Quantum Physics in One Dimension: Basic Training in Condensed Matter Physics

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Lecture Notes: Spring 2018

Preface

About Basic Training

Basic Training in Condensed Matter physics is a modular team taught course offered by the theorists in the Cornell Physics department. It is designed to expose our graduate students to a broad range of topics. Each module runs 2-4 weeks, and require a range of preparations. This module, "Greens Function Approach to Transport," is designed for students who have completed a standard one semester graduate solid state physics course. It assumes no prior exposure to many-body Greens functions, but requires a knowledge of second quantization.

Prior Topics

2006 Random Matrix Theory (Piet Brouwer)
Quantized Hall Effect (Chris Henley)
Disordered Systems, Computational Complexity, and Information Theory (James Sethna)
Asymptotic Methods (Veit Elser)

2007 Superfluidity in Bose and Fermi Systems (Erich Mueller)
Applications of Many-Body Theory (Tomas Arias)
Rigidity (James Sethna)
Asymptotic Analysis for Differential Equations (Veit Elser)

2008 Constrained Problems (Veit Elser)
Quantum Optics (Erich Mueller)
Quantum Antiferromagnets (Chris Henley)
Luttinger Liquids (Piet Brouwer)

2009 Continuum Theories of Crystal Defects (James Sethna)
Probes of Cold Atoms (Erich Mueller)
Competing Ferroic Orders: the Magnetoelectric Effect (Craig Fennie)
Quantum Criticality (Eun-Ah Kim)

2010 Equation of Motion Approach to Many-Body Physics (Erich Mueller)
Dynamics of Infectious Diseases (Chris Myers)
The Theory of Density Functional Theory: Electronic, Liquid, and Joint (Tomas Arias)
Nonlinear Fits to Data: Sloppiness, Differential Geometry and Algorithms (James Sethna)

2011 Practical Density Functional Theory (Tomas Arias)
Semiclassical Methods (Erich Mueller)
Ginzburg-Landau Theory of Superconductivity (Chris Henley)
Continuum Quantum Monte Carlo Methods in Chemistry and Physics (Cyrus Umrigar)

2012 Quantum 14-15 puzzle: mechanics of a single hole in a spin-1/2 lattice (Veit Elser)
The science of writing Science (Eun-Ah Kim)
Feynman diagrams and *ab-initio* computation of excited states in condensed matter (Tomas Arias)
Probes of Cuprate Superconductors (theorist's view) (Chris Henley)
Physics of Life (Jane Wang)

2013 Many Body Field Theory (Erich Mueller)
The Theory of Density Functional Theory: Electronic, Liquid, and Joint (Tomas Arias)
Conformal Symmetry (Andre Leclair)
Nonlinear Fits to Data: Sloppiness, Differential Geometry and Algorithms (James Sethna)

2014 Rigidity (James Sethna)
Practical Density Functional Theory (Tomas Arias)
Quantum Monte-Carlo (Cyrus Umrigar)
Geometry in Quantum Mechanics (Erich Mueller)

2018 Greens Function approach to Transport (Erich Mueller)
Dynamical Systems that solve hard problems (Viet Elser)
Variational and Projector Monte Carlo Methods for Physics and Chemistry (Cyrus Umrigar)
Sloppy Models (Jim Sethna)

Contents

Chapter 1

Introduction – Mar 13, 2019

A. Reading

To learn more about Fermi liquid theory, and how it breaks down in 1D, see Giamarchi, "Quantum Physics in One Dimension", chapter 1. To see some cartoon reasoning about "quasiparticles," see Mattuck,"A Guide to Feynman Diagrams in the Many Body Problem."

B. What is special about 1D

This section takes some of the highlights from Giamarchi's "Quantum Physics in One Dimension," chapter 1. Note, my treatment will be quite a bit simpler than Giamarchi's: Though if you have the background, Giamarchi tells a very nice story.

There are at least two arguments for the importance of studying 1D systems. The first, which is emphasized in Giamarchi's chapter, is that 1D is special. The traditional approaches from 3D break down in 1D. Understanding the breakdown sheds light on the original problem. The unusual physics of 1D also acts as a template for other exotic phenomena. A second important part of 1D systems is that we can actually calculate things in 1D. The approaches are technically beautiful.

B.1. Fermi Liquid Theory

To appreciate the special properties of 1D systems, it is useful to learn a bit about 3D systems. Giamarchi assumes that the reader already knows about

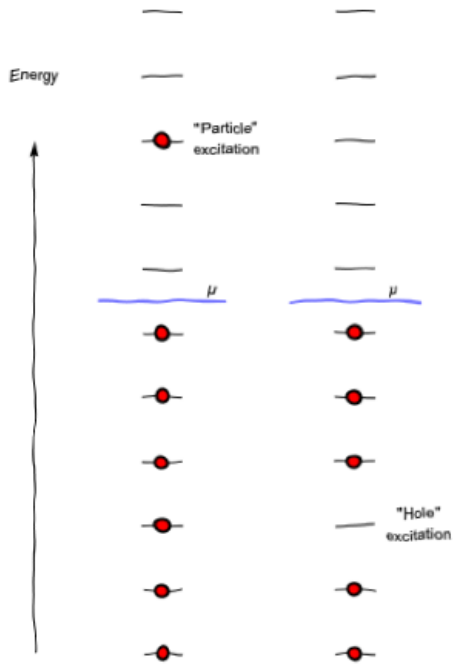
them, and does a fairly technical review of their properties. I will just briefly go over the highlights.

The theory of typical 3D electron systems is "Fermi Liquid Theory." This approach was first put forth phenomenologically, and tested experimentally. Later its theoretical foundations were established. These theoretical foundations are one of the crowning achievements of many-body theory.

