P7654 HW2

Due [Questions 1-4: Wednesday Jan 30, 2013, Question 5: Friday Feb 1

Depending on your quantum mechanics background (in particular, how well you know scattering theory), this problem set could be easy or "hard". By hard, I mean there could be a lot of concepts coming at you. I will assume that most of the students are in this latter category. I would like you to complete problems 1-4 by Wednesday, and problem 5 by Friday. I will have office hours on both Tuesday and Thursday at 2:30 – meet at 514A.

Problem 1. Equation of State within the Hartree-Fock approximation In class we derived a perturbation expansion for the free energy of a weakly interacting gas, described by Hamiltonian

$$\begin{aligned} H &= \sum_k \epsilon_k a_k^\dagger a_k + \frac{1}{2\Omega}\sum_{kpq} V_q a_k^\dagger a_p^\dagger a_{p+q} a_{k-q}, \quad (1) \\ &= H_0 + H_1 \quad (2) \end{aligned}$$

it is easy enough to add spin, but for simplicity we will leave it out. In free space the dispersion is

$$\epsilon_k = \frac{k^2}{2m} - \mu,$$

but we can just as well take ϵ_k to correspond to some sort of lattice dispersion. The interactions are parameterized by V_q which is the Fourier transform of the real space potential, and Ω is the volume of space, which could be removed or changed to another power depending on your Fourier conventions. I like these Fourier conventions because they make the thermodynamic limit straightforward

$$\frac{1}{\Omega}\sum_k \to \int \frac{dk}{(2\pi)^d}.$$

The free energy is

$$e^{-\beta F} = \mathrm{Tr} e^{-\beta \mathrm{H}}.$$

Here we will calculate the lowest order correction to the equation of state. On general grounds, if you are only going to sum a finite number of terms in the perturbation expansion, you might as well stop at this first correction. If the interactions are very weak, then all further contributions are negligable. If the interactions are strong, then you need all of the terms.

Note that we do need to use "diagrams" or "field theory" to do this calculation – it is just first order perturbation theory – but nonetheless we will frame it in terms of diagrams. If you take a solid state physics class you will probably see it done using a variational method. It therefore is not really that good of an example of the utility of many-body field theory – rather it is a warmup. Question 5 is a better example.

1.1. Expand the free energy to first order in H_1. Write the resulting expression in terms of an integral of the expectation value of H_1. (We did this in class)

1.2. Use Wick's theorem to write this first order correction to the Free Energy as the sum of two terms, corresponding to the two contractions of this term.

1.3. These expressions are easy to evaluate, as all of the field operators are at the same time. Write the resulting integrals in terms of the equilibrium occupations $n_k = \langle a_k^\dagger a_k \rangle$.

1.4. Specialize to the case of a point interaction $V_k = \text{const} = V_0$. Calculate the first order interaction shift to the Free energy. You should find that for fermions it vanishes – this is because two identical fermions can never be in the same place, so they never see a point interaction. For bosons, however, you will get a finite result. Express the shift in terms of the density $n = (1/\Omega)\sum_k n_k$. Verify that the shift is extensive.

Problem 2. Scattering Phase Shifts This should be a reminder of things from your quantum class. If you already know this, then feel free to skip this question. I am just assigning this because if you don't know this stuff, you will have some conceptual issues with the next questions. Our eventual goal (in question 5) is to derive a non-trivial result, namely the superfluid transition temperature of a Fermi gas with strong local interactions. It is hopeless to do that until we know how to describe scattering off of a local potential. We also need to do this question so you can understand which of the phenomena in question 5 are due to many-body physics, and which are just two-body physics coming from a 3D point scatterer.

Consider a spherically symmetric potential in three dimensions, $V(r)$, with $V(r) = 0$ for $r > r_0$. We are going to solve the single particle Schrodinger equation

$$\left[\frac{-\nabla^2}{2m} + V(r)\right]\psi(r) = E\psi(r).$$

2.1. Assuming that the wavefunction $\psi(\mathbf{r})$ is spherically symmetric, $\psi(\mathbf{r}) = u(r)/r$. Write down the time independent Schrodinger equation for $u(r)$.

