P7654 HW3

Due Wednesday Feb 13, 2013

Problem 1. Polarization Bubble as a Photon Selfenergy Here is a slightly backward way about thinking about optical absorption/index of refraction. Nobody does things this way, but I thought it would be instructive.

We are going to consider light coupled to some system. Generically we can write

$$H = \sum_k cka_k^\dagger a_k + \sum_k \left(\Lambda_k^\dagger a_k + a_k^\dagger \Lambda_k\right) + H_s \tag{1}$$

where H_s is the system Hamiltonian, and Λ_k is some operator which acts on the system. As a concrete example you could have the system being a single two-level atom at the origin, in which case

$$H_s = \epsilon\sigma_z \tag{2}$$

$$\Lambda_k = \lambda\sigma_-. \tag{3}$$

Another concrete example could be an electron gas, where one might take the photon to couple to the electron density:

$$\Lambda_k = \lambda \sum_p p\psi_{p+k/2}^\dagger \psi_{p-k/2}. \tag{4}$$

Regardless, one defines the photon self-energy via the expression

$$\left(G^{-1}(\omega)\right)_{kk'} = \left(G_0^{-1}(\omega)\right)_{kk'} - \Sigma_{kk'}(\omega). \tag{5}$$

where

$$\left(G_0^{-1}\right)_{kk'} = \delta_{kk'}(\omega - ck) \tag{6}$$

is the free photon Greens function. In the case of a translationally invariant medium, this simplifies to

$$G(k,\omega) = \frac{G_0(k,\omega)}{1 - \Sigma(k,\omega)G_0(k,\omega)}, \tag{7}$$

where $G_0(k,\omega) = 1/(\omega - ck)$.

1.1. In terms of a time ordered correlation function of the Λ operator, what is $\Sigma(k,\omega)$ to lowest order in the photon-system coupling?

Typically $\Sigma(k,\omega)$ is small compared to optical photon energies. This means G will have a pole somewhere near $\omega = ck$. It is therefore reasonable to approximate Σ by its "on-shell" value,

$$\Sigma(k,\omega) \approx \Sigma(k,ck) = \Pi_k + i\Gamma_k/2. \tag{8}$$

For the remainder of this question we will work within this approximation.

1.2. Optical Absorption: If I start at time $t = 0$ with a single photon with momentum k. In terms of Γ_k, what is the probability that after a time t I will still have a photon in that mode *and the system has not been excited*?

1.3. Index of Refraction: I can consider adding a state consisting of a wavepacket at position r with width σ and momentum k_0:

$$|r, k_0, \sigma\rangle = \int \frac{d^3k}{(2\pi\sigma^2)^{3/2}} e^{-\sigma^2(\mathbf{k}-\mathbf{k_0})^2/2} e^{i\mathbf{k}\cdot\mathbf{r}} a_k^\dagger |\Psi\rangle, \tag{9}$$

where Ψ is the ground state of the "system" without photons. After time t one imagines this wavefunction should evolve into something of the form

$$|t\rangle = \int \frac{d^3k}{(2\pi\sigma^2)^{3/2}} e^{-i\phi_k t - \gamma_k t} e^{-\sigma^2(\mathbf{k}-\mathbf{k_0})^2/2} e^{i\mathbf{k}\cdot\mathbf{r}} a_k^\dagger |\Psi\rangle + |\text{incoherent}\rangle \tag{10}$$

where $|\text{incoherent}\rangle$ represents terms where the system is excited in some way. Come up with an argument relating ϕ_k and γ_k to the self-energy. This gives you an intuitive meaning for the self-energy.

Problem 2. Polarization bubble as response to classical disturbance: This is the more typical way of thinking of optical absorption/index of refraction.

Consider driving a system via a classical electromagnetic field. Typically we can write the Hamiltonian as

$$H = \sum_k \left(\Lambda_k^\dagger \alpha_k(t) + \alpha_k^*(t)\Lambda_k\right) + H_s, \tag{11}$$

where the c-number $\alpha_k(t)$ is proportional to the Fourier transform of the electric field. In equilibrium we will assume $\langle\Lambda\rangle = 0$. We should think of α_k as the classical version of the a_k in the last problem.

2.1. For weak fields, the work done on the system by the electromagnetic field will be quadratic in the α's. Express the power dissipated P in terms of a correlation function of the Λ's.

2.2. The "net loss rate of photons" should be $P/(ck)$. How does this compare with the last problem?

Problem 3. Doppler Line-shape of an ideal gas of two-level atoms. Consider a gas of two-level systems with Hamiltonian

$$H = \sum_k (\epsilon_k + \nu) b_k^\dagger b_k + \epsilon_k a_k^\dagger a_k, \tag{12}$$

where a_k and b_k annihilate atoms in the two states. Lets take $\epsilon_k = k^2/2m - \mu$, and let ν be in a frequency range where we can imagine either $\beta\nu \gg 1$ or $\beta\nu \ll 1$. We are going to try to pass electomagnetic waves through this system.