The central premise of Fermi Liquid Theory is that the excitations of a 3D electron gas are in 1-to-1 correspondence with those of a non-interacting gas, but these quasiparticles

- have a different effective mass
- have a finite lifetime (the lifetime scales as $\tau \propto 1/E^2$, so low energy excitations are long-lived)
- are weakly interacting (even if the interactions between the bare electrons are strong.

The mental picture is that for a Fermi gas, the excitations are:



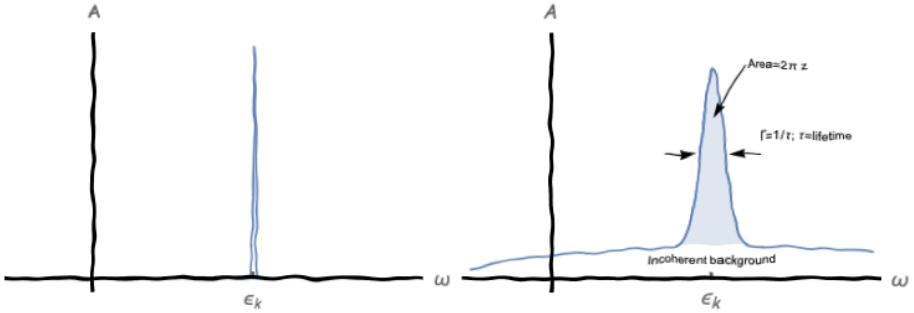
Imagine starting with these states and adiabatically turning on the interactions. The resulting state is a "quasiparticle" or "quasihole." One imagines that the electron (or hole) becomes surrounded by a screening cloud.

Multiple representations are always useful, and one concrete picture, is to consider the operator $\hat{c}_k$ which annihilates a quasiparticle with momentum k. This operator can be expressed in terms of the original fermion operators $\hat{a}_k$ as

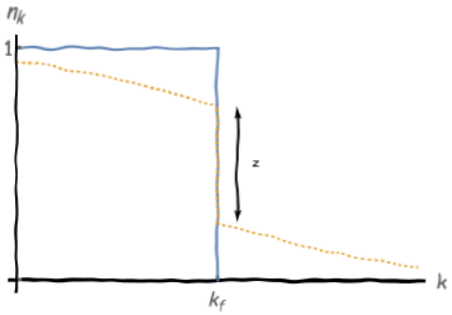
$$\hat{c}_k = \sqrt{z}\hat{a}_k + \sum_{pq} f_{kpq}\hat{a}_{k+p-1}\hat{a}_p^\dagger\hat{a}_q + \cdots \tag{1.1}$$

That is, the quasiparticle operator includes a linear combination of all terms which have total momentum k and total particle number 1. The square of the coefficient of the single-particle term, z, is known as the "quasi-particle renormalization factor," and physically corresponds to the square overlap of the single particle state and the quasiparticle state. The central premise of Fermi liquid theory is that z is non-zero.

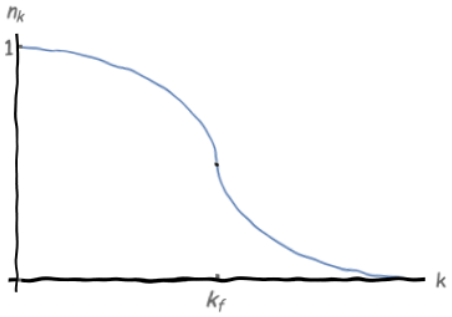
Another useful picture to have in mind is the "single particle spectral density." This is the quantity which is measured in photo-emission, or tunneling. Imagine you try to inject an electron with momentum k and energy ω. In an ideal gas, you will only succeed if $\omega = k^2/2m$. In an interacting gas, you instead expect that your spectrum (the probability of success) will be bimodal (or multimodal). You can add the particle, leading to a final state which contains a single quasiparticle: this will give a Lorentzian contribution to the spectrum. The width of the Lorentzian is related to the lifetime of the quasiparticle. The area of the peak is related to the quasiparticle renormalization factor. Alternatively you can inject the electron and create a state with some number of particle-hole excitations. This will give a broad background. Pictorially, the graph on the left represents the spectral density of a noninteracting Fermi gas, while the one on the right is a Fermi liquid:



The other story you will commonly see in the textbooks is the distribution function n_k, which represents the number of fermions with momentum k. In a zero temperature non-interacting gas, $n_k = 1$ for $k < k_f$, and $n_k = 0$ for $k > k_f$. In particular, there is a discontinuity at $k = k_f$. In a Fermi liquid, the discontinuity remains, and $n_{k_f^+} - n_{k_f^-} = z$:

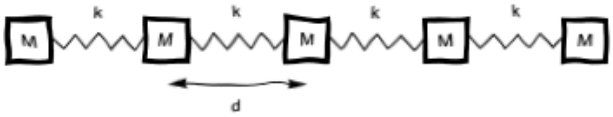


In 1D we will find that $z = 0$, but there will still be a singularity at k_f: the occupations will have a kink there: $n_k \sim n_0 - (k - k_f)^\alpha$ for $k > k_f$, and $n_k \sim n_0 + (k_f - k)^\alpha$ for $k < k_f$:



C. From "Coupled Masses" to "Classical Field Theory"

One of the most dramatic results that we will derive is that the excitations of a 1D fermi gas are exhausted by "phonons." To understand what this means, it is useful to remind ourselves where phonons come from. We will begin with a 1D model of masses connected by springs:



If we want to be able to take the continuum limit, then the masses and the spring constants are related to the equilibrium separation by:

$$\lambda = \frac{M}{d} \tag{1.2}$$

$$\kappa = kd. \tag{1.3}$$

Here λ is the linear mass density, and κ is the stiffness.

