# Basic Training – Geometry HW1

**Due Wednesday April 16, 2013**

**Problem 1. Symmetries and Equivalence Classes** [This is not supposed to be challenging – it is just a little thinking to reinforce what was said in lecture.]

Suppose we have a spin-1/2 particle which sits on a ring. We want to classify the possible Hamiltonians by thinking about which magnetic field configurations can be mapped into one-another, with the requirement that the magnetic field never goes to zero on the ring. The point is that adding constraints (symmetries) to our Hamiltonians can change the topology of the space of Hamiltonians.

**1.1.** If the magnetic field is restricted to point along the $\hat{z}$ directions, how many different equivalence classes of Hamiltonians are there? How are they characterized?

**1.2.** If the magnetic field is restricted to point in the $\hat{x} - \hat{y}$ plane, how many different equivalence classes of Hamiltonians are there? How are they characterized?

**1.3.** If the magnetic field is free to point in any direction, how many different equivalence classes of Hamiltonians are there? How are they characterized?

**Problem 2. Parallel Transport of a vector on a sphere** [This one is straightforward – well worth doing.]

Consider the tangent vector to the sphere at the north pole $\mathbf{v} = (\hat{\mathbf{x}} + \hat{\mathbf{y}})/\sqrt{2}$. I parallel transport it, dragging it along the great circle where $y = 0$ until I get to the equator. What direction is the vector now pointing in? [Hint: What rotation matrix takes us to the equator. Apply that to $\vec{\mathbf{v}}$.]

**Problem 3. Euler Characteristic for a Sphere** [This one is a little more tedious – but instructive.]

Consider the parameterization of the sphere $\mathbf{r} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$ with $0 < \theta < \pi$ and $0 < \phi < 2\pi$. We define a curvature "two-form" as

$$d\omega = \mathbf{n} \cdot \frac{\partial \mathbf{n}}{d\theta} \times \frac{\partial \mathbf{n}}{d\phi} \, d\theta d\phi \qquad (1)$$

$$= \Omega d\theta d\phi, \qquad (2)$$

where $\mathbf{n} = \mathbf{r}$ is the normal vector. [A two-form is just an object meant to be integrated over two dimensions. Similarly $dx$ is a "one-form".] Explicitly calculate $\Omega$ and perform the integral

$$\Phi = \int_0^{\pi} d\theta \int_0^{2\pi} d\phi \, \Omega \qquad (3)$$

**Problem 4. Euler Characteristic for a Torus** [This one is more of a pain. You may have diminishing returns here.]

Consider the parameterization of a torus $\mathbf{r} = (A\sin(\theta) + B\cos(\phi), B\sin(\phi), A\cos(\theta))$.

**4.1.** Find the normal vector $\hat{n}$ as a function of $\theta$ and $\phi$. I usually do this by noting that $\hat{n}$ is aligned with $\partial_\theta \mathbf{r} \times \partial_\phi \mathbf{r}$.

**4.2.** Find the curvature

$$\Omega = \mathbf{n} \cdot \frac{\partial \mathbf{n}}{d\theta} \times \frac{\partial \mathbf{n}}{d\phi}. \tag{4}$$

**4.3.** Use symmetry to perform the integral

$$\Phi = \int_0^\pi d\theta \int_0^{2\pi} d\phi \, \Omega \tag{5}$$

**Problem 5. Bonus: Derivation of parallel transport for small displacements** [I am probably doing this the hard way, but thats OK. Plowing through a hard calculation is better training than seeing a cute trick that only works for some special cases. On the other hand, I am not sure what you actually learn by plowing through this particular calculation. I mainly put it here since I spent two full days deriving it, and it seems a shame to waste all that work. Only do this if you really want to.]

Let $\hat{n}$ and $\hat{n}'$ be two neighboring normal vectors. Let $\hat{e}_1, \hat{e}_2$ and $\hat{e}_1', \hat{e}_2'$ be the basis vectors for the tangent space at these two locations. We want to know how a vector $\mathbf{v} = v_1\hat{e}_1 + v_2\hat{e}_2$ transforms when we move from the first to second point. In particular, we define a rotation $R$ such that $R\hat{n} = \hat{n}'$. This does not uniquely determine $R$, but we will take "parallel transport" to mean that we rotate by the smallest angle possible: That is we take $R$ to be a rotation about $\vec{\theta} = \hat{n} \times \hat{n}'$. Thus we write $\mathbf{v}' = Rv = v_1'\hat{e}_1' + v_2'\hat{e}_2'$. If $\hat{n}$ and $\hat{n}'$ are close together, we can write

$$\hat{n}' = R\hat{n} = (1 - \alpha^2/2 - \beta^2/2)\hat{n} + \alpha\hat{e}_1 + \beta\hat{e}_2, \tag{6}$$

plus terms which are cubic or higher order in the small parameters $\alpha, \beta$.