Since $V(r) = 0$ for $r > r_0$, this reduces to a free Schrodinger equation for $r > r_0$. All of the details of the potential can be replaced by a boundary condition at $r = r_0$, namely

$$\left.\frac{u'}{u}\right|_{r=r_0} = f(r_0, E), \tag{3}$$

where $f(r_0, E)$ is a well behaved function of E as $E \to 0$. [I.E. it should have a nice Taylor expansion which has a finite radius of convergence.] We will be interested in low energy scattering, so we will replace $f(r_0, E)$ with $f(r_0, 0)$.

2.2. Show that if one is only interested in the wavefunction for $r > r_0$, that the solutions to the free Schrodinger equation with boundary condition (3) can be reproduced by using a free Schrodinger equation with boundary conditions at $r = r_0$. This boundary condition has the form

$$\left.\frac{u'}{u}\right|_{r=0} = -\frac{1}{a}, \tag{4}$$

where a, the s-wave scattering length, has dimensions of length. Write a in terms of $f(r_0, 0)$.

[Hint – since we are looking at the limit of low energy scattering you can linearize the wavefunction about r_0.]

2.3. Find a for a hard sphere potential of radius r_0.

2.4. Show that the free Schrodinger equation, with boundary condition (4), has a bound state if and only if $a > 0$. (Yes, this sign is right, the result is indeed counter-intuitive.) What is the energy of the bound state?

2.5. For $r > r_0$ we can write $u(r) = \sin(kr + \delta_k)$. How is the phase shift δ_k related to $f(0, E)$? Linearize δ_k about $k = 0$, and relate it to a.

Lets now calculate the scattering length for a delta-function potential

$$V(r) = V_0\delta(r) = \frac{V_0}{\Omega}\sum_k e^{ik\cdot r}$$

It will be convenient to solve the Schrodinger equation in momentum space.

2.6. Define $\psi(r) = \frac{1}{\Omega}\sum_k e^{ik\cdot r}\psi_k$. Fourier transform the time independent single particle Schrodinger equation to write an equation for ψ_k

With the delta-function interaction, the interaction term $V(r)\psi(r) = \delta(r)V_0\psi(0) = \frac{V_0\psi(0)}{\Omega}\sum_k e^{ik\cdot r}$, is a constant in momentum space. Lets call $\Delta = V_0\psi(0)$.

2.7. Find ψ_k in terms of Δ and E. [Assume $E = -E_b < 0$ – we will look for bound states. That way you do not have to worry about dividing by zero.]

2.8. We can also write

$$\Delta = \frac{V_0}{\Omega}\sum_k \psi_k.$$

Insert your expression for ψ_k into this formula. Δ should appear on both sides of the equation and can be cancelled. Write the resulting equation, which involves E, $\epsilon_k = k^2/2m$, V_0 and Ω. Solve for $1/V_0$ in terms of the other quantities.

In three dimensions, the right hand side of this equation is formally $-\infty$, telling you that there are no bound states for delta-function potential in 3D unless $V_0 = 0^-$. This is weird but true. If this was a quantum mechanics class I would have us explore this a bit more. Here we will accept this, and define

$$\frac{1}{V} = \frac{1}{V_0} + \frac{1}{\Omega}\sum_k \frac{1}{\epsilon_k}.$$

The quantity V will parameterize the deviation of V_0 from zero.

2.9. Using your equation for $1/V_0$, find an equation for $1/V$. The sum should now be convergent. Convert your sum into an integral, and find a relationship between V and $E_b = -E$.

2.10. Using your knowledge of how E_b is related to a, give an expression for V in terms of the scattering length.

Problem 3. Lipman-Schwinger approach to scattering Here we are going to rederive some of the key results of question 2 in a diagrammatic framework. In order to mirror the language we will use in problem 5, we will couch it as a two-body problem instead of a one-body problem – question 2 was simply in the center of mass frame (so in comparing results with those in this section you should replace m with the reduced mass $m/2$). We will consider a Hamiltonian for two-component fermions with point interactions

$$H = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma} + \frac{V_0}{\Omega} \sum_{kpq} a^\dagger_{k\uparrow} a^\dagger_{p\downarrow} a_{p-q\downarrow} a_{k+q\uparrow}$$

