D. Homework 2 – Due Jan 31

Problem 6. Do the second notebook on using tensors in Mathematica. – no need to hand anything in.

Problem 7. For Credit: A gas of fermions hopping on a 1D lattice can be mapped onto a spin model – and then analyzed using the techniques in this module. The approach for doing this is the "Jordan-Wigner" transform. For simplicity we will work with spinless Fermions, but adding spin is straightforward. Note, it is a non-local transformation, so local order in the spin language may be non-local in the Fermi language, and vice-versa.

Consider a single spin 1/2, and define the down-spin state to be $|0\rangle$ – which will map onto the absence of a Fermion, while the up-spin state $|1\rangle$ will be the presence of a fermion. It is natural to define

$$f^\dagger = \sigma^+ \tag{2.32}$$

$$f = \sigma^- \tag{2.33}$$

where $\sigma^\pm$ are Pauli operators on the spin basis.

7.1. Express σ_z in terms of f and $f^\dagger$.

Solution 7.1.

$$\sigma_z = 2f^\dagger f - 1 \tag{2.34}$$

Problem 7. cont...

7.2. Show that f and $f^\dagger$ obey fermion anticommutation relations.

Solution 7.2. If we use the basis $(|1\rangle, |0\rangle)$,

$$\{f, f^\dagger\} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \left\{\begin{matrix} 0 & 0 \\ 1 & 0 \end{matrix}\right)\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{2.35}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.36}$$

Problem 7. cont...

7.3. If we have more sites, we run into the problem that the f's defined this way commute on different sites. To fix this, we define

$$a_j^\dagger = (-1)^{\sum_{k<j} f_k^\dagger f_k} f_j^\dagger \quad (2.37)$$
$$= (-1)^{N_{<j}} f_j^\dagger \quad (2.38)$$
$$a_j = (-1)^{\sum_{k<j} f_k^\dagger f_k} f_j \quad (2.39)$$
$$= (-1)^{N_{<j}} f_j, \quad (2.40)$$

where $N_{<j}$ is the total number of particles on sites to the left of j. Show that these a's obey fermionic anticommutation relations.

Solution 7.3. Suppose $i > j$, then

$$a_i a_j^\dagger = (-1)^N_{<i} f_i (-1)^N_{<j} f_j^\dagger \quad (2.41)$$
$$= (-1)^N_{[j,i)} f_i f_j^\dagger \quad (2.42)$$

where $N_{[j,i)}$ is the number of particles on sites j through $i-1$. Conversely

$$a_j^\dagger a_i = (-1)^N_{<j} f_j^\dagger (-1)^N_{<i} f_i \quad (2.43)$$
$$= (-1)^N_{[j,i)-1} f_i f_j^\dagger. \quad (2.44)$$

Adding these gives 0.
The same argument works when $j < i$. When $i = j$ the $(-1)^N$ terms cancel.

Problem 7. cont...

7.4. Write $a_i^\dagger a_{i+1}$ in terms of the f's.

Problem 7. cont...

7.5. Map the XXZ model onto a Fermi model,

$$H = \sum_j J_x(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + J_z(\sigma_j^z \sigma_{j+1}^z). \tag{2.45}$$

Solution 7.5. Recall: $\sigma^x = \sigma^+ + \sigma^- = f + f^\dagger$ and $\sigma^y = (\sigma^+ - \sigma^-)/i$, so

$$(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) = (f_j + f_j^\dagger)(f_{j+1} + f_{j+1}^\dagger) + \frac{f_j - f_j^\dagger}{i} \frac{f_{j+1} - f_{j+1}^\dagger}{i} \tag{2.46}$$

$$= 2 f_j f_{j+1}^\dagger + 2 f_j^\dagger f_{j+1} \tag{2.47}$$

$$= 2(f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}). \tag{2.48}$$

If we write this in terms of the a's

$$(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) = 2((-1)^{N_j} a_{j+1}^\dagger a_j + a_j^\dagger (-1)^{N_j} a_{j+1}) \tag{2.49}$$

