth(\beta\omega_0/2) + \frac{F^2}{2\left(\omega_0^2 - \omega^2\right)^2}. \tag{9}$$

As $\hbar \to 0$ the result approaches the familiar resonance peak for a forced harmonic oscillator.

Problem 2. Sophisticated Calculation of linear response of harmonic oscillator

Recall from class

$$\chi^R(t) = i\theta(t)\langle [x(t), x(0)] \rangle \tag{10}$$

where the expectation values are taken in the absence of the probe.

2.1. Write the equations of motion for the operators $\hat{x}(t)$ and $\hat{p}(t)$ (in the absence of any perturbation).

Solution 2.1. These are given by $i\partial_t \hat{A} = \left[\hat{A}, \hat{H}\right]$, and so

$$\begin{aligned} i\partial_t x(t) &= ip(t) \\ i\partial_t p(t) &= -i\omega_0^2 x(t) \end{aligned} \tag{11}$$

2.2. The solution of these equations will be of the form

$$\hat{x}(t) = a(t)\hat{x}(0) + b(t)\hat{p}(0). \tag{12}$$

Find $a(t)$ and $b(t)$.

Solution 2.2. This leads to the classical equations of motion,

$$\begin{aligned} x(t) &= x(0)\cos(\omega_0 t) + \frac{p(0)}{\omega_0}\sin(\omega_0 t) \\ p(t) &= -\omega_0 x(0)\sin(\omega_0 t) + p(0)\cos(\omega_0 t). \end{aligned} \tag{13}$$

2.3. Substitute this result into Eq. (10). Use the computation relationships. Compare with the result in the previous question.

Solution 2.3.

$$[x(t), x(0)] = \left[x(0)\cos(\omega_0 t) + \frac{p(0)}{\omega_0}\sin(\omega_0 t), x(0)\right] = -\frac{i}{\omega_0}\sin(\omega_0 t) \tag{14}$$

so

$$\chi^R(t) = \theta(t)\frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2\omega_0}. \tag{15}$$

The same result.

Problem 3. Equilibrium fluctuations of harmonic oscillator

We know that in thermal equilibrium

$$\chi^<(\omega) = 2f(\omega)\mathrm{Im}\chi^R(\omega),$$

where $f(\omega) = 1/(e^{\beta\omega} - 1)$. In the time domain,

$$\chi^<(t) = \langle x(0)x(t)\rangle.$$

3.1. Use your explicit expression for $\chi^R(\omega)$ to calculate the equilibrium RMS equilibrium fluctuations $\sigma = \sqrt{\langle x(t)^2\rangle}$.

Solution 3.1.

$$\int \mathrm{dx}\, \frac{f(x)}{x-a} = \tag{16}$$

Using

$$\chi^R(\omega) = \frac{1}{4\omega_0}\left[\frac{1}{\omega+\omega_0} - \frac{1}{\omega-\omega_0}\right] = P - \frac{i\pi}{4\omega_0}\left[\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right] \tag{17}$$

we have

$$\begin{aligned} \chi^<(\omega) &= \frac{\pi}{2\omega_0} f(\omega)\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right] \\ \chi^<(t) &= \frac{1}{4\omega_0}\left[e^{-i\omega_0 t} f(\omega_0) - e^{i\omega_0 t} f(-\omega_0)\right] \end{aligned} \tag{18}$$

and so

$$\left\langle x(t)^2 \right\rangle = \left\langle x(0)^2 \right\rangle = \chi^<(0) = \frac{1}{4\omega_0}\coth(\beta\omega_0/2). \tag{19}$$

Problem 4. Lindhard Response function Here we give an elementary "semiclassical" derivation of the density response function of an ideal gas. In the next question, we will give a diagrammatic argument.

We will start with the phase space distribution function $f(r,p,t)$ – this is the density of particles in phase space. I.E. $f(r,p,t)d^3r d^3p$ is the number of particles in a small element of phase space $d^3r d^3p$ at time t. Clearly the very existence of this function is at odds with quantum mechanics. We won't worry about such minor issues.

The equation of motion obeyed by this distribution function is the Boltzmann equation

$$(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_r - (\nabla_r V)\cdot\nabla p) f(r,p,t) = 0.$$

We will imagine that $V(r,t)$ is an arbitrary function of space and time. It turns out that this is a good description as long as V varies slowly in space compared to the interparticle spacing, and slowly in time compared to the "Fermi energy."