where $\sigma = \uparrow / \downarrow$ is the spin index, and $\epsilon_k = k^2/2m$. Since this is a point interaction, I have only included interactions between non-alike fermions. Our one "field theory" trick is to look at is the resolvent:

$$R(k,\omega) = \sum_q \langle a_{k/2+q\uparrow} a_{k/2-q\downarrow} \frac{1}{\omega - H} a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} \rangle, \tag{5}$$

where the expectation value is taken in the vacuum state containing no particles. This resolvent is nothing but a trace of a zero temperature two-particle greens function in frequency and momentum space. It is useful because it is related to the two-particle density of state $\rho_2(\omega, k)$ via

$$\rho_2(\omega, k) = \frac{1}{\pi} \mathrm{Im} R(k, \omega). \tag{6}$$

This density of states tells us how many two-particle states have energy E and momentum k.

3.1. Prove Eq. (6). Hint: Use $1/(\omega - \epsilon) = P/(\omega - \epsilon) \pm i\pi\delta(\omega - \epsilon)$.

Following a similar approach to what we did in class for the free energy, we will calculate a perturbation expansion for R in powers of V_0. We will sum this expansion to all orders. For simplicity we will specialize to the case $k = 0$, but the $k \neq 0$ case is not much harder. The calculation is fairly straightforward. One tricky bit is that if we want to get a finite scattering length, then V_0 is formally 0^-. This is just an quirk of treating the potential as zero range, and the same problem came up in the last question. Another tricky bit is that the real part of R is formally infinite. This is not a big deal, as this real part has no physical significance.

3.2. Write a sum for $R_0(\omega) = R_0(0, \omega)$ in the case where $V_0 = 0$. [The subscript 0 on R_0 represents the fact that there are no interactions.] This sum should be familiar. Convert the sum into an integral. Argue that the real part is infinite.

3.3. Define $\Pi_0 = \sum_k \frac{-1}{2\epsilon_k}$, and $\Pi(\omega) = R_0(\omega) - \Pi_0(\omega)$. Clearly Π_0 is infinite, on the other hand $\Pi(\omega)$ is a nice analytic function in the complex ω plane, except on the positive real axis, where it has a branch cut.

Take ω to be real and negative. Perform the integral to calculate $\Pi(\omega)$.

3.4. Calculate $\rho_2(\omega, 0)$ for the non-interacting case. Does this match what you know from elementary arguments? [Remember in the center of mass frame this is a 1-body problem, so (up to a factor of $\sqrt{2}$) this should agree with the single particle density of states from elementary statistical mechanics.

3.5. We can expand the resolvant in a power series in

$$H_1 = \frac{V_0}{\Omega} \sum_{kpq} a^\dagger_{k\uparrow} a^\dagger_{p\downarrow} a_{p-q\downarrow} a_{k+q\uparrow},$$

formally writing

$$\frac{1}{\omega - H} = \frac{1}{\omega - H_0} + \frac{1}{\omega - H_0} H_1 \frac{1}{\omega - H_0} + \frac{1}{\omega - H_0} H_1 \frac{1}{\omega - H_0} H_1 \frac{1}{\omega - H_0} + \cdots \tag{7}$$

By construction H_1 is always acting on a two-particle state, with total momentum 0. Thus the only terms in H_1 which matter are those with $p = -k$. Moreover, since we are acting on two-particle states, we can insert a vacuum state in the middle, and use

$$H_1 \to \frac{V_0}{\Omega} \left(\sum_k a^\dagger_{k\uparrow} a^\dagger_{-k\downarrow} |\mathrm{vac}\rangle \right) \left(\sum_q \langle \mathrm{vac}| a_{k\downarrow} a_{-k\uparrow} \right)$$

The state $|\mathrm{vac}\rangle$ is the vacuum containing no particles. A potential that can be decomposed into a creation term times an annihilation term is known as "separable."

Substitute this form for H_1 into Eq. (7), and write the resolvent $R(\omega)$ as a power series in $R_0(\omega)$.

