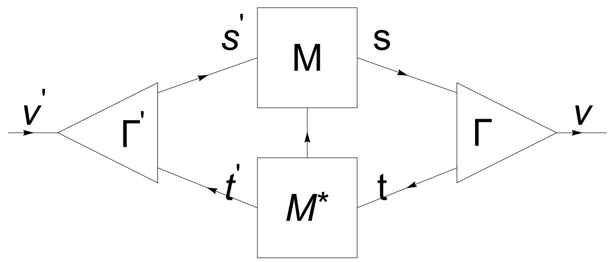
This is a rank 4 tensor, each index taking on one of two values. It is however, natural to group the left two indices and the right two indices to write this as a 4×4 matrix. That is, we create a matrix whose rows correspond to $st = 00, 01, 10, 11$. One usually just does this in one's head, but one can also formally add graphical "combiners"

$E_I =$ 

(1.23)

where the rank 3 tensors have non-zero elements:

$$\Gamma^{s_0 t_0}_{v_{00}} = \Gamma^{s_1 t_0}_{v_{10}} = \Gamma^{s_0 t_1}_{v_{01}} = \Gamma^{s_1 t_1}_{v_{11}} \tag{1.24}$$

It is just a trivial relabeling of two indices as one composite index. In fact, when we write computer programs, we will make functions which exactly do that.

The reason for calculating E_I is that if we use periodic boundary conditions the norm is just $\langle\psi|\psi\rangle = \mathrm{Tr}\,(E_I)^N$. That means if λ is the largest eigenvalue of E_I, the norm is $\langle\psi|\psi\rangle = \lambda^N$. The state can then be normalized by dividing each M by $1/\sqrt{\lambda}$.

I. HW 1 – Due : Jan 29

Problem 1. (For Credit) Consider the "spin singlet": $\uparrow\downarrow - \downarrow\uparrow$. Write this as a matrix product state. Hint the matrices are 1×2 and 2×1 (so I guess this could be called a "vector product state.")

Solution 1.1.

$$|\uparrow\downarrow - \downarrow\uparrow\rangle = \begin{pmatrix} |\uparrow\rangle & -|\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\downarrow\rangle \\ |\uparrow\rangle \end{pmatrix} \tag{1.25}$$

Problem 2. (For Credit) The "GHZ" or "cat" state of five spins is $|\uparrow\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$. Write this as a matrix product state.

Solution 2.1.

$$|\psi\rangle = \begin{pmatrix} |\uparrow\rangle & |\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle & \\ & |\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle & \\ & |\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle & \\ & |\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle \\ |\downarrow\rangle \end{pmatrix} \tag{1.26}$$

Problem 3. (For Credit) The "W"-state is $|\uparrow\downarrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\downarrow\uparrow\rangle$, Write this as a matrix product state. Hint: It is the same as what we used for the single-particle state.

Solution 3.1.

$$|\psi\rangle = \begin{pmatrix} |\downarrow\rangle_1 & |\uparrow\rangle_1 \end{pmatrix} \begin{pmatrix} |\downarrow\rangle_2 & |\uparrow\rangle_2 \\ 0 & |\downarrow\rangle_2 \end{pmatrix} \begin{pmatrix} |\downarrow\rangle_3 & |\uparrow\rangle_3 \\ 0 & |\downarrow\rangle_3 \end{pmatrix} \begin{pmatrix} |\downarrow\rangle_4 & |\uparrow\rangle_4 \\ 0 & |\downarrow\rangle_4 \end{pmatrix} \begin{pmatrix} |\uparrow\rangle_5 \\ |\downarrow\rangle_5 \end{pmatrix} \tag{1.27}$$

Problem 4. (For Credit) The ferromagnetic 1D transverse field Ising Model is a spin model, defined by a Hamiltonian

$$H = \sum_j \left[-J\sigma_z^j \sigma_z^{j+1} - h\sigma_x^j \right]. \tag{1.28}$$

Here σ_z and σ_x are the regular Pauli matrices. We can get a simple understanding of how this model works through a variational calculation. The simplest variational wavefunction we can use is a product: $|\psi\rangle = \otimes_j |\psi_j\rangle$ – where $|\psi_j\rangle$ is a two-component spinor. In the $\hat{z}$ basis we can parametrize $|\psi_j\rangle$ as

$$|\psi_j\rangle = \cos(\theta/2)|\uparrow\rangle + \sin(\theta/2)|\downarrow\rangle, \tag{1.29}$$

so that

$$\begin{aligned} \langle\psi_j|\sigma_z^j|\psi_j\rangle &= \cos(\theta) && (1.30) \\ \langle\psi_j|\sigma_x^j|\psi_j\rangle &= \sin(\theta). && (1.31) \end{aligned}$$

4.1. Show that up to boundary terms (which are irrelevant in the thermodynamic limit),

$$\langle\psi|H|\psi\rangle = -JN\cos^2\theta - hN\sin(\theta), \tag{1.32}$$

where N is the total number of sites.

