G. Homework 3 – Due Feb 5

Problem 12. For Credit: Take any image. Use the SVD on it. Make a plot of the singular values λ_j as a function of j. Truncate to a small number of singular values. How many do you need to keep for the image to look good? By how much did you compress the image. [Note that you won't do as well as *.jpg*, unless you do an image which has a lot of horizontal and vertical lines – like a flag or a Mondrian painting.]

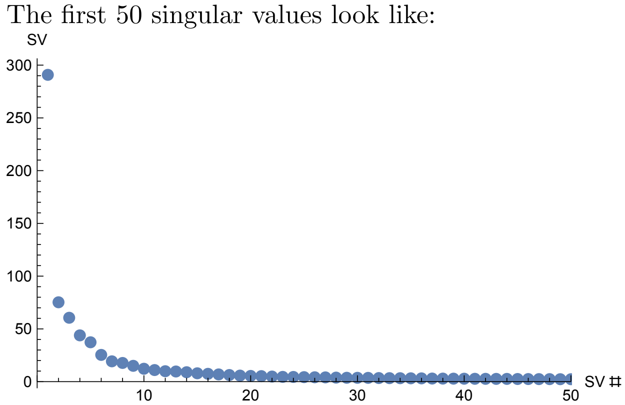
Solution 12.1. I used Mathematica, with the following code on a 400×600 image of a cat:

```
cat = Import["cat.jpg"]
cd = ImageData[ColorConvert[cat, "GrayScale"]]
catsvd = SingularValueDecomposition[cd, 400];
```

To graph the singular values, I did

```
catsvdplot =
 ListPlot[Tr[catsvd[[2]], List], PlotRange -> {{0, 50}, All},
  AxesLabel -> {"SV #", "SV"}, PlotStyle -> PointSize[Large]]
```

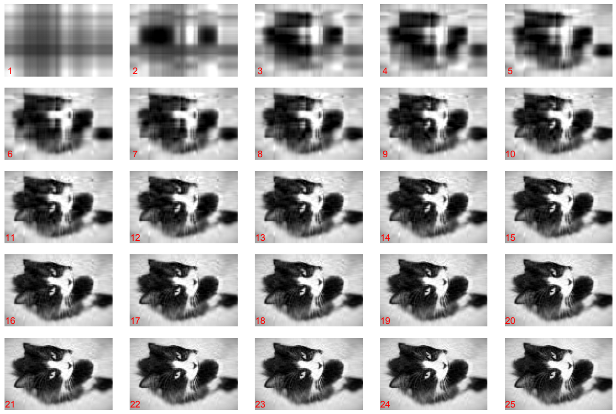
The first 50 singular values look like:



As you can see, for this particular image the singular values fall off pretty quickly. To make a panel showing how the cat looks as a function of the number of singular values

```
anel = GraphicsGrid[
  ArrayReshape[
   Table[
     Graphics[{
       Raster[
         catsvd[[1, All, ;; trunc]].catsvd[[2, ;; trunc, ;; trunc]].
           Transpose[catsvd[[3, All, ;; trunc]]]],
       Red, Text[trunc, {30, 30}]}], {trunc, 1, 25}],
     {5, 5}]]
```

which gives



Problem 13. For Credit A 'cat' state is $|\psi\rangle = |\uparrow\uparrow\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\downarrow\downarrow\rangle$. Find its entanglement entropy if it is split in two

Solution 13.1. There are two approaches, either writing down the Schmidt decomposition, or calculating the reduced density matrix. The density matrix approach is more familliar, so lets do that. Tracing over the right half of the system, the (non-normalized) density matrix is

$$\rho_L = |\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|. \tag{3.29}$$

Clearly this has two equal eigenvalues, so $S = \log(2)$.
The alternative approach is to simply write down the Schmidt decomposition

$$|\psi\rangle = |\uparrow\uparrow\rangle|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle|\downarrow\downarrow\downarrow\rangle \tag{3.30}$$

The Schmidt values are 1 and 1. After normalizing they are $1/\sqrt{2}$ and $1/\sqrt{2}$, which gives the same result.