3.6. Sum the power series.

3.7. Since $\Pi(\omega)$ is real for negative ω, $R(\omega)$ will also be real. Since the imaginary part of R corresponds to the density of states, this would imply that there are no bound states. The loophole is that $R(\omega)$ could have a pole. From the condition $R(\omega)^{-1} = 0$ find the energy of the bound state. Write the result in terms of V_0 and Π_0.

3.8. Using the result from problem 2,

$$E_b = \frac{\hbar^2}{2 m_r a_s^2},$$

where $m_r = m/2$, relate V_0 to the scattering length and the infinite constant R_0. [You know which branch of the square root to use, since a positive scattering length corresponds having a bound state.]

Note 1: In your quantum mechanics class you may have done essentially this same calculation in terms of solving the Lipmann-Schwinger equation. This treatment in terms of the resolvent is equivalent. You might want to look at your notes and compare.

Problem 4. Diagramatic derivation of Lipman-Schwinger equation

Here we just connect the calculation in the previous section to the Feynman Rules introduced in lecture. We first note that our Resolvent is nothing but a frequency space Greens function. We first use the fact that we are taking a vacuum expectation value to write:

$$\begin{aligned} R(\omega) &= \sum_q \langle a_{k/2+q\uparrow} a_{k/2-q\downarrow} \frac{1}{\omega - H} a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} \rangle \\ &= \sum_q \langle a_{k/2+q\uparrow} a_{k/2-q\downarrow} \frac{1}{\omega - H} a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} - a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} \frac{1}{\omega - H} a_{k/2+q\uparrow} a_{k/2-q\downarrow} \rangle. \end{aligned}$$

The second term we added is just a fancy name for zero. For the first term we take the branch where

$$\frac{1}{\omega - H} = \frac{1}{i}\int dt e^{i(\omega - H)t}\theta(t).$$

For the second term we instead use

$$\frac{1}{\omega - H} = -\frac{1}{i}\int dt e^{i(\omega - H)t}\theta(-t)$$

to yield $R(\omega) = \int dt e^{i\omega t} R(t)$ with

$$\begin{aligned} R(t>0) &= \sum_q \langle a_{k/2+q\uparrow} a_{k/2-q\downarrow} e^{-iHt} a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow}\rangle \\ R(t<0) &= \sum_q \langle a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} e^{iHt} a_{k/2+q\uparrow} a_{k/2-q\downarrow}\rangle \end{aligned}$$

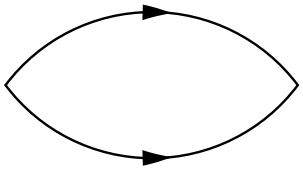
But since the vacuum state has zero energy we can just as well write this as

$$\begin{aligned} R(t>0) &= \sum_q \langle e^{iHt} a_{k/2+q\uparrow} a_{k/2-q\downarrow} e^{-iHt} a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow}\rangle \\ R(t<0) &= \sum_q \langle a^\dagger_{k/2-q\downarrow} a^\dagger_{k/2+q\uparrow} e^{iHt} a_{k/2+q\uparrow} a_{k/2-q\downarrow} e^{-iHt}\rangle, \end{aligned}$$

which we can recognize as a time ordered response function. We can either work with these, or if we really want to shoe-horn it into our old approach, we can write the vacuum expectation value as a finite temperature expectation value for $\mu < 0$ and $T \to 0$. We would probably also take $t \to i\tau$. We will see this in the next section though.

Expanding the H's in powers of H_1 and using Wick's theorem gives us a diagrammatic expansion similar to the one we had for the Free energy. The same Feynman Rules apply here as applied there. As before, we will focus on the case $k = 0$, the more general case is a straightforward extension.