As argued in problem 1, the phase shift of the waves will be related to the real part of a polarization bubble, while the imaginary part will give the absorption. The relevant bubble here is

$$\Pi(r, t) = \frac{1}{i}\langle T b^\dagger(r, t) a(r, t) a^\dagger(0, 0) b(0, 0)\rangle. \tag{13}$$

3.1. Using the techniques you have developed in previous homeworks, write an expression for $\Gamma(k,\omega) = 2\mathrm{Im}\Pi(k,\omega)$ in terms of an integral over the momentum in the bubble. Physically this corresponds to the net absorption. Thus your answer should be the difference of an absorption term and an emission term.

3.2. The Doppler effect comes about from the fact that in the energy delta-function there is momentum dependence. If you neglect this momentum dependence the line becomes a delta-function. What frequency of light is absorbed? [Don't overthink it – you would have given this answer without doing any calculation.]

3.3. Take the limit $\beta\nu \gg 1$, in which case one can set all occupation numbers involving b's to zero. Take the classical limit, where the occupation numbers for the a's are $n_k = e^{-\beta\epsilon_k}$. In this limit you can do the momentum integrals (including the momentum dependence in the integral).

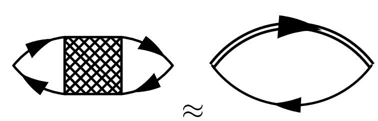
In particular, plot the absorption spectrum $\Gamma(k = \omega/c, \omega)$ as a function of ω. What sets the line width?

We can add the natural linewidth from spontaneous emission by using the following propegator for the b atoms (when the imaginary part of $\omega > 0$):

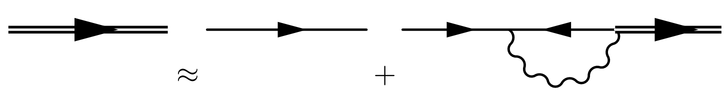
$$G_b(k,\omega) = \frac{1}{\omega - \epsilon_k - \nu - i\gamma}. \tag{15}$$

3.4. Neglecting the Doppler effect, but including spontaneous emission, calculate the absorption spectrum.

As a final aside, the decay of the b atoms comes from the coupling to the electromagnetic fiel. Thus we can write this approximation to the polarization bubble is



where the propegator for the b atoms is



You summed this series in the last homework set.

Problem 4. Electromagnetically induced transparency We now consider a gas of three-level atoms.

$$\bar{H}_0 = \sum_k (\epsilon_k + \nu_b) b_k^\dagger b_k + (\epsilon_k + \nu_a) a_k^\dagger a_k + (\epsilon_k + \nu_c) \bar{c}_k^\dagger \bar{c}_k, \tag{16}$$

where $\nu_a \approx \nu_c$, but ν_b is much larger. If I send in light which couples to the $a - b$ transition, then we just recover the results of the previous problem. I want to see how this $a - b$ polarization bubble is changed when I add a laser which resonantly couples b and c. In other words, lets add to this a term

$$\bar{H}' = \sum_k \Omega_c \left[e^{i(\nu_c - \nu_b)t} b_{k+q}^\dagger \bar{c}_k + e^{-i(\nu_c - \nu_b)t} \bar{c}_k^\dagger b_{k+q} \right]. \tag{17}$$

We will neglect recoil, taking $q = 0$.

The standard approach to dealing with such time dependent coupling terms is to do a canonical transformation

$$\bar{c}_k(t) = e^{-i(\nu_c - \nu_b)t} c_k(t). \tag{18}$$

The Hamiltonian in terms of the c operators is

$$H = \bar{H} - \sum_k (\nu_c - \nu_b) c_k^\dagger c_k. \tag{19}$$

4.1. Verify that this transformation is Cannonical. IE. Show that the following two sets of equations of motion are identical:

$$i\partial_t c_k = [c_k, H] \tag{20}$$

$$i\partial_t \bar{c}_k = [\bar{c}_k, \bar{H}]. \tag{21}$$

4.2. Find the propegator $G_{bb}(k, \omega)$. [Use whatever method you like – one approach is to sum a series in Ω_c.] Where does this have poles?

4.3. What is the selfenergy $\Sigma_{bb}(k, \omega)$?

4.4. We can add spontaneous emission by taking

$$\Sigma \to \Sigma + i\gamma/2.$$

What is $B_{bb}(k, \omega)$ in that case?

4.5. Calculate the imaginary part of the polarization bubble $\Gamma(k, \omega) = 2\mathrm{Im}\Pi(k, \omega)$ where,

$$\Pi(r, t) = \frac{1}{i} \langle T b^\dagger(r, t) a(r, t) a^\dagger(0, 0) b(0, 0) \rangle. \tag{22}$$

Assume that the temperature is small compared to $\nu_b - \nu_a$, so you can neglect the occupation numbers for the b states. Furthermore, neglect Doppler broadenning.

4.6. Sketch the absorption spectrum $I(\omega) = \Gamma(k = \omega/c, \omega)$ as a function of ω. Hypothesize why this is known as "electromagnetically induced transparency".

Aside: While I will not have you do it, the real value of this approach is that you can readily add in Doppler broadenning.