To first order in $V(r,t)$ we want to know what the density is at position r' and time t'. Since there are no interactions, this should be doable – the only hard work is book-keeping. We begin by writing $f(r,p,t) = f_0(r,p,t) + \delta f(r,p,t)$ where

$$f_0(r,p,t) = \frac{1}{e^{\beta(p^2/2m-\mu)} \mp 1}.$$

In the classical limit we can just use $f_0(r,p,t) = e^{-\beta(p^2/2m-\mu)}$. For most of this problem we will not worry about the exact form.

Linearizing the Boltzmann equation, we get

$$(\partial_t + \frac{\mathbf{p}}{m}\cdot\nabla_r)\delta f = ((\nabla_r V)\cdot\nabla p) f_0 \tag{20}$$

4.1. Fourier transform Eq. (20) and solve for

$$\delta f(k,p,\omega) = \int dr dt e^{-i(kr-\omega t)} \delta f(r,p,t).$$

Solution 4.1. Substituting $\delta_f(r,p,t) = \int \frac{\mathrm{d}^3\mathrm{k}\mathrm{d}\omega}{(2\pi)^4} e^{i(kr-\omega t)} \delta f(k,p,\omega)$ and similar for V, and noting $f_0(r,p,t) = f_0(p)$, we have

$$\left(-i\omega + i\frac{\mathbf{p}}{m}\cdot\mathbf{k}\right)\delta f(k,p,\omega) = (i\mathbf{k}V(k,\omega))\cdot(\nabla_p f_0) \tag{21}$$

or

$$\delta f(k,p,\omega) = \frac{\mathbf{k}\cdot\nabla_p f_0}{\left(-\omega + \frac{\mathbf{p}}{m}\cdot\mathbf{k}\right)} V(k,\omega). \tag{22}$$

If we integrate this expression over p we can find the density fluctuations:

$$\delta n(k,\omega) = \int \frac{dp}{(2\pi)^3} \delta f(k,p,\omega).$$

The resulting expression will be of the form

$$\delta n(k,\omega) = \chi(k,\omega) V(k,\omega),$$

which by now should feel comfortable.

4.2. What is $\chi(k,\omega)$? Don't do the p-integral yet.

Solution 4.2. Integrating over p on both sides we find

$$\chi(k,\omega) = \int \frac{\mathrm{dp}}{(2\pi)^3} \frac{\mathbf{k}\cdot\nabla_p f_0}{\left(-\omega + \frac{\mathbf{p}}{m}\cdot\mathbf{k}\right)}. \tag{23}$$

If you double k and double ω, you should find that $\chi(k,\omega)$ is unchanged. This means that

$$\chi(k,\omega) = \chi(\omega/k)$$

There are lots of things one can do with this function. The most important result is what happens for zero temperature Fermions, where

$$\nabla_p f_0 = -\hat{p}\delta(|p| - k_F).$$

4.3. By writing the p integral in spherical coordinates, find $\chi(k,\omega)$ for a zero temperature Fermi gas. It may help organize things if you note (by dimensional analysis)

$$\chi(k,\omega) = k_f m f\left(\frac{\omega m}{k p_f}\right),$$

and use the result

$$\int_{-1}^{1} dc \frac{c}{x-c} = x \log\left(\frac{x+1}{x-1}\right) - 2$$

Plot the real and imaginary parts of $f(x)$. Don't worry too much about which branch of the log to use – just be consistent.

Solution 4.3. Rewrite our equation, for zero temperature fermions,

$$\begin{aligned}\chi(k,\omega) &= \int \frac{\mathrm{p}^2 \mathrm{dpd}(\cos\theta)}{(2\pi)^2} \frac{|\mathbf{k}|\cos\theta\delta(p-k_f)}{\left(\omega - \frac{1}{m}p|\mathbf{k}|\cos\theta\right)} \\ &= k_f m \int \frac{\mathrm{d}(\cos\theta)}{(2\pi)^2} \frac{\cos\theta}{(\omega m/k_f k - \cos\theta)}\end{aligned} \tag{24}$$

to find

$$f(x) = x \log\left(\frac{x+1}{x-1}\right) - 2. \tag{25}$$

Letting Mathematica pick the log branch, we have the graph pictures in Fig. 1.