The zeroth order term has only one contraction:

$R_0 =$ 

This is nothing but the statement that

$$\begin{aligned} R_0(t>0) &= \sum_q \langle e^{iH_0 t} a_{k/2+q\uparrow} e^{-iH_0 t} a^\dagger_{k/2+q\uparrow}\rangle \langle e^{iH_0 t} a_{k/2-q\downarrow} e^{-iH_0 t} a^\dagger_{k/2-q\downarrow}\rangle \\ R_0(t<0) &= \sum_q \langle a^\dagger_{k/2+q\uparrow} e^{iH_0 t} a_{k/2+q\uparrow} e^{-iH_0 t}\rangle \langle a^\dagger_{k/2-q\downarrow} e^{iH_0 t} a_{k/2-q\downarrow} e^{-iH_0 t}\rangle \end{aligned} \tag{8}$$

4.1. Write $R_0(t)$ in terms of $G_{0\sigma}(k,t) = -i\langle T a_{k\sigma}(t) a^\dagger_{k\sigma}(0)\rangle = -i\theta(t)\langle a_{k\sigma}(t) a^\dagger_{k\sigma}(0)\rangle + i\theta(-t)\langle a^\dagger_{k\sigma}(0) a_{k\sigma}(t)\rangle$.

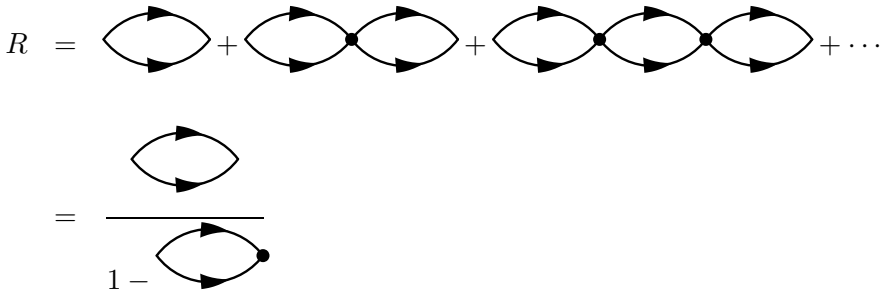
4.2. Use this result to write $R_0(\omega)$ as a convolution of $G_0(\omega)$. Someone experienced with diagrams would immediately write down this expression without going through the exercise of going to the time domain. [Of course in this particular case, the calculation is easier in the time domain.]

4.3. Substitute $G_0(\omega) = 1/(\omega - \epsilon_k)$. Use a partial fraction expansion, and the residue theorem to recover the result in question 3.2,

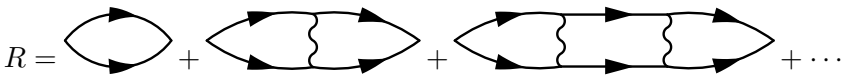
$$R_0(\omega) = \sum_k \frac{1}{\omega - 2\epsilon_k} \tag{9}$$

4.4. As an alternative way to calculate $R_0(\omega)$, write the explicit time domain expression for $R_0(t)$. Fourier transform this result. Often I find that this approach is quicker than doing a contour integral.

Note: I won't ask you to do it (since it is just the calculation you already did in the last problem) but now the full calculation of R can be done diagrammatically, you just sum the series



where the dot represents the point interaction, and in the diagrammatic language the dot is assigned the value V_0/Ω. Sometimes this series is instead drawn



In the next question we will see how to do this calculation in the "Matsubara" formalism at finite temperature and density.

Problem 5. Superfluid transition temperature within the Nozieres-Schmidt Rink approximation

You are aware that a Fermi gas with attractive interactions is unstable towards superfluidity at low temperatures. Here we will calculate the equation of state in the normal state, and find the phase transition as a thermodynamic singularity. You may find it useful to look at the original paper where this result was first given: P. Nozières and S. Schmitt-Rink, J. Low Temp. Phys. 59, 195 (1985). `http://link.springer.com/article/10.1007/BF00683774` I caution you however that there is a lot in that paper, and we are just doing a subset of it here. That paper has a great description of the "physics." Here we will predominantly worry about the mathematics.

We are going to start by thinking of the "BEC" limit. By this, we will think about what happens when the interactions become so strong that there is a two-body bound state. Physically, when the temperature is of

order the binding energy, pair form. At a lower temperature they Bose condense. The Bose condensation condition is the superfluid transition, and will show up as a singularity in the Free energy.

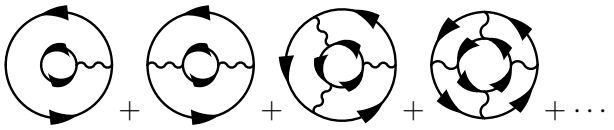
How do we capture the physics of these pairs? Well, we know how to get bound states in the 2-body problems. The idea of Nozieres and Schmidt-Rink is to include the same set of diagrams in the many-body problem. As before we get a series we can sum.

