Basic Training – Geometry HW2

Due Friday April 18, 2013

Problem 1. Landau Quantization in Free Space[This is a standard problem from grad quantum mechanics – so if you have done it before, don't feel obliged to waste your time. Though it can't hurt to review this for the next questions.]

We want to solve the single particle quantum mechanics problem for a charged particle in a magnetic field. Using the Landau gauge $\mathbf{A} = Bx\hat{\mathbf{y}}$ and converting to units where the length are measured in terms of the "magnetic length" and energies in terms of the cyclotron frequency, the Schrodinger equation reads

$$E\psi = -\frac{\partial_x^2 + (\partial_y - ix)^2}{2}\psi. \tag{1}$$

1.1. Making the ansatz $\psi(x,y) = e^{iky}\phi(x)$, write a differential equation for $\phi(x)$.

1.2. Recognizing the resulting equation as corresponding to a 1D Schrodinger equation for a simple harmonic oscillator, write a complete set of solutions and their energies.

An important feature here is that the spectrum is highly degenerate. Classically this corresponds to the fact that the frequency of cyclotron motion is independent of the center of the circle. Since any linear combination of degenerate states are also an eigenstate, there is more than one way to write the eigenstates.

Problem 2. Landau Quantization on a "torus" We want to solve

$$E\psi = -\frac{\partial_x^2 + (\partial_y - ix)^2}{2}\psi \tag{2}$$

with periodic boundary conditions. There is a little bit of a subtlety in that a naive introduction of "periodic boundary conditions" ends up inserting unwanted fluxes.

We can interpret Eq. (2) geometrically. It is just the regular Laplacian, but with a different connection. In particular, it says that if we have a complex number ψ at position (x,y), and parallel transport it a small distance to $(x+dx,y)$, it transforms as

$$\psi \to \psi \tag{3}$$

but if we transport it to $(x+dx,y)$

$$\psi \to e^{ixdy}\psi. \tag{4}$$

2.1. Consider the condition $\psi(x+L_x,y)=\psi(x,y)$, along with the picture that when one crosses the line $x=L_x$ one returns to the line $x=0$. We will explicitly show that this is an unnatural boundary condition.

Consider parallel transport in a clockwise direction starting at the point $x=L_x-\epsilon, y=y_0$, moving down to $x=L_x-\epsilon, y=y_0-L_y$, over to $x=L_x+\epsilon, y=y_0-L_y$, up to $x=L_x+\epsilon, y=y_0$, then back to $x=L_x-\epsilon, y=y_0$. If ϵ is small, we get no contributions from horizontal parts. According to our periodic boundary conditions, we can do the upward part of the journey at $x=0$.

What is the total phase Φ accumulated?

The solution to this problem is to take $\psi(x+L_x,y)=e^{iL_xy}\psi(x,y)$. By this I mean, we put a *connection* on the bonds between $x=L_x-\epsilon$ and $x=0$, and specify that if we parallel transport across we get a phase e^{iL_xy}. The signs on these things always confuse me. Regardless, when you do your loop, you now get contributions from the horizontal bonds which cancel the contributions from the vertical ones.

2.2. Suppose we require $\psi(x+L_x,y)=e^{iL_xy}\psi(x,y)$, and $\psi(x,y+L_y)=\psi(x,y)$. Barring singularities, then we must have that the total phase accumulated must be a multiple of 2π when going around a loop $(0,0)\to(L_x,0)\to(L_x,L_y)\to(0,L_y)$. This is because we could replace this path with an equivalent one with zero area. [This is nothing but our argument in class about the total curvature of a closed surface must be a multiple of 2π.] What constraint does this put on L_x and L_y?

2.3. Argue that the function

$$\psi_m(x,y)=\sum_n e^{2\pi i(m+nN)y/L_y-(x-2\pi m/L_y-nL_x)^2/2}, \tag{6}$$

with $L_xL_y=2\pi N$. This is clearly in the lowest Landau level, since it is the sum of the functions you found in your first problem. Show that

$$\psi(x+L_x,y) = e^{iL_xy}\psi(x,y) \tag{7}$$

$$\psi(x,y+L_y) = \psi(x,y). \tag{8}$$

These N functions span the lowest landau level on a torus.

2.4. Look up the definition of the "Jacobi Theta functions". Relate Eq. (6) to these functions.

Mathematica knows about these functions. Make a density plot of $|\psi|^2$ for the case $L_x=L_y=\sqrt{2\pi}$. Plot over several periods so you can see what is going on. Also make a plot of the phase of ψ.