The dynamical variables of this set of masses are the displacements from

equilibrium, s_i, and the momenta p_i. The equations of motion read:

$$M\frac{\partial^2 s_j}{\partial_t^2} = k(s_{j+1} + s_{j-1} - 2s_j). \tag{1.4}$$

This is solved by making the ansatz $s_j = e^{i(qdj-\omega t}$, where $x_j = dj$ is the equilibrium position of the $j'th$ mass. Substituting this into our equations of motion gives

$$\begin{aligned} \omega &= \sqrt{\frac{k}{M}} 2\sin(qd/2) & (1.5)\\ &\rightarrow q\sqrt{\frac{kd}{M/d}} = q\sqrt{\frac{\kappa}{\lambda}}. & (1.6) \end{aligned}$$

We teach that argument in PHYS 2218 at Cornell, and it should feel pretty familiar. One interesting way to think about things is to introduce the continuous function $s(x)$, where $s_j = s(x_j)$. In the continuum limit this is a well-defined procedure, and if you want to be pedantic there are formal ways of defining $s(x)$. We can relate $s(x)$ to the local mass density. The separation between two neighboring masses is

$$\Delta_j = d + sj + 1 - s_j = d\left(1 + \frac{\partial s}{\partial x}\right). \tag{1.7}$$

That is if s is uniform, then the separation is d. If $\partial s/\partial x > 0$ then the separation is greater than d. If you know something about elasticity, you would recognize $\partial s/\partial x$ as the 1D version of the strain tensor. Regardless, the mass density is

$$\begin{aligned} \rho &= \frac{M}{\Delta} = \frac{M}{d}\frac{1}{1+\frac{\partial s}{\partial x}} & (1.8)\\ &\approx \lambda\left(1 - \frac{\partial s}{\partial x}\right). & (1.9) \end{aligned}$$

Thus, we can identify $-\partial s/\partial x$ as the fractional change in the density. Remember this – we will see this structure again. Note that s is independent of the discretization.

The classical Lagrangian associated with this system is

$$\begin{aligned} L &= \sum_j \frac{M}{2}\dot{s}_j^2 - \frac{k}{2}(s_{j+1} - s_j)^2 & (1.10)\\ &= \sum_j d\frac{M}{2d}\dot{s}_j^2 - d\frac{kd}{2}\left(\frac{s_{j+1}-s_j}{d}\right)^2 & (1.11)\\ &= \int dx\left[\frac{\lambda}{2}\left(\frac{\partial s}{\partial t}\right)^2 - \frac{\kappa}{2}\left(\frac{\partial s}{\partial x}\right)^2\right]. & (1.12) \end{aligned}$$

The conjugate variable to $s(x)$ is the momentum density

$$\pi(x) = \frac{\delta L}{\delta \dot{s}(x)} = \lambda \dot{s}(x). \tag{1.13}$$

It has units of momentum per unit length. That is, the momentum of the j'th mass is $p_j = d\pi(x_j)$.

The classical Hamiltonian is

$$H = \int dx \frac{1}{2\lambda}\pi^2 + \frac{\kappa}{2}(\partial_x s)^2. \tag{1.14}$$

D. Quantum Coupled Masses

The quantum version of this problem is essentially identical to the classical problem. It is convenient to think about the operators $\hat{s}_j$ and $\hat{p}_j$, with commutation relations

$$[\hat{s}_i, \hat{p}_j] = i\hbar\delta_{ij}. \tag{1.15}$$

We can use our classical definitions of the fields

$$\hat{s}_j = \hat{s}(x_j) \tag{1.16}$$

$$\hat{p}_j = d\hat{\pi}(x_j). \tag{1.17}$$

It is straightforward to argue that

$$[\hat{s}(x), \hat{\pi}(y)] = i\hbar\delta(x-y). \tag{1.18}$$

We note that the density operator is $\hat{\rho}(x) = \lambda(1 - \hat{s}'(x))$.

E. HW 1 – Due March 20

Problem 1.1. Number Phase Uncertainty: *This problem makes concrete some of the algebra that we will encounter in the lectures.*
In your linear algebra class you may have learned about the polar-decomposition: It is the statement that any matrix can be written as the product of a Hermitian and Unitary matrix. The name comes from noting that a unitary matrix can be written as the exponential of a Hermitian matrix, so it looks like the polar decomposition of a complex number. In other words, the decomposition is:

$$M = e^{i\theta}R \tag{1.19}$$

where

$$R^\dagger = R \tag{1.20}$$

$$\theta^\dagger = \theta. \tag{1.21}$$

1.1.1. Two-level system: To make this very concrete, thing about a system with two states, labeled $|0\rangle$ and $|1\rangle$. Typically we order these as $|1\rangle, |0\rangle$, but for our purposes it will be better to write them as $|0\rangle, |1\rangle$. In this basis, with this order, the "lowering" operator is

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{1.22}$$

Show that

$$\{a, a^\dagger\} = 1, \tag{1.23}$$

and hence a can be thought of as a Fermion lowering operator (where we interpret the two states as "full" and "empty").

1.1.2. Show that

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{1.24}$$

and that the first matrix is Unitary, and the second Hermitian. Thus we write

$$e^{i\theta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{1.25}$$

$$R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.26}$$

1.1.3. Calculate the anticommutator $\{e^{i\theta}, R^\dagger R\}$.

1.1.4. Find the matrix θ. Hint: Mathematica has a function `MatrixLog`. Alternatively you can write θ as a linear combination of the identity matrix and the Pauli matrices, and solve for the coefficients. Note, because log is multivalued, the answer is not unique.

Problem 1.2. Number-phase uncertainty for a bosonic mode:
The bosonic problem is a bit more complicated because the Hilbert space is infinite. One strategy is to work in a truncated space, where you limit the occupation to be $0, 1, 2, \cdots n$, so the matrices are all finite. We can then take $n \to \infty$ at the end.