As I have emphasized before, the field theory is a language, and it doesn't tell you what approximation to use. The NSR technique captures the fact that two-body bound states can form. It misses out on other physics (such as charge density waves). As with many calculations in many-body physics, it is an uncontrolled approximation. Its principle value is in the qualitative insight it gives.

We again consider a Hamiltonian of the form

$$H = \sum_{k\sigma} \epsilon_k a^\dagger_{k\sigma} a_{k\sigma} + \frac{V_0}{\Omega} \sum_{kpq} a^\dagger_{k\uparrow} a^\dagger_{p\downarrow} a_{p-q\downarrow} a_{k+q\uparrow}$$

denoting these two terms as H_0 and H_1. We are going to take terms in the perturbation expansion for the free energy which involve all "multiple scatterings of two particles". These "pair ring diagrams" look like:

$\delta F =$  $+ \cdots$

We will implicitly assume that the spins on the inner right are "down" and on the outer ring "up. These are exactly the ladder graphs we used in the 2-body problem, just "wrapped up." By construction we will be able to exactly reproduce the physics of two-particles in free space. This approximation is the simplest many-body extension of those results. Note, if we stopped at the first term we would have "Hartree-Fock." There is no "Fock" term because the spins on the two propagators are distinct.

We now have 5 tasks:

1. Express each ring diagram in terms of the interaction V_0 and the elementary particle-particle bubble (which looks like R_0)
2. Calculate the multiplicity of the ring diagrams
3. Sum the series
4. Calculate R_0
5. Find the thermodynamic instability

To start with, lets consider a typical ring diagram appearing in the perturbative expansion of the partition

function,

$R_4 =$ 

For now lets not worry about the multiplicity (which we discuss shortly). This diagram has 4 powers of V_0, so its expression as an intergral must begin with

$$\left(\frac{-\beta V_0}{\Omega}\right)^4 \int_0^\beta d\tau_1 d\tau_2 d\tau_3 d\tau_4$$

Next we have a sum over a bunch of k's. We could write down the most general form, then start taking contractions. A little thought however makes it easier. At each vertex, the total momentum going in equals the total momentum going out. Thus the center of mass momentum of the "two particles" going around the ring is fixed. We need to sum over this global momentum, K, as well as the four relative momenta $k_1, \ldots k_4$. With the point interaction all of these are weighted equally, so we get

$$\begin{aligned}
\tfrac{1}{\Omega}\textstyle\sum_K \tfrac{1}{\Omega^4}\sum_{k_1\cdots k_4} \quad & \langle T\psi_{K/2+k_1}(\tau_1)\psi^\dagger_{K/2+k_1}(\tau_2)\rangle\langle T\psi_{K/2-k_1}(\tau_1)\psi^\dagger_{K/2-k_1}(\tau_2)\rangle \\
\times \quad & \langle T\psi_{K/2+k_2}(\tau_2)\psi^\dagger_{K/2+k_2}(\tau_3)\rangle\langle T\psi_{K/2-k_2}(\tau_2)\psi^\dagger_{K/2-k_2}(\tau_3)\rangle \\
\times \quad & \langle T\psi_{K/2+k_2}(\tau_3)\psi^\dagger_{K/2+k_2}(\tau_4)\rangle\langle T\psi_{K/2-k_2}(\tau_3)\psi^\dagger_{K/2-k_2}(\tau_4)\rangle \\
\times \quad & \langle T\psi_{K/2+k_2}(\tau_4)\psi^\dagger_{K/2+k_2}(\tau_1)\rangle\langle T\psi_{K/2-k_2}(\tau_4)\psi^\dagger_{K/2-k_2}(\tau_1)\rangle
\end{aligned}$$

5.1. Express R_4 in terms of $R_0(\tau)$, which is defined by Eq. 8, with $t \to -i\tau$.

This expression is simpler in frequency space, writing

$$R_0(\tau) = \frac{1}{\beta}\sum_{\omega_n} e^{i\omega_n\tau} R_0(i\omega_n) \tag{10}$$

where $\omega_n = 2\pi n/\beta$.