**5.1.** Fine the vector $\vec{\theta}$ in the bases $\hat{e}_1, \hat{e}_2$. Note, as an orthonormal basis $\hat{e}_1 \times hate_2 = \hat{n}, \hat{e}_2 \times \hat{n} = \hat{e}_1, \hat{n} \times \hat{e}1 = \hat{e}_2$.

**5.2.** Generically

$$rot2R\hat{e}_1 = (1 - \delta^2/2 - \gamma^2/2)\hat{e}_1 + \gamma e_2 + \delta\hat{n} \tag{8}$$

$$R\hat{e}_2 = (1 - \epsilon^2/2 - \tau^2/2)\hat{e}_2 + \epsilon e_1 + \tau\hat{n} \tag{9}$$

plus higher order terms. Given that these are orthogonal $R\hat{e}_1 \cdot R\hat{e}_2 = 0$. Neglecting cubic and higher terms, what relationship does this give between $\gamma, \epsilon, \tau$ and $\delta$? You should find that $\epsilon$ and $\gamma$ are higher order than $\tau$ and $\delta$. It is tempting to neglect $\epsilon$, and $\gamma$, but it turns out we need them. We can however neglect $\epsilon^2$ and $\gamma^2$.

**5.3.** Using the simplified expressions

$$\begin{aligned} R\hat{e}_1 &= (1-\delta^2/2)\hat{e}_1 + \gamma e_2 + \delta\hat{n} && (11) \\ R\hat{e}_2 &= (1-\tau^2/2)\hat{e}_2 + \epsilon e_1 + \tau\hat{n} && (12) \end{aligned}$$

and the fact that $R\hat{n} \cdot R\hat{e}_1 = R\hat{n} \cdot R\hat{e}_2 = 0$, relate $\delta$ to $\alpha$ and $\tau$ to $\beta$.

Since $\gamma$ and $\epsilon$ are small, you can neglect terms like $\gamma\beta$ or $\alpha\delta$. Also neglect terms which are cubic or higher.

**5.4.** At this point you should have deduced

$$\begin{aligned} R\hat{n} &= (1-\alpha^2/2-\beta^2/2)\hat{n} + \alpha\hat{e}_1 + \beta\hat{e}_2, && (15) \\ R\hat{e}_1 &= (1-\alpha^2/2)\hat{e}_1 - \alpha\hat{n} + \gamma e_2 && (16) \\ R\hat{e}_2 &= (1-\beta^2/2)\hat{e}_1 - \beta\hat{n} + \epsilon e_1 && (17) \\ & && (18) \end{aligned}$$

By looking at the cubic terms when you set $R\hat{n} \cdot R\hat{e}_1 = R\hat{n} \cdot R\hat{e}_2 = 0$, show that $\epsilon = \gamma = -\alpha\beta/2$. (There is probably a more elegant way to do this, but this is how I did it.)

**5.5.** Show that

$$Re_1 = e_2 \times n' - \frac{1}{2}(e_2 \cdot n')(n \times n'). \qquad (19)$$

Hint: First write out $e_2 \times n'$. Note that this is $Re_1$ plus an extra term. This extra term can be related to $n \times n'$. Finally use $\beta = e_2 \cdot n'$.

**5.6.** Find the analogous expression for $Re_2$

**5.7.** We now want to find the coefficients

$$\begin{aligned} R\hat{e}_1 &= ue_1' + ve_2' && (20) \\ R\hat{e}_2 &= -ve_1' + ue_2', && (21) \end{aligned}$$

where

$$\begin{aligned} u &= e_1' \cdot R\hat{e}_1 = e_2' \cdot R\hat{e}_2 && (22) \\ v &= e_2' \cdot R\hat{e}_1 = -e_1' \cdot R\hat{e}_2 && (23) \end{aligned}$$

which then gives

$$\begin{pmatrix} v_1' \\ v_2' \end{pmatrix} = \begin{pmatrix} u & v \\ -v & u \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \qquad (24)$$

By using the cyclic invariance of the triple product, we know:

$$\begin{aligned} e_1' \cdot (e_2 \times n') &= e_2 \cdot (n' \times e_1') && (25) \\ &= e_2 \cdot e_2'. && (26) \end{aligned}$$

Using tricks like this, calculate $u$ and $v$.