1.2.1. Polar decomposition: In this finite space, the annihilation operator is

$$a = \begin{pmatrix} 0 & \sqrt{1} & 0 & & \\ 0 & 0 & \sqrt{2} & & \\ \vdots & & \ddots & & \\ 0 & & & 0 & \sqrt{n-1} \\ 0 & & \cdots & & 0 \end{pmatrix}. \tag{1.27}$$

Show that

$$a = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & & & & \\ 0 & \dots & & & 1 \\ 1 & \dots & & & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & \sqrt{2} & 0 & \cdots \\ \vdots & & & & \\ 0 & \dots & & & \sqrt{n} \end{pmatrix}. \tag{1.28}$$

and that the first matrix is Unitary, and the second Hermitian.

1.2.2. Phase Matrix: The matrix θ is quite ugly here – so I won't ask you to calculate it. However it is straightforward to calculate

$$[e^{i\theta}, R^\dagger R]. \tag{1.29}$$

A way to make the arithmetic simpler is to write

$$e^{i\theta} = \sum_{j=1}^{n} |j-1\rangle\langle j| + |n\rangle\langle 0| \tag{1.30}$$

$$R^\dagger R = \sum j = 0^n |j\rangle j \langle j|. \tag{1.31}$$

You will find

$$[e^{i\theta}, R^\dagger R] = e^{i\theta} + X. \tag{1.32}$$

What is the matrix X?

1.2.3. In the limit $n \to \infty$, the matrix X involves a state with infinitely many particles – and we typically argue that it can be neglected. If we do so, then $[e^{i\theta}, R^\dagger R] \approx e^{i\theta}$. Under this approximation, infer

$$[\theta, R^\dagger R]. \tag{1.33}$$

Problem 1.3. Coherent States

Consider the following two harmonic oscillator Hamiltonians, where we have taken dimensionless units ($\hbar, m, \omega$ all are unity):

$$H_1 = \frac{p^2}{2} + \frac{1}{2}x^2 \tag{1.34}$$

$$H_2 = \frac{p^2}{2} + \frac{1}{2}(x - x_0)^2. \tag{1.35}$$

Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be the ground states. Clearly

$$|\psi_2\rangle = e^{ipx_0}|\psi_1\rangle, \tag{1.36}$$

where $\exp(ipx_0)$ is a translation operator.

We introduce the ladder operators:

$$a = \frac{1}{\sqrt{2}}(x + ip) \tag{1.37}$$

$$a^\dagger = \frac{1}{\sqrt{2}}(x - ip). \tag{1.38}$$

Show that $|\psi_2\rangle$ is an eigenstate of a. Find the eigenvalue.

Hint: What is $a|\psi_1\rangle$? What is $[a, e^{ipx_0}]$?

Problem 1.4. Quantum Fluctuations in a 1D harmonic Chain

Consider the 1D chain described by Hamiltonian:

$$H = \sum_j \frac{p_j^2}{2m} + \frac{1}{2}k(x_{j+1} - x_j)^2, \tag{1.39}$$

where $[x_i, p_j] = i\hbar\delta_{ij}$, and where j runs from 1 to N. For concreteness we will use periodic boundary conditions (so x_{n+1} is identified with x_1.

We can diagonalize this Hamiltonian by the transformation:

$$x_j = \sum_q e^{iqj} u_q (a_q + a_{-q}^\dagger) \tag{1.40}$$

$$p_j = \sum_q e^{-iqj} v_q \frac{a_q - a_{-q}^\dagger}{i}.$$

The sums run over $q = 2\pi n/N$, with n running from $-N/2$ to $N/2$. It turns out that we can choose u_q and v_q real. Furthermore, since x_j and p_j are Hermitian, we need $u_{-q} = u_q$ and $v_{-q} = v_q$. We want the operators a_q to obey

$$[a_q, a_{\bar{q}}^\dagger] = \delta_{q,\bar{q}} \tag{1.41}$$

1.4.1. Show that if Eq. (1.41) is satisfied, then

$$[a_q \pm a_{-q}^\dagger, a_{q'} \pm a_{-q'}^\dagger] = 0 \tag{1.42}$$

for any q, q'. Hence the requirements $[x_i, x_j] = 0$ and $[p_i, p_j] = 0$ is automatically satisfied by this ansatz

1.4.2. Show that if Eq. (1.41) is satisfied, then

$$[a_q + a_{-q}^\dagger, \frac{a_{q'} - a_{-q'}^\dagger}{i}] = 2i\delta_{q,q'}. \tag{1.43}$$

for any q, q'.

1.4.3. Use that last result to show that $[x_i, p_j] = i\hbar\delta_{ij}$ if and only if

$$\begin{aligned} u_q &= \sqrt{\frac{\hbar}{2N}}\delta_q \\ v_q &= \sqrt{\frac{\hbar}{2N}}\frac{1}{\delta_q}, \end{aligned} \tag{1.44}$$

where δ_q is aribtrary (with $\delta_{-q} = \delta_q$).

1.4.4. Substitute the ansatz, Eq. (1.40) into the Hamiltonian, using Eq. (1.44) to write the Hamiltonian as a function of the raising/lowering operators and δ_q. The resulting expression should be of the form

$$\begin{aligned} H &= \sum_q (a_q a_{-q} + a^\dagger_{-q} a^\dagger_q)\frac{\hbar}{2}\left[A_q\delta_q^2 - \frac{B_q}{\delta_q^2}\right] \\ &\quad + (a_q a^\dagger_q + a^\dagger_{-q} a_{-q})\frac{\hbar}{2}\left[A_q\delta_q^2 - \frac{B_q}{\delta_q^2}\right]. \end{aligned} \tag{1.45}$$

Find B_q and A_q. We then set

$$\delta_q^4 = \frac{B_q}{A_q}, \tag{1.46}$$

so that

$$H = \sum_q \left(a^\dagger_q a_q + \frac{1}{2}\right)\hbar\omega_q. \tag{1.47}$$

Find ω_q.