5.2. By using your knowledge of the antiperiodicity of the Fermion Greens functions, explain why $\omega_n = 2\pi n/\beta$.

5.3. What is R_4 in terms of $R_0(i\omega_n)$?

5.4. What is the generic m'th order ring diagram R_m in terms of $R_0(i\omega_n)$? Having worked out R_4, you should be able to instantly write it down.

Next we have to calculate the multiplicity. First of all, the m'th order diagram appears with a factor of $(1/m!)$. Next, we note that there are vertices at m times. We arbitrarily take one to be t_1. There are then $(m-1)!$ ways to order the other ones so you form a ring. Thus R_m appears with a factor of $1/m$ in front of it, leading to the expression:

$$\beta\delta F = \sum_{\omega_n}\sum_K\sum_m \frac{1}{m}\left(\frac{V_0 R_0(i\omega_n)}{\Omega}\right)^m .$$

5.5. Sum the series to simplify the expression for $\beta\delta F$.

[In comparing with Nozieres and Schmidt-Rink, they call $\chi = V_0 R_0/\Omega$.] Now we just have to calculate $R_0(i\omega_n)$. Our result should reduce to the expression in Eq. 9 in the limit of zero density (ie. the two particle limit). Just as in the zero temperature case there are two approaches.

5.6. Write the explicit expression for $R_0(\tau)$ in the time domain (don't do the sum over q). The only τ dependence comes from θ-functions and exponentials. You can then readily do the Fourier transform.

Here is the harder way to do the calculation. Strangely, this is the way that appears in all the textbooks. Given that $R_0(K,\tau) = \frac{1}{\Omega}\sum_q G_0(K/2+q,\tau)G_0(K/2-q,\tau)$, the Fourier transform is a convolution:

$$R_0(K, i\omega_n) = \frac{1}{\beta}\sum_\nu \frac{1}{\Omega}\sum_q \frac{1}{i\omega_n - i\nu - \epsilon_{K/2+q}}\frac{1}{i\nu - \epsilon_{K/2-q}}. \tag{11}$$

5.7. What are the values that ν are summed over?

The function $f(z) = 1/(e^{\beta z}+1)$ has poles with residue $1/\beta$ when $z = (2m+1)\pi/\beta$. Thus by the residue theorem, we can rewrite the sum in Eq. (11) as an integral in the complex plane:

$$R_0(K, i\omega_n) = \frac{\beta}{2\pi i}\oint dz f(z)\frac{1}{\Omega}\sum_q \frac{1}{i\omega_n - z - \epsilon_{K/2+q}}\frac{1}{z - \epsilon_{K/2-q}},$$

where the contour is circumnavigated in the counterclockwise direction, and includes all of the poles of f.

5.8. Calculate this contour integral by closing the contour at infinity.

At this point we are almost done. We have an explicit expression for R_0 in terms of a sum over momenta q. We have a further expression for the free energy in terms of the sum over k and ω of a function of R_0. These sums are readily done numerically. Even without doing them, however, we can see some interesting physics. In particular, the free energy has a singularity when $1 - \frac{\beta V_0 R_0(K,i\omega_n)}{\Omega} = 0$ for some term in the sum. A singularity in the free energy corresponds to a phase transition. It turns out that this is the pairing phase transition, and the term which first diverges is the one with $K = 0$ and $i\omega_n = 0$. This gives the "Thouless" criterion for superconductivity:

$$\frac{1}{V_0} = \frac{1}{\Omega}R_0(0,0).$$

5.9. Rewrite this expression in terms of the scattering length.

Unlike the two particle problem, we actually find that this equation can be solved for both positive and negative scattering length. For positive scattering length, this just corresponds to BEC of pairs. For negative scattering length this instability is a non-trivial many-body effect. The remarkable result of Nozieres and Schmidt-Rink is that these two forms of condensation are continuously connected. In a future homework we might explore this further: taking the derivative of the free energy with respect to μ in order to get an expression for the density. Regardless, at this point you probably have enough knowledge to make a stab at reading the original paper.