In the first term, we can replace the N_j with zero, since it is to the left of a a_j operator. Similarly, in the second term we can replace the N_j with zero since it is to the right of a $a_j^\dagger$ operator. Consequently,

$$(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) = 2(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}). \tag{2.50}$$

The other term is easy, and we have

$$H = \sum_j 2J_x(a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}) + J_z(2a_j \dagger a_j - 1)(2a_{j+1}^\dagger a_{j+1} - 1), \tag{2.51}$$

which is a model of spinless fermions hopping on a 1D lattice, with a nearest-neighbor interaction and a chemical potential.

Problem 7. cont...

7.6. Map the transverse field Ising model onto a Fermi model, using the basis perpendicular to the one we used in the last homework

$$H = \sum_j -h\sigma_j^z - J\sigma_j^x\sigma_{j+1}^x. \tag{2.52}$$

The resulting Hamiltonian has the structure of a model of superconductivity. (It is a 1D p-wave superconductor, sometimes refered to as the "Kitaev chain" – Alexi Kitaev has a famous paper where he shows that it has "Majorana" edge modes.)

Solution 7.6. Following the arguments in the last problem,

$$\begin{aligned}
\sigma_j^x\sigma_{j+1}^x &= (f_j + f_j^\dagger)(f_{j+1} + f_{j+1}^\dagger) && (2.53)\\
&= (a_j + a_j^\dagger)(-1)^{N_j}(a_{j+1} + a_{j+1}^\dagger) && (2.54)\\
&= (a_j^\dagger - a_j)(a_{j+1}^\dagger + a_{j+1}) && (2.55)\\
&= a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1}a_j && (2.56)
\end{aligned}$$

Which gives

$$H = \sum_j -h(2a_j^\dagger a_j - 1) - J(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1}a_j). \tag{2.57}$$

This model can be solved exactly using a Bogoliubov transformation.

7.7. In the last homework we distinguished the phases of the transverse field Ising model by the order parameter $\langle\sigma^x\rangle$ (recall we have rotated by 90 degrees). Show that this is a non-local operator in the Fermi language. Hence the Fermi analog of the ordered phase is topological – in fact it is an example of a symmetry protected topological phase.

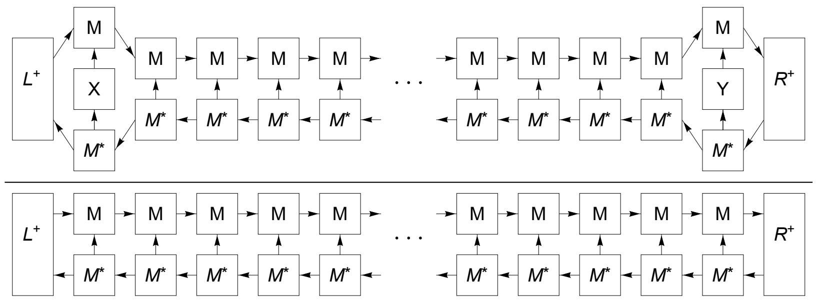
Solution 7.7. This is straightforward,

$$\langle\sigma_j^x\rangle = \langle(-1)^{\sum_{i<j} N_i}(a_j + a_j^\dagger)\rangle. \tag{2.58}$$

This is non-local because it depends on an infinite number of sties.

Problem 8. (For Credit) Derive Eq. (2.22).

Solution 8.1. Following the argument for the local operator,

$\langle X_i Y_j \rangle =$  (2.59)

In the denominator there are $j-i+1$ factors of E_I, so the denominator is $\lambda_+^{j-i+1}\langle L^+|R^+\rangle$. In the numerator there are $j-i-1$ factors of E_I. We use the decomposition in Eq. (2.29),

$$E_I = \sum_j \frac{|R_j\rangle \lambda_j \langle L_j|}{\langle L_j|R_j\rangle}. \tag{2.60}$$