1.4.5. The ground state of this system will have $a_q|GS\rangle = 0$ for all a_q. Use this to write

$$\langle (x_i - x_j)^2\rangle = \sum_q u_q^2\, 4\sin^2\left(\frac{q(i-j)}{2}\right) \tag{1.48}$$

$$\to \frac{N}{2\pi}\int dq\, u_q^2\, 4\sin^2\left(\frac{q(i-j)}{2}\right), \tag{1.49}$$

where we converted the sum to an integral by noting that the separation between q's is $2\pi/N$.

If $|i-j| \gg 1$, the sin function in Eq. (1.48) is rapidly oscillating, and one might naively expect that one could replace it with $1/2$. Show that if you do, the integral is divergent. (Hint, the problem happens as $q \to 0$). [You don't need to do it, but If you work a bit harder, you can even argue that the divergence should be $\langle (x_i - x_j)^2\rangle \sim \log|i-j|$.]

Regardless, his means that $\langle (x_i - x_j)^2\rangle \to \infty$ as $|i-j| \to \infty$. Physically, this means that the quantum fluctuations disorder the chain. If you did the same calculation in 3D, you would instead find the integral is convergent.

Chapter 2

"Constructive" Bosonization – Mar 15, 2019

In this lecture we derive a quite remarkable result: The excitations of an interacting 1D Fermi system can be put in a 1-to-1 correspondence with those of a harmonic chain. I note that it is an obvious statement that *among* the excitations of the electrons are "plasmons" – corresponding to density waves in the electron fluid. This is true in any dimension. The special thing about 1D is that *all* excitations can be written in terms of these phonons.

We will also first neglect spin, then add it later. Our treatment is similar to Giamarchi chapter 2 – but maybe even closer to Delft and Schoeller's article "Bosonization for Beginners..." which is on the arXiv.

A. Linearizing

The first step will be to linearize about the Fermi surface. It turns out some of our results do not depend on this approximation, but I think it is best to do the simplest treatment first. The idea is that we imagine a setting where the interactions are weak. In that case the ground state of the interacting system will look a lot like the ground state of the non-interacting system, with the exception of structure near the Fermi points. Thus it suffices to model physics

near the Fermi points. In other words we write the kinetic energy as

$$\begin{aligned} H &= H_R + H_L + H' && (2.1)\\ H_R &= \sum_{k \,\text{near}\, k_f} v_f(k-k_f) c_k^\dagger c_k && (2.2)\\ H_L &= \sum_{k \,\text{near}\, -k_f} v_f(k_f-k) c_k^\dagger c_k && (2.3)\\ H' &= \sum_{\text{other}\, k} \epsilon_k c_k^\dagger c_k. && (2.4) \end{aligned}$$

We will come up with a model of H_R and H_L. We won't worry about H', as it will not be important to the physics we are after. Clearly if the interactions are strong, this argument will break down and we will need to include H'.

It makes sense to shift the origin of momenta, and write

$$\begin{aligned} a_{rk} &= c_{k-k_f} && (2.5)\\ a_{\ell k} &= c_{k+k_f}. && (2.6) \end{aligned}$$

when there is no possibility of confusion, we will leave off the r, ℓ indices. We refer to these two "species" of particles as "right movers" and "left movers". I have been a little vague about what range of k we are summing over. In principle it doesn't matter – if all the action is happening at the Fermi surface, I can include as many states as I want.

B. Density Operator

In this section we will see that the Fourier transform of the density acts as a ladder operator. There are a couple tricky things here. The first is just keeping track of L's and 2π's.

The best mnemonic that I have for the factors is to start with the electron field operator – which is something like

$$\psi(x) = \frac{1}{\sqrt{L}} \sum_k c_k e^{ikx}, \tag{2.7}$$

where $k = 2\pi n/L$. We can verify that I have the factors right by looking at the density

$$\rho(x) = \psi^\dagger(x)\psi(x) = \frac{1}{L} \sum_{kq} e^{i(k-q)x} c_q^\dagger c_k. \tag{2.8}$$

In particular, if we have a homogeneous gas $\langle c_q^\dagger c_k \rangle$ is zero unless $k = q$ and $k < k_f$, and we get

$$\rho(x) = \frac{N}{L}. \tag{2.9}$$

The Fourier transform of the density operator is

$$\rho_p = \sum_q a^\dagger_{q+p} a_q. \tag{2.10}$$

That is

$$\rho(x) = \frac{1}{L}\sum_p e^{ipx}\rho_p. \tag{2.11}$$

Perhaps I should define ρ_p with a factor of $1/\sqrt{L}$ in it. In the end it doesn't matter – as long as you are consistent. This convention makes the following arithmetic a bit easier.

Clearly if $p \ll k_f$, we can write

$$\rho_p = \rho^R_p + \rho^L_p + \rho^{\rm other}_p, \tag{2.12}$$

where

$$\rho^R_p \;\; = \;\; \sum_q c^\dagger_{r,q+p} c_{rq}, \tag{2.13}$$

and a similar expression for ρ^L. We will analyze the properties of ρ^R_p.

