Basic Training – Geometry HW4

Due Wednesday May 7, 2014

Both of these problems are a bit ugly. I literally spent days trying to come up with simplified versions, but ended up returning to these.

Problem 1. Artificial Magnetic Field

As discussed in lecture, there are cold atom experiments which engineer a Schrodinger Equation

$$i\partial_t \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix} = \begin{pmatrix} -\frac{\nabla^2}{2} - \lambda x & \Omega(x)e^{iky} \\ \Omega(x)e^{-iky} & -\frac{\nabla^2}{2} + \lambda x \end{pmatrix} \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}. \tag{1}$$

In the semiclassical approximation, the spin adiabatically follows the field as the particle moves about: $|\psi\rangle = \phi|\xi\rangle$, where $|\xi\rangle$ is a two-component unit spinor, and ϕ is a scalar wavefunction. This yields an effective equation of motion:

$$i\partial_t \phi = -\frac{1}{2}|\nabla - iA|^2\phi + V\phi. \tag{2}$$

Here

$$V^2 = (\lambda x)^2 + \Omega(x)^2 \tag{3}$$

$$A = \langle\xi|i\nabla|\xi\rangle. \tag{4}$$

The effective vector potential $\mathbf{A}$ depends on the choice of gauge, but $\mathbf{B} = \nabla \times \mathbf{A}$ is gauge invarient. Calculate $\mathbf{B}$.

Hint: This is a bit messy, and is inhomogeneous. The experiment is at: `http://www.nature.com/nature/journal/v462/n7273/abs/nature08609.html`

Problem 2. Tight-binding model with spin-orbit coupling Consider the tight binding Hamiltonian

$$\begin{aligned} H &= \sum_{i,\sigma,\tau} \left[a^\dagger_{i,\sigma} a_{i-\hat{x},\tau}(-t\delta_{\sigma\tau} + i\alpha(S_y)_{\sigma\tau}) + a^\dagger_{i,\sigma} a_{i-\hat{y},\tau}(-t\delta_{\sigma\tau} - i\alpha(S_x)_{\sigma\tau}) \right] + \mathrm{HC} \\ & +\epsilon \sum_i (a^\dagger_{i\uparrow} a_{i\uparrow} - a^\dagger_{i\downarrow} a_{i\downarrow}). \end{aligned} \tag{5}$$

Although it is not the most physical limit, we will work with the special case where $t = 0$. This is really just for simplicity. Write this Hamiltonian in momentum space via

$$a_{i,\sigma} = \sum_k e^{ik\cdot r} a_{k,\sigma}, \tag{6}$$

which yields

$$H = \sum_k \left(a^\dagger_{k\uparrow}, a^\dagger_{k\downarrow} \right) \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a_{k\uparrow} \\ a_{k\downarrow} \end{pmatrix}. \tag{7}$$

2.1. Setting $t = 0$, find the matrix

$$H_k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{8}$$

2.2. Make a 3D plot of the the bandstructure $E(k)$ for the case $\epsilon = 0, \alpha = 1$.

2.3. Make a 3D plot of the the bandstructure $E(k)$ for the case $\epsilon = 1, \alpha = 0.5$.

2.4. Write an expression for the curvature for the lowest band

$$\Omega(k_x, k_y) = i \left[(\partial_{k_x} \langle\psi|) \left(\partial_{k_y} |\psi\rangle \right) - \left(\partial_{k_y} \langle\psi| \right) (\partial_{k_x} |\psi\rangle) \right] \tag{9}$$

where $|\psi\rangle$ is the eigenvector of H_k.

[Note, this will be fairly ugly – don't waste time trying to make it look nice. Feel free to just use a computer algebra system to produce this.]

2.5. Plot $\Omega(kx, ky)$ for the case $\epsilon = 1, \alpha = 0.5$.