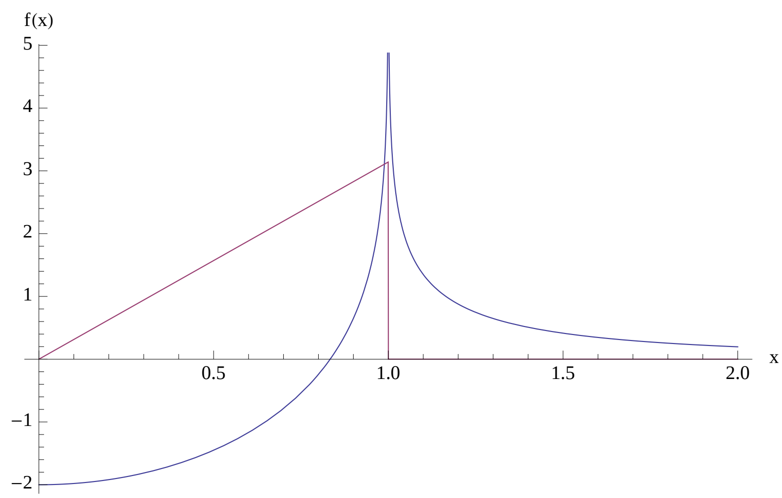


Figure 1: The real (blue) and imaginary (magneta-ish) parts of $f(x)$.

4.4. For large ω, this response function falls off as the square of frequency $\chi \sim \omega^{-2}$. Verify this result, and calculate the coefficient.

Solution 4.4. Using $\log[1+x] \approx x - x^2/2 + x^3/3$, we have

$$f(x) = x \log\left(\frac{1+1/x}{1-1/x}\right) - 2 \approx \frac{2}{3}\frac{1}{x^2} \tag{26}$$

and

$$\chi \approx \frac{2}{3}\frac{k_f^3}{m}\left(\frac{k}{\omega}\right)^2. \tag{27}$$

We will use this in Question 6

Problem 5. Lindhard function from single particle Greens Functions Here we will again calculate the density response function of a non-interacting gas, but this time we will use the formalism developed in class:

$$\chi^R(r,t) = i\theta(t)\langle[\rho(r,t),\rho(0,0)]\rangle \tag{28}$$

where

$$\rho(r,t) = \psi^\dagger(r,t)\psi(r,t).$$

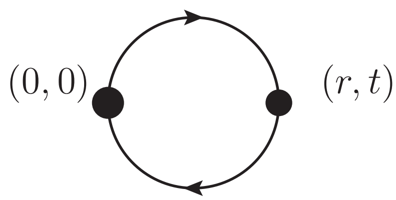
It is often more convenient to work with the time-ordered response function

$$\chi(r,t) = \langle T\rho(r,t)\rho(0,0)\rangle.$$

5.1. Apply Wick's theorem, and come up with a "diagram" for χ. Why is this called a "particle-hole bubble".

Solution 5.1.

$$\chi(r,t) = \left\langle \psi^\dagger(r,t)\,\psi(r,t)\right\rangle\left\langle \psi^\dagger(0,0)\,\psi(0,0)\right\rangle \pm \left\langle T\psi^\dagger(r,t)\,\psi(0,0)\right\rangle\left\langle T\psi^\dagger(0,0)\,\psi(r,t)\right\rangle \tag{29}$$



There are many ways to calculate this bubble. One can use Matsubara sums (see Coleman's text) but the simplest is actually to work with Eq. (28).

5.2. Using Wick's Theorem, express

$$\Pi(r,t) = \langle[\rho(r,t),\rho(0,0)]\rangle$$

in terms of $G^>(r,t)$ and $G^<(r,t)$.

Solution 5.2. Recalling $G^>(r,t) = \left\langle \hat\psi(r,t)\,\hat\psi^\dagger(0,0)\right\rangle$, $G^<(r,t) = \left\langle \hat\psi^\dagger(0,0)\,\hat\psi(r,t)\right\rangle$, we have simply

$$\begin{aligned}\Pi(r,t) &= \left\langle \hat\psi^\dagger(r,t)\,\hat\psi(r,t)\,\hat\psi^\dagger(0,0)\,\hat\psi(0,0)\right\rangle - \left\langle \hat\psi^\dagger(0,0)\,\hat\psi(0,0)\,\hat\psi^\dagger(r,t)\,\hat\psi(r,t)\right\rangle \\ &= \left\langle \hat\psi^\dagger(r,t)\,\hat\psi(0,0)\right\rangle\left\langle \hat\psi(r,t)\,\hat\psi^\dagger(0,0)\right\rangle - \left\langle \hat\psi^\dagger(0,0)\,\hat\psi(r,t)\right\rangle\left\langle \hat\psi(0,0)\,\hat\psi^\dagger(r,t)\right\rangle \\ &= G^<(-r,-t)\,G^>(r,t) - G^<(r,t)\,G^>(-r,-t)\,.\end{aligned} \tag{30}$$

Recalling Feynman's interpretation of antiparticles as particles moving backwards in time, we can see this again as a particle-antiparticle diagram.