Now we come to our first profound observation. Suppose $|\psi\rangle$ is an eigenstate of H^R:

$$H|\psi\rangle = E|\psi\rangle. \tag{2.14}$$

(Note I have left off the R.) Applying ρ^R_p to $|\psi\rangle$ will give a superposition of terms where various right-moving particle have had their momentum increased by p. But the energy of a given particle is proportional to p – so we have increased the energy by $v_f p$. In other words

$$H\rho_p|\psi\rangle = (E + v_f p)\rho_p|\psi\rangle. \tag{2.15}$$

Putting these two expressions together, we have $H\rho_p|\psi\rangle = \rho_p(H + v_f p)|\psi\rangle$. This statement is true for all eignenstates, and hence

$$[\rho_p, H] = -v_f p \rho_p. \tag{2.16}$$

This is exactly the requirement for a ladder operator. Actually – as you see ρ_p appears on both the left and right hand side of that expression, so this is actually the requirement to be a constant times a ladder operator. Below, we will show that the correct normalization is (for $p > 0$),

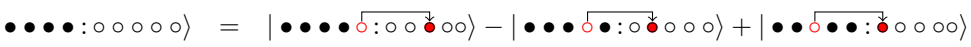
$$b^\dagger_p \;\; = \;\; \frac{2\pi}{L}\frac{1}{p}\rho_p \tag{2.17}$$

$$b_p \;\; = \;\; \frac{2\pi}{L}\frac{1}{p}\rho_{-p}, \tag{2.18}$$

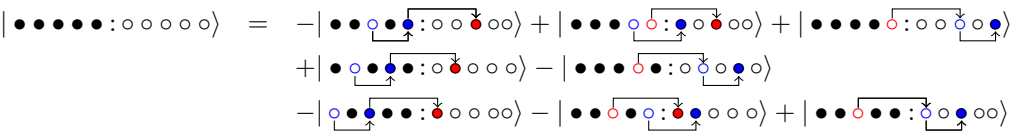
where these obey the standard relations $[b_p, b_q^\dagger] = \delta_{p-q}$.

To establish this result, we need to understand what these operators do to states. In particular we want to look at how $\rho_p\rho_q|\psi\rangle$ differs from $\rho_q\rho_p|\psi\rangle$. To make the notation simpler, let us write momenta in units of $2\pi/\hbar$, which is the spacing between momentum states.

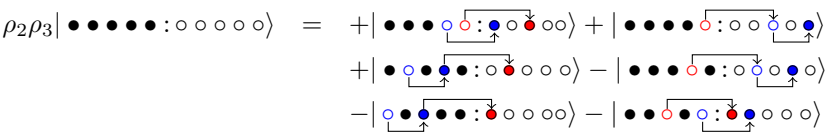
Lets start with a Fermi sea $|\psi\rangle = |\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle$. Applying ρ_3 to this gives three terms

$\rho_3|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle$ = 

Following this with ρ_2 gives a total of 8 terms:

$\rho_2\rho_3|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle$ = 

For what it is worth, two of these cancel, leaving

$\rho_2\rho_3|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle$ = 

Remarkably, if you do these two operators in the opposite order ,you get exactly the same terms, with exactly the same signs. Even more remarkably, the same thing happens on any initial state. I will prove this below, but it is worth playing with this to convince yourself. You will find, however, that if $p = -q$, then they don't commute.

$$\begin{aligned}\rho_{-3}\rho_3|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle &= \rho_{-3}\left[|\bullet\bullet\bullet\bullet\circ:\circ\circ\bullet\circ\circ\rangle - |\bullet\bullet\bullet\circ\bullet:\circ\bullet\circ\circ\circ\rangle + |\bullet\bullet\circ\bullet\bullet:\bullet\circ\circ\circ\circ\rangle\right] \\ &= |\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle + |\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle + |\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle \\ &= 3|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle\end{aligned}$$

but

$$\rho_3\rho_{-3}|\bullet\bullet\bullet\bullet\bullet:\circ\circ\circ\circ\circ\rangle = 0.$$

Having established plausibility, we now prove the general result $[\rho_p, \rho_q] = q\delta_{p+q}$ – or in real units $[\rho_p, \rho_q] = q(L/2\pi)\delta_{p+q}$. We start with

$$[\rho_p, \rho_q] = \sum_{k,k'} [a_{k+p}^\dagger a_k, a_{k'+q}^\dagger a'_k]. \tag{2.19}$$

This commutator is zero unless $k'+q = k$ or $k+p = k'$. We need to be somewhat careful about the bounds on the sum, but for the moment lets not worry about it,

$$[\rho_p, \rho_q] = \sum_k a^\dagger_{k+p} a_{k-p} - \sum_{k'} a'_{k+q} a_{k'-p}. \tag{2.20}$$

We shift indices on the sum, and everything cancels – except the boundary terms. In the case $p \neq -q$, we actually don't need to worry about the boundary terms, they are things like $a^\dagger_{k+p} a_{k-p}$ for k well inside the Fermi surface. Any states we act on will have both those states occupied, and hence such terms do nothing. When $p = -q$ these expressions are $a^\dagger_{k-q} a_{k-p}$, which is 1 if we are well inside the Fermi surface, but -1 if we are well outside. A little thought should convince you that $[\rho_{-q}, \rho_q] = q$ (or actually $[\rho_{-q}, \rho_q] = qL/2\pi$). [When I have a chance I'll write it in more detail, but I already went through it in class, and am running out of time in typing up the notes.]