Therefore,

$$E_I^n = \sum_j \frac{|R_j\rangle \lambda_j^n \langle L_j|}{\langle L_j|R_j\rangle}. \tag{2.61}$$

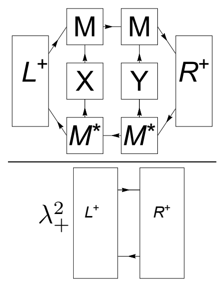
In the present case there are just two terms in the sum: either $+$ or $-$, so we have

$$\langle X_i Y_j \rangle = \frac{\langle L^+|E_X|R^+\rangle\langle L^+|E_Y|R^+\rangle\lambda_+^{j-i-1}/\langle L^+|R^+\rangle + \langle L^+|E_X|R^-\rangle\langle L^-|E_Y|R^+\rangle\lambda_-^{j-i-1}/\langle L^-|R^-\rangle}{\lambda_+^{j-i+1}\langle L^+|R^+\rangle} \tag{2.62}$$

Where E_X and E_Y are the tensors formed from $MXM^\dagger$ and $MYM^\dagger$. A little reordering gives the desired expression.

Problem 9. (For Credit) In the lecture notes we produced an expression for $\langle X_i Y_j \rangle$ when $|i-j|>1$. Find a similar result for the cases where $j = i+1$ and $j = i$.

Solution 9.1. When $j = i+1$,

$\langle X_i Y_{i+1} \rangle =$ 

(2.63)

When $j = i$ this is just a special case of the on-site expectation value, and we can use that expression.

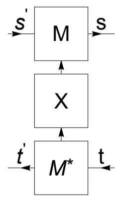
Problem 10. (For Credit) Here you will complete the calculation of the properties of the matrix product state described by Eq. (2.1). We will take ϵ to be real throughout. I recommend using a computer algebra system.

10.1. Show that the right eigenvectors of the E_I in Eq. (2.18), corresponding to $\lambda_- = 1-\epsilon^2$ and $\lambda_+ = 1+\epsilon^2$ are $(1-\epsilon^2, 0, 0, \epsilon^2-1), (1+\epsilon^2, 2\epsilon, 2\epsilon, 1+\epsilon^2)$. Similarly, show that the left eigenvectors are $(-1,0,0,1), (1,0,0,1)$.

Solution 10.1. This is just matrix multiplication.

Problem 10. cont...

10.2. Calculate

$E_X =$ 

(2.64)

as a 4×4 matrix, where $X = \sigma_x$ is the Pauli matrix. (Same indices as we used for E_I).

Solution 10.2. The non-zero elements of E_X are

$$E^{s_0 s_0'}_{t_1 t_1'} = M^{\uparrow}_{s_0 s_0'} (M^{\downarrow}_{t_1 t_1'})^* = 1 \tag{2.65}$$

$$E^{s_0 s_1'}_{t_1 t_1'} = M^{\uparrow}_{s_0 s_1'} (M^{\downarrow}_{t_1 t_1'})^* = \epsilon \tag{2.66}$$

$$E^{s_0 s_0'}_{t_1 t_0'} = M^{\uparrow}_{s_0 s_0'} (M^{\downarrow}_{t_1 t_0'})^* = \epsilon \tag{2.67}$$

$$E^{s_0 s_1'}_{t_1 t_0'} = M^{\uparrow}_{s_1 s_1'} (M^{\downarrow}_{t_1 t_0'})^* = \epsilon^2 \tag{2.68}$$

$$E^{s_1 s_1'}_{t_0 t_0'} = M^{\downarrow}_{s_1 s_1'} (M^{\uparrow}_{t_0 t_0'})^* = 1 \tag{2.69}$$

$$E^{s_1 s_0'}_{t_0 t_0'} = M^{\downarrow}_{s_1 s_0'} (M^{\uparrow}_{t_0 t_0'})^* = \epsilon \tag{2.70}$$

$$E^{s_1 s_1'}_{t_0 t_1'} = M^{\downarrow}_{s_1 s_1'} (M^{\downarrow}_{t_0 t_1'})^* = \epsilon \tag{2.71}$$

$$E^{s_1 s_0'}_{t_0 t_1'} = M^{\downarrow}_{s_1 s_0'} (M^{\downarrow}_{t_0 t_1'})^* = \epsilon^2. \tag{2.72}$$