Recall

$$G^<(k,\omega) = n_k 2\pi\delta(\omega-\epsilon_k) \quad (31)$$
$$G^>(k,\omega) = (1\pm n_k)2\pi\delta(\omega-\epsilon_k). \quad (32)$$

Also, note that one can Fourier transform Eq. (28) to get

$$\chi(k,\omega) = \int \frac{dz}{2\pi}\frac{1}{\omega-z}\Pi(k,\omega)$$

5.3. Write $\chi(k,\omega)$ as an integral over momenta. If you linearize this result for small k, you will reproduce your results from the semiclassical arguments.

Solution 5.3. In Fourier space, $\Pi(k,\omega)$ is

$$\Pi(k,\omega) = \int \frac{\mathrm{d}\eta}{2\pi}\int\frac{d^3k}{(2\pi)^3} G^<(k-p/2,\eta-\omega/2)\,G^>(k+p/2,\eta+\omega/2) - G^<(k+p/2,\eta+\omega/2)\,G^>(k-p/2,\eta-\omega/2)\,. \quad (33)$$

Using the explicit expressions, we have

$$\Pi(k,\omega) = \int \frac{\mathrm{d}\eta}{2\pi}\int\frac{d^3k}{(2\pi)^3}\left(n_{k-p/2}(1\pm n_{k+p/2}) - n_{k+p/2}(1\pm n_{k-p/2})\right)\times 2\pi\delta(\eta-\omega/2-\epsilon_{k-p/2})2\pi\delta(\eta+\omega/2-\epsilon_{k+p/2}) \quad (34)$$

One of the delta functions can be used to get rid of the η integral – that still leaves one more,

$$\Pi(k,\omega) = \int\frac{d^3k}{(2\pi)^3}\left(n_{k-p/2}-n_{k+p/2}\right)2\pi\delta(\omega-(\epsilon_{k-p/2}-\epsilon_{k+p/2})). \quad (35)$$

Problem 6. Random Phase Approximation Now we will use our previous result to approximate the density response function of an interacting gas. We will first do this by continuing our semiclassical argument.

The idea is that we will include interactions in our Boltzmann equation, by taking the intuitive approximation

$$V(r,t) = V_x(r,t) + \int dr' U(r-r')n(r'),$$

where $U(r)$ is the inter-particle interactions and V_x stands for ”eXternal”. This is the ”Hartree” approximation, where each particle interacts with the ”average density.” Substituting this into our Boltzmann equation, and linearizing, one has

$$(\omega - \mathbf{p}\cdot\mathbf{k}m)\delta f(k,p,\omega) = [V_x(k)+U(k)\delta n(k)]\,(k\cdot\nabla_p)f_0(p,\omega)$$

As before, we solve for δf to get

$$\delta f(k,p,\omega) = [V_x(k)+U(k)\delta n(k)]\,\frac{(k\cdot\nabla_p)f_0(p,\omega)}{\omega-\mathbf{p}\cdot\mathbf{k}m}.$$

Summing over p gives

$$\delta n(k,t) = \chi_0(k,\omega)\left[V_x(k) + U(k)\delta n(k)\right]$$

where χ_0 is the susceptability of the noninteracting gas. A little straightforward manipulation then gives

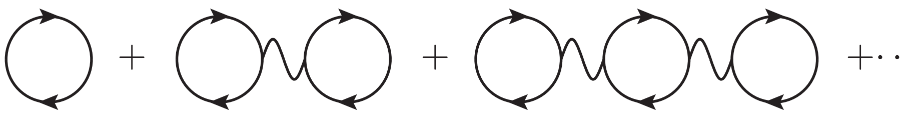
$$\chi(k,\omega) = \frac{\chi_0(k,\omega)}{1 - U(k)\chi_0(k,\omega)}$$

6.1. Write a set of diagrams which corresponds to this statement.

Solution 6.1. Rewriting

$$\chi = \frac{\chi_0}{1 - U\chi_0} = \sum_{n=0}^{\infty} \chi_0 \left(U\chi_0\right)^n \tag{36}$$

we can interpret the sum as



6.2. Plasmons Suppose our particles are interacting with a Coulomb potential $U(k) = e^2/k^2$. Use the large ω approximation you derived for χ_0 to show that χ has a simple pole. This corresponds to a propegating mode. What is its frequency?

Solution 6.2. Using the large ω approximation we have simply

$$\chi\left(k,\omega\right) = \left[1 - \frac{2}{3}\frac{k_f^3}{m}\left(\frac{e}{\omega}\right)^2\right]^{-1} \frac{2}{3}\frac{k_f^3}{m}\left(\frac{k}{\omega}\right)^2 \tag{37}$$

which has a simple pole at

$$\omega = \sqrt{\frac{2}{3}\frac{e^2 k_f^3}{m}}. \tag{38}$$