We can therefore define ladder operators (for $p¿0$),

$$\hat{b}_p = \sqrt{\frac{2\pi}{Lp}} \rho_{-p} \tag{2.21}$$

$$\hat{b}^\dagger_p = \sqrt{\frac{2\pi}{Lp}} \rho_p. \tag{2.22}$$

We further have

$$\begin{aligned} [\hat{b}_p, \hat{b}^\dagger_p] &= \delta_{pp'} \\ [\hat{b}_p, H_R] &= v_f p \hat{b}_p \end{aligned} \tag{2.23}$$

For left movers we would instead get (for $p < 0$)

$$\hat{b}_{p,\ell} = \sqrt{\frac{2\pi}{L|p|}} \rho_{-p,\ell} \tag{2.24}$$

$$\hat{b}^\dagger_{p,\ell} = \sqrt{\frac{2\pi}{L|p|}} \rho_{p,\ell}. \tag{2.25}$$

We further have

$$\begin{aligned} [\hat{b}_{p,\ell}, \hat{b}^\dagger_{p,\ell}] &= \delta_{pp'} \\ [\hat{b}_{p,\ell}, H_L] &= v_f |p| \hat{b}_{p,\ell} \end{aligned} \tag{2.26}$$

Thus the Hamiltonian for the non-interacting system reads

$$\hat{H} = \sum_{p>0} v_f p \left[\hat{b}^\dagger_{pr} \hat{b}_{pr} + \hat{b}^\dagger_{p-\ell} \hat{b}_{-p\ell} \right] + \hat{\Delta} \tag{2.27}$$

where $\hat{\Delta}$ commutes with all of the $\hat{b}$ operators.

Claim: The only operators which commute with all of the $\hat{b}$'s and conserve the number of left and right movers are $\hat{N}_L$ and $\hat{N}_R$ – the total number of left and right movers – and functions of these.

In other words, given any state, say

$$|\psi\rangle = |\bullet\bullet\circ\bullet\circ\circ\bullet\circ\circ\circ\rangle. \tag{2.28}$$

The claim is that $|\psi\rangle$ can be expressed as a linear superposition of $\hat{b}$'s acting on

$$|\psi_0\rangle = |\bullet\bullet\bullet\bullet\circ\circ\circ\circ\circ\circ\rangle. \tag{2.29}$$

This claim is equivalent to the statement that the number of ways of making orthogonal states of energy E by acting on the ground state with b's is the same as the number of ways of making such a state by adding an equal number of particles and holes. Both constructions make orthogonal states – hence the Hilbert spaces will have the same dimension – and must be identical.

It is fun to do the counting for the first few states. For example, it is clear that there is a unique state with energy 1. There are two states with energy 2: $|\bullet\bullet\bullet\circ:\circ\bullet\circ\circ\rangle$ and $|\bullet\bullet\circ\bullet:\bullet\circ\circ\circ\rangle$. These are linear combinations of $b_1^\dagger b_1^\dagger|\psi_0\rangle$ and $b_2^\dagger|\psi_0\rangle$.

In general, the fact that these numbers are always the same is equivalent to the "pentagonal number theorem" – which is at the core of the Jacobi triple product formula, which can also be considered a Jacobi Theta function identity (relating it to Wierstrauss functions). I won't go through the proof of the Jacobi identiy, but will show how to reduce this physics problem to that result.

First, the number of ways of using the boson operators to generate a state of energy m is given by the number of partitions of the integer m. That is the number of ways of choosing integers which add up to n. For example, 2=1+1 or 2=2, has two ways. We write $p(2) = 2$. On the other hand 4=3+1=2+2+2+1+1=1+1+1+1 has 5 ways. We will define the generating function

$$f(x) = \sum_m p(n)x^m. \tag{2.30}$$

Standard generating function tricks give

$$f(x) = \prod_k \frac{1}{1-x^k}. \tag{2.31}$$

[I'll go through these in class.] This is a famous function in number theory.

Lets denote the number of ways of distributing an equal number of particles and holes to produce energy m as $s(m)$. We define

$$g(x) = \sum_m s(m)x^m. \tag{2.32}$$

The claim is that $f(x) = g(x)$. Calculating $g(x)$ is hard, so we introduce $s(m,n)$ as the number of ways of distributing particles and holes to produce energy m, where the number of particles minus the number of holes is n. We define

$$G(x,y) = \sum_{nm} s(m,n) x^m y^n. \tag{2.33}$$

Clearly $g(x)$ is the y^0 term in the series. Formally we can write G as the product of generating functions for each mode

$$G(x,y) = \left[(1+xy)(1+x^2y)(1+x^3y)\cdots\right] \times \left[(1+y^{-1})(1+xy^{-1})(1+x^2y^{-1})\cdots\right]. \tag{2.34}$$

We then use the "Jacobi triple product" identity, which says

$$G(x,y) = f(x) \sum_n x^{n(n-1)/2} y^n. \tag{2.35}$$

Taking the $y = 0$ term of both side establishes the proof.

C. Klein factors

There is one other set of operators we will need, namely the "Klein factors" $F_r^\dagger$ and $F_\ell^\dagger$, which are unitary operators which just shift the number of left and right movers by 1. They are defined as the unique operators which commute with the b's, and shift the Fermi sea:

$$F_r^\dagger |\bullet\bullet\bullet\bullet : \circ\circ\circ\circ\rangle_R = (-1)^{N_L} |\bullet\bullet\bullet\bullet : \bullet\circ\circ\circ\rangle_R \tag{2.36}$$

$$F_r |\bullet\bullet\bullet\bullet : \circ\circ\circ\circ\rangle_R = (-1)^{N_L} |\bullet\bullet\bullet\circ : \circ\circ\circ\circ\rangle_R \tag{2.37}$$

$$F_\ell^\dagger |\bullet\bullet\bullet\bullet : \circ\circ\circ\circ\rangle_L = |\bullet\bullet\bullet\bullet : \bullet\circ\circ\circ\rangle_L \tag{2.38}$$

$$F_\ell |\bullet\bullet\bullet\bullet : \circ\circ\circ\circ\rangle_L = |\bullet\bullet\bullet\circ : \circ\circ\circ\circ\rangle_L \tag{2.39}$$

On other states, it similarly shifts everything by one:

$$F_r^\dagger |\bullet\bullet\circ\bullet : \circ\bullet\circ\circ\rangle_R = (-1)^{N_L} |\bullet\bullet\bullet\circ : \bullet\circ\bullet\circ\rangle_R. \tag{2.40}$$

It should be clear that they do indeed commute with the b's, and

$$F_r^\dagger F_r = F_r F_r^\dagger = F_\ell^\dagger F_\ell = F_\ell F_\ell^\dagger = 1 \tag{2.41}$$

$$\{F_r, F_\ell\} = 0. \tag{2.42}$$

The factor of $(-1)^{N_L}$ was chosen so that this last anticommutation relationship would hold.