Which then can be reshaped into

$$\begin{pmatrix} E^{s_0 s_0'}_{t_0 t_0'} & E^{s_0 s_0'}_{t_1 t_0'} & E^{s_1 s_0'}_{t_0 t_0'} & E^{s_1 s_0'}_{t_1 t_0'} \\ E^{s_0 s_0'}_{t_0 t_1'} & E^{s_0 s_0'}_{t_1 t_1'} & E^{s_1 s_0'}_{t_0 t_1'} & E^{s_1 s_0'}_{t_1 t_1'} \\ E^{s_0 s_1'}_{t_0 t_0'} & E^{s_0 s_1'}_{t_1 t_0'} & E^{s_1 s_1'}_{t_0 t_0'} & E^{s_1 s_1'}_{t_1 t_0'} \\ E^{s_0 s_1'}_{t_0 t_1'} & E^{s_0 s_1'}_{t_1 t_1'} & E^{s_1 s_1'}_{t_0 t_1'} & E^{s_1 s_1'}_{t_1 t_1'} \end{pmatrix} = \begin{pmatrix} & \epsilon & \epsilon & \\ & 1 & \epsilon^2 & \\ & \epsilon^2 & 1 & \\ & \epsilon & \epsilon & \end{pmatrix}. \tag{2.73}$$

Problem 10. cont...

10.3. Show that

$$\langle L^+|E_X|R^+\rangle = 8\epsilon^2 \tag{2.74}$$

Solution 10.3.

$$\begin{aligned}\langle L^+|E_X|R^+\rangle &= \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \epsilon \\ 1 & \epsilon^2 \\ \epsilon^2 & 1 \\ \epsilon & \epsilon \end{pmatrix} \begin{pmatrix} 1+\epsilon^2 \\ 2\epsilon \\ 2\epsilon \\ 1+\epsilon^2 \end{pmatrix} && (2.75) \\ &= 8\epsilon^2. && (2.76)\end{aligned}$$

Problem 10. cont...

10.4. Show that $\langle L^+|R^+\rangle = 2(1+\epsilon^2)$.

Solution 10.4. This is just vector multiplication

Problem 10. cont...

10.5. Calculate $\langle \sigma_x \rangle$.

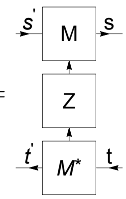
Solution 10.5.

$$\langle \sigma_x \rangle = \frac{1}{\lambda_+} \frac{\langle L^+|E_X|R^+\rangle}{\langle L^+|R^+\rangle} = \frac{4\epsilon^2}{(1+\epsilon^2)^2}. \tag{2.77}$$

This result makes sense: It is bounded below 0, which it achieves at $\epsilon = 0, \infty$. The former corresponds to the ferromagnetic state, the latter the antiferromagnet. It is bounded above by 1, which it achieves when $\epsilon = 1$.

Problem 10. cont...

10.6. Calculate

$E_Z =$ 

(2.78)

as a 4×4 matrix, where $Z = \sigma_Z$ is the Pauli matrix.

Solution 10.6.

$$E_Z = \begin{pmatrix} 1 & -\epsilon^2 \\ \epsilon & -\epsilon \\ \epsilon & -\epsilon \\ \epsilon^2 & -1 \end{pmatrix}, \tag{2.79}$$

Problem 10. cont...

10.7. Show that

$$\begin{aligned}
\langle L_+|E_Z|R_+\rangle &= 0 && (2.80)\\
\langle L_+|E_Z E_Z|R_+\rangle &= 2(1+\epsilon^2)^2(1-\epsilon^2) && (2.81)\\
\langle L_+|E_Z|R_-\rangle &= 2(1-\epsilon^4) && (2.82)\\
\langle L_-|E_Z|R_+\rangle &= -2(1-\epsilon^4) && (2.83)\\
& && (2.84)
\end{aligned}$$

Solution 10.7. This is just matrix multiplication

10.8. Calculate the correlation function $\langle\sigma_z^i\sigma_z^j\rangle$ in this state. Separately consider the case $|i-j|=1$ and $|i-j|>1$.