Problem 14. For Credit The "W"-state is $|\uparrow\downarrow\downarrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\downarrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\downarrow\rangle + |\downarrow\downarrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\downarrow\downarrow\uparrow\rangle$. Find its entanglement entropy if it is split between the second and third state.

Solution 14.1. Again, we have two approaches, tracing over the last 3 sites gives

$$\rho_L = (|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)(\langle\uparrow\downarrow| + \langle\downarrow\uparrow|) + 3|\downarrow\downarrow\rangle\langle\downarrow\downarrow|. \tag{3.31}$$

This has two eigenvalues 2 and 3. Normalized this is 2/5 and 3/5, so

$$S = -\frac{2}{5}\log\frac{2}{5} - \frac{3}{5}\log\frac{3}{5} \tag{3.32}$$
$$= 0.67. \tag{3.33}$$

Probably the notable thing is that it is smaller than $\sqrt{2}$.
The alternative approach is to write

$$|\psi\rangle = (|\uparrow\downarrow + \downarrow\uparrow)(|\downarrow\downarrow\downarrow\rangle) + (|\downarrow\downarrow\rangle)(|\uparrow\downarrow\downarrow\rangle + |\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\rangle) \tag{3.34}$$

which after properly normalizing gives the same result.

Problem 15. Bonus

15.1. Write a program which will take in the matrix describing a uniform infinite matrix product state, and give you the transfer matrix, and its eigenvalues. Test it with with the wavefunctions from section A of this chapter. Verify the single-particle wavefunction yields a non-diagonalizable transfer matrix, and the chiral state yields one with complex eigenvalues.

Solution 15.1. After loading in all of our "NamedTensor" routines, I wrote the following

```
MakeTransfer[namedtensor_, leftindex_, rightindex_, physindex_] :=
 Module[{
   transfer = Contract[
       Combine[namedtensor, namedtensor],
       disambiguate[{physindex, physindex}]]},
  transfer = CombineIndices[transfer, disambiguate[{leftindex, leftindex}]];
  transfer = CombineIndices[transfer, disambiguate[{rightindex, rightindex}]]
  ]
```

[I am showing off there by using "disambiguate" – a less fancy version would just hard-code the names of the indices.] I generated the free particle matrix via

```
freeparticle = {"tensor" -> SparseArray[
                        {{1, 1, 1} -> 1, {1, 2, 2} -> x, {2, 2, 1} -> 1}, {2, 2, 2}],
                       "names" -> {"l", "r", "\[Sigma]"}}
```

and the transfer matrix via

```
freetransfer=MakeTransfer[freeparticle, "l", "r", "\[Sigma]"]
```

If you look at it with `MatrixForm` you will see that it is already in a Jordan form. Alternatively you can run `JordanDecomposition` on it.
For the chiral matrix I used

```
chiral = {"tensor" -> SparseArray[
                     {{1, 2, 3} -> 1, {2, 3, 1} -> 1, {3, 1, 2} -> 1}, {3,  3, 3}],
              "names" -> {"l", "r", "\[Sigma]"}}
charaltransfer=MakeTransfer[chiral, "l", "r", "\[Sigma]"]
Eigenvalues[chiraltransfer]
```

As stated in class, the three non-zero eigenvalues are the cube root of unity – two of which are complex.

15.2. What physical property does the single particle state have that makes the transfer matrix non-diagonalizable?

Solution 15.2. This is somewhat ambiguously worded, and there are likely many answers. I think the main feature is that we have an infinite sum of events, where each event only occurs a finite number of times. That finiteness is what gives the Jordan form.

15.3. What physical property does the chiral state have that gives the transfer matrix complex eigenvalues?

Solution 15.3. The reason for the complex eigenvalues is that there is something oscillatory going on: the state has periodicity 3, and hence the cube root of unity shows up.