Solution 4.1. This is trivial. Each spin is independent, and there are N terms.

Problem 4. cont...

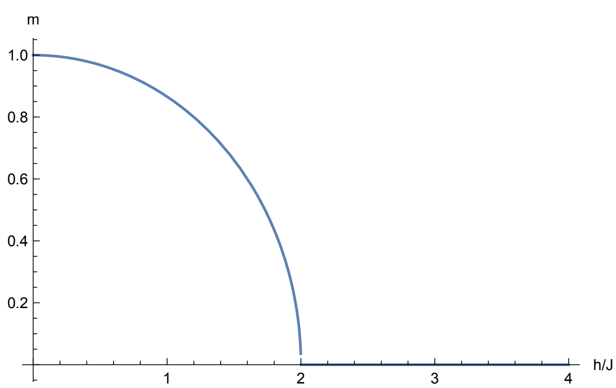
4.2. Write $x = \sin(\theta)$. Minimize $\langle\psi|H|\psi\rangle$ with respect to x (with the constraint that $-1 < x < 1$). Make a plot of the magnetization $m = \langle\sigma_z^j\rangle = \sqrt{1-x^2}$ as a function of the ratio h/J.
You should see two phases – a "ferromagnetic" phase where $m \neq 0$, and a "paramagnetic" one where $m = 0$. This is the simplest example of what is referred to as a "quantum phase transition." Note, the mean-field theory over-estimates the stability of the ordered phase, so you should not take the numbers too seriously.

Solution 4.2. The we scale the energy, and write

$$\bar{E} = \frac{E}{JN} = -(1-x^2) - \frac{h}{J}x. \tag{1.33}$$

The slope $d\bar{E}/dx = 2x - h/J$ vanishes at $x = h/(2J)$. If $h/2J < 1$ the energy is minimized at $x = h/2J$, otherwise it is minimized at $x = 1$. Thus the magnetization is

$$m = \begin{cases} \sqrt{1 - \frac{h^2}{4J^2}} & h < 2J \\ 0 & h > 2J \end{cases} \tag{1.34}$$



Problem 5. Challenge – not for Credit
If I have N hard core bosons on M sites, and use the algorithm in Sec. D for parameterizing the states, how big is the m'th matrix? As we will see later, this is related to the entanglement entropy of a generic state. Hint: the m'th matrix in this product has d_1 rows and d_2 columns. Express d_1 and d_2 as sums over binomial coefficients. Separately consider the cases $m < N$ and $m > N$.
For $m < N$ you should be able to do the sums. For $m > N$ there is no closed form.

Solution 5.1. The matrix has d_1 rows and d_2 columns.
Case 1: If $m < N$ then

$$d_1 = \sum_{n=0}^{N} \begin{pmatrix} m-1 \\ n \end{pmatrix} \tag{1.35}$$

and

$$d_2 = \sum_{n=0}^{N} \begin{pmatrix} m \\ n \end{pmatrix}. \tag{1.36}$$

The logic is that each row corresponds to a different configuration of the previous $m-1$ sites, while each column corresponds to a configuration of the first m sites. We therefore just count the number of ways of putting n particles on $m-1$ sites, then sum over n.
These sums are elementary, and $d_1 = 2^{m-1}$ and $d_2 = 2^m$. You can verify that this works for the case $N = 2$ that we explicitly did in class.
Case 2: If $m > N$, then we have to modify this slightly, since the configuration with N particles just gives a single row. Therefore

$$d_1 = 1 + \sum_{n=0}^{N-1} \begin{pmatrix} m-1 \\ n \end{pmatrix} \tag{1.37}$$

and

$$d_2 = 1 + \sum_{n=0}^{N-1} \begin{pmatrix} m \\ n \end{pmatrix}. \tag{1.38}$$

As stated in the question there is no closed form for these sums.