Solution 10.8. The nearest-neighbor correlation as

$$\begin{aligned}
\langle\sigma_z^i\sigma_z^{i+1}\rangle &= \frac{\langle L_+|E_Z E_Z|R_+\rangle}{\lambda_+^2\langle L_+|R_+\rangle} && (2.85)\\
&= \frac{1-\epsilon^2}{1+\epsilon^2}. && (2.86)
\end{aligned}$$

This makes sense. It is bounded above by 1 – which is reached when $\epsilon=0$ – which is the Ferromagnetic state. It is bounded below by -1 – which is reached when $\epsilon=\infty$. It vanishes at $\epsilon=1$.
For larger distances, we use

$$\begin{aligned}
\langle\sigma_z^i\sigma_z^{i+d}\rangle &= \left(\frac{\lambda_-}{\lambda_+}\right)^{d-1}\frac{\langle L_+|E_Z|R_-\rangle\langle L_-|E_Z|R_+\rangle}{\lambda_+^2\langle L_+|R_+\rangle\langle L_-|R_-\rangle} && (2.87)\\
&= \left(\frac{1-\epsilon^2}{1+\epsilon^2}\right)^{d+1} && (2.88)
\end{aligned}$$

Problem 10. cont...

10.9. Calculate the expectation value $\langle H \rangle$ in this state, where H is given by Eq. (1.28). Minimize with respect to ϵ to optimize the wavefunction. Plot the resulting $E/(NJ)$ as a function of h/J. Compare it with the mean field prediction. Note this ansatz has the opposite problem of the product state we previously used – it overestimates the stability of paramagnetic state – so you should find the phase transition at smaller h/J.
Hint: Define $y = \epsilon^2/(1+\epsilon^2)$. The minimization is easier in terms of y. Note, $0 < y < 1$.

Solution 10.9. Plugging in our previous results

$$\bar{E} = \frac{E}{NJ} = -\frac{1-\epsilon^2}{1+\epsilon^2} - \frac{h}{J}\frac{4\epsilon^2}{(1+\epsilon^2)^2} \tag{2.89}$$

$$= 2y - 1 - 4\frac{h}{J}y(1-y). \tag{2.90}$$

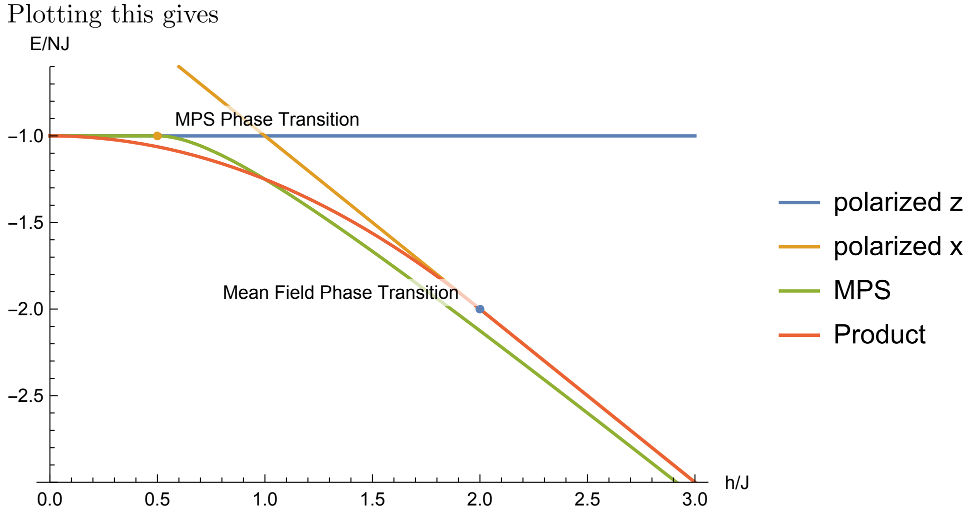
The slope is $\partial\bar{E}/\partial y = 2 - 4(h/J) + 8(h/J)y$, which vanishes at $y = 1/2(1 - J/2h)$. When $J/h > 2$, the minimum is at $y = 0$, and this ansatz gives the phase transition at $h/J = 1/2$. Substituting that back in gives

$$\bar{E}_{MPS} = \begin{cases} -1 & h < J/2 \\ -\frac{h}{J} - \frac{J}{4h} & h > J/2 \end{cases} \tag{2.91}$$

For comparison, the mean field result is

$$\bar{E}_{MF} = \begin{cases} -1 - \frac{1}{4}\frac{h}{J} & h < 2J \\ -\frac{h}{J} & h > 2J \end{cases} \tag{2.92}$$

Plotting this gives



Problem 11. (Challenge – not for Credit) We can describe both the ordered and disordered phases with the two-parameter matrix product state:

$$|\psi\rangle = \cdots \begin{pmatrix} |\uparrow\rangle & \epsilon\eta|\downarrow\rangle \\ \epsilon|\uparrow\rangle & \eta|\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle & \epsilon\eta|\downarrow\rangle \\ \epsilon|\uparrow\rangle & \eta|\downarrow\rangle \end{pmatrix} \begin{pmatrix} |\uparrow\rangle & \epsilon\eta|\downarrow\rangle \\ \epsilon|\uparrow\rangle & \eta|\downarrow\rangle \end{pmatrix} \cdots \tag{2.93}$$

Where ϵ and η are real and positive.
It would be nice to repeat the previous problem with this state. Unfortunately the algebra gets too messy for paper and pencil – so this will be a partially numerical problem. I used Mathematica for it – which was pretty efficient. If you don't already have good familiarity with Mathematica, it is probably more trouble than it is worth. I also recommend naming your variables x and y instead of ϵ and η. Will make entering easier in the computer.

11.1. Generate the transfer matrix E_I, using the same basis as the lecture.

Solution 11.1. Following the same logic from lecture:

$$E_I = \begin{pmatrix} 1 & & & \epsilon^2\eta^2 \\ \epsilon & & & \epsilon\eta^2 \\ \epsilon & & & \epsilon\eta^2 \\ \epsilon^2 & & & \eta^2 \end{pmatrix} \tag{2.94}$$

Problem 11. cont...

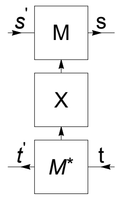
11.2. Write six different functions in some computer language that take ϵ and η produce each of: $\lambda_+(\epsilon,\eta), \lambda_-(\epsilon,\eta), R_+(\epsilon,\eta), R_-(\epsilon,\eta), L_+(\epsilon,\eta), L_-(\epsilon,\eta)$. The latter 4 are length 4 vectors. You can debug this by noting that when $\eta = 1$ you should get the same results as for the last question (with the caution that the eigenvectors are only defined up to a multiplicative constant).

Solution 11.2. I used Mathematica:

```
EI = {{1, 0, 0, x^2 y^2}, {x, 0, 0, x y^2}, {x, 0, 0, x y^2}, {x^2, 0, 0, y^2}}
res = Eigensystem[EI]
les = Eigensystem[Transpose[EI]]
lambdap = res[[1, 4]]
lambdam = res[[1, 3]]
rp = res[[2, 4]]
lp = les[[2, 4]]
rp = res[[2, 3]]
lp = les[[2, 3]]
```

Problem 11. cont...

11.3. Calculate

$E_X =$ 

(2.95)

as a 4×4 matrix, where $X = \sigma_x$ is the Pauli matrix.

Solution 11.3. The non-zero elements of E_X are

$$EX^{s_0 s_0'}_{t_1 t_1'} = M^{\uparrow}_{s_0 s_0'} (M^{\downarrow}_{t_1 t_1'})^* = \eta \tag{2.96}$$
$$EX^{s_0 s_1'}_{t_1 t_1'} = M^{\uparrow}_{s_0 s_1'} (M^{\downarrow}_{t_1 t_1'})^* = \epsilon\eta \tag{2.97}$$
$$EX^{s_0 s_0'}_{t_1 t_0'} = M^{\uparrow}_{s_0 s_0'} (M^{\downarrow}_{t_1 t_0'})^* = \epsilon\eta \tag{2.98}$$
$$EX^{s_0 s_1'}_{t_1 t_0'} = M^{\uparrow}_{s_0 s_1'} (M^{\downarrow}_{t_1 t_0'})^* = \epsilon^2\eta \tag{2.99}$$
$$EX^{s_1 s_1'}_{t_0 t_0'} = M^{\downarrow}_{s_1 s_1'} (M^{\uparrow}_{t_0 t_0'})^* = \eta \tag{2.100}$$
$$EX^{s_1 s_0'}_{t_0 t_0'} = M^{\downarrow}_{s_1 s_0'} (M^{\uparrow}_{t_0 t_0'})^* = \epsilon\eta \tag{2.101}$$
$$EX^{s_1 s_1'}_{t_0 t_1'} = M^{\downarrow}_{s_1 s_1'} (M^{\uparrow}_{t_0 t_1'})^* = \epsilon\eta \tag{2.102}$$
$$EX^{s_1 s_0'}_{t_0 t_1'} = M^{\downarrow}_{s_1 s_0'} (M^{\uparrow}_{t_0 t_1'})^* = \epsilon^2\eta. \tag{2.103}$$

Which then can be reshaped into

$$\begin{pmatrix} EX^{s_0 s_0'}_{t_0 t_0'} & EX^{s_0 s_0'}_{t_1 t_0'} & EX^{s_1 s_0'}_{t_0 t_0'} & EX^{s_1 s_0'}_{t_1 t_0'} \\ EX^{s_0 s_0'}_{t_0 t_1'} & EX^{s_0 s_0'}_{t_1 t_1'} & EX^{s_1 s_0'}_{t_0 t_1'} & EX^{s_1 s_0'}_{t_1 t_1'} \\ EX^{s_0 s_1'}_{t_0 t_0'} & EX^{s_0 s_1'}_{t_1 t_0'} & EX^{s_1 s_1'}_{t_0 t_0'} & EX^{s_1 s_1'}_{t_1 t_0'} \\ EX^{s_0 s_1'}_{t_0 t_1'} & EX^{s_0 s_1'}_{t_1 t_1'} & EX^{s_1 s_1'}_{t_0 t_1'} & EX^{s_1 s_1'}_{t_1 t_1'} \end{pmatrix} = \begin{pmatrix} & \epsilon\eta & \epsilon\eta & \\ & \eta & \epsilon^2\eta & \\ & \epsilon^2\eta & \eta & \\ & \epsilon\eta & \epsilon\eta & \end{pmatrix}. \tag{2.104}$$

Problem 11. cont...

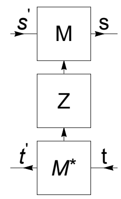
11.4. Write a computer program that will calculate Calculate $\langle\sigma_x\rangle$ as a function ϵ and η.

Solution 11.4. My code:

```
EX = {{0, x y, x y, 0}, {0, y, x^2 y, 0}, {0, x^2 y, y, 0}, {0, x y, x y, 0}}
expsigx = Expand[lp.EX.rp]/Expand[( lambdap lp.rp)]  // Simplify
```

Problem 11. cont...

11.5. Calculate

$E_Z =$ 

(2.105)

as a 4×4 matrix, where $Z = \sigma_Z$ is the Pauli matrix.

Solution 11.5.

$$E_Z = \begin{pmatrix} 1 & -\epsilon^2\eta^2 \\ \epsilon & -\epsilon\eta^2 \\ \epsilon & -\epsilon\eta^2 \\ \epsilon^2 & -\eta^2 \end{pmatrix} \tag{2.106}$$

Problem 11. cont...

11.6. Write functions that will calculate $\langle \sigma_z^i \rangle$ and $\langle \sigma_z^i \sigma_z^{i+1} \rangle$ as a function of ϵ and η.

Solution 11.6. My code:

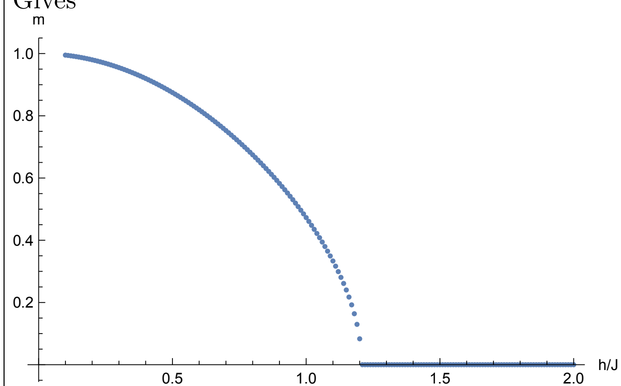
```
EZ = {{1, 0, 0, -x^2 y^2}, {x, 0, 0, -x y^2}, {x, 0, 0, -x y^2}, {x^2, 0, 0, -y^2}}
expsigz = Expand[lp.EZ.rp]/Expand[( lambdap lp.rp)]  // Simplify
expsigzsigz = Expand[lp.EZ.EZ.rp]/Expand[( lambdap^2 lp.rp)]  // Simplify
```

11.7. Combine your results to make a function that will give the energy $\bar{E} = \langle H \rangle/(JN)$ from Eq. (1.28, when given ϵ, η, and h/J. Feed this into a minimization routine, so that you optimize the parameters. Plot the resulting $\langle \sigma_z \rangle$ as a function of h/J.

Solution 11.7. My code:

```
en[x_, y_, h_] = -expsigzsigz - h expsigx;
ic[h_] = If[h > 1, {{y, 0.99}, {x, Sqrt[(2 h - 1)/(2 h + 1)]}}, {{x, 0.99}, {y, (2 - Sqrt[4 - h^2])/h}
opt[h_?NumericQ] := FindMinimum[en[x, y, h], Evaluate@ic[h]]
splot = ListPlot[Table[{h, ez /. opt[h][[2]]}, {h, 0.1, 2, 0.01}], AxesLabel -> {"h/J", "m"}]
```

Gives



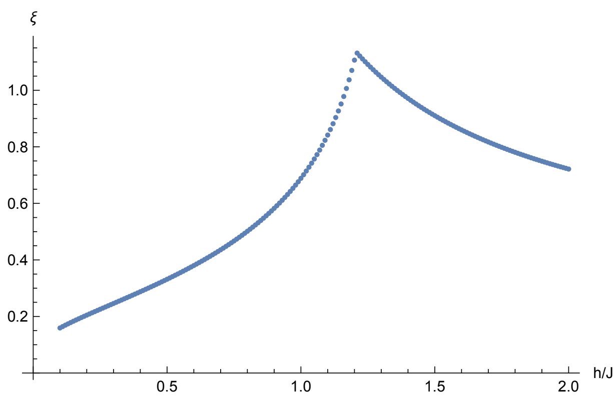
The exact result (which you can get by the techniques in question 7.6) has the phase transition at $h/J = 1$, so this is closer than the previous two theories, but is far from exact.

Problem 11. cont...

11.8. Plot the correlation length as a function of h/J.

Solution 11.8. I generated the plot with

```
xiplot = ListPlot[Table[{h, 1/Log[lambdap/lambdam] /. opt[h][[2]]},
                                    {h, 0.1, 2, 0.01}], AxesLabel -> {"h/J", "\[Xi]"}]
```



In the exact theory the correlation length will diverge at the phase transition.