D. Fermion Operators

We can now write the fermion operators in terms of the phonons. In particular

$$\begin{aligned}\psi(x) &= \frac{1}{\sqrt{L}}\sum_k c_k e^{ikx} \qquad (2.43)\\ &= \frac{e^{ik_f x}}{\sqrt{L}}\sum_q a_{rq}e^{iqx} + \frac{e^{-ik_f x}}{\sqrt{L}}\sum_q a_{\ell q}e^{iqx} + \text{stuff away from Fermi surface.}\end{aligned}$$

In the obvious notation, we call the first two terms ψ_r and ψ_ℓ.

We will now construct ψ_r in terms of the b's, by looking at the commutation relations. Recall

$$\rho_r(p) = \sum_k c^\dagger_{r\,k+p}c_{r\,k} \qquad (2.44)$$

Straightforward arithmetic gives

$$[\rho(p), c_q] = -c_{q-p}, \qquad (2.45)$$

and hence

$$\begin{aligned}[\rho_r(p), \psi_r(x)] &= -\frac{1}{\sqrt{L}}\sum_q c_{r\,q-p}e^{iqx} \qquad (2.46)\\ &= -e^{ipx}\psi_r(x). \qquad (2.47)\end{aligned}$$

We now use that

$$[\rho_r(p), \rho_r(-p)] = -p\frac{L}{2\pi}, \qquad (2.48)$$

to conclude that

$$\psi_r(x) = (\cdots)\exp\left(\frac{2\pi}{Lp}\rho_r(-p)e^{ipx}\right), \qquad (2.49)$$

where the term in parentheses commutes with $\rho_r(p)$. The only expression consistent with this (for all p), and which does the right thing to the number of particles is

$$\begin{aligned}\psi_r(x) &= AF_r\exp\sum_p \frac{2\pi}{Lp}e^{ipx}\rho_r(-p) \qquad (2.50)\\ &= AF_r\exp\sum_{p>0}\sqrt{\frac{2\pi}{Lp}}\left[e^{ipx}b_p - e^{-ipx}b^\dagger_p\right], \qquad (2.51)\end{aligned}$$

where A is a constant.

It is traditional to define

$$\begin{aligned}\varphi_r(x) &= i\frac{2\pi}{L}\sum_{p>0}\frac{e^{ipx}}{p}\rho_r(-p) \qquad (2.52)\\ &= i\sqrt{\frac{2\pi}{L}}\sum_{p>0}\frac{e^{ipx}}{\sqrt{p}}b_p, \qquad (2.53)\end{aligned}$$

and

$$\begin{aligned}\phi_r(x) &= \varphi_r(x) + \varphi_r^\dagger(x) && (2.54)\\ &= i\frac{2\pi}{L}\sum_p \frac{e^{ipx}}{p}\rho_r(-p). && (2.55)\end{aligned}$$

Then

$$\psi_r(x) = AF_r e^{-i\phi_r(x)} \quad (2.56)$$

In addition to being a cute way of writing things, we can note an interesting property of the field $\phi_r(x)$, in particular

$$\begin{aligned}\partial_x\phi_r(x) &= -\frac{2\pi}{L}\sum_p e^{ipx}\rho_r(-p) && (2.57)\\ &= -2\pi\rho_r(x). && (2.58)\end{aligned}$$

Thus ϕ_r has exactly the same meaning as the displacement field that we introduced for the mass chain.

Now Giamarchi makes life more complicated by introducing two other fields via:

$$\begin{aligned}\phi_r(x) &= \phi(x) - \theta(x) && (2.59)\\ \phi_\ell(x) &= -\phi(x) - \theta(x). && (2.60)\end{aligned}$$

The reason for doing this is that the physical density is the sum of the right-moving and left-moving densities (plus stuff from other modes). Thus $\partial_x\phi$ has the meaning of the total density. The other field θ involves the mis-match of the left-moving and right-moving densities. It is clearly related to the momentum density, and will find

$$\begin{aligned}\rho(x) &= -\frac{1}{\pi}\partial_x\phi(x) && (2.61)\\ \pi(x) &= \frac{1}{\pi}\partial_x\theta(x). && (2.62)\end{aligned}$$

Indeed, it is a straightforward exercise, left as a homework exercise, that these fields obey the right commutation relations.

E. Free Hamiltonian

Lets now consider how the Hamiltonian looks in terms of these operators. First, as already argued, up to a constant, the free particle Hamiltonian is

$$\hat{H}_0 = \sum_{p>0} v_f p\left[\hat{b}_{pr}^\dagger\hat{b}_{pr} + \hat{b}_{-p\ell}^\dagger\hat{b}_{-p\ell}\right] + v_f\frac{2\pi}{L}(\hat{N}_R^2 + \hat{N}_L^2) - \mu(\hat{N}_R + \hat{N}_L), \quad (2.63)$$

where we have added a chemical potential. This, and the field operators, are more conveniently described by

$$b_{p>0} = b_{rp} \tag{2.64}$$

$$b_{p<0} = b_{\ell p}. \tag{2.65}$$

The first term in Eq. (2.63) is then

$$\hat{H}_0^b = \sum_p v_f p b_p^\dagger b_p, \tag{2.66}$$

where the superscript b reminds us that we have left off the $\hat{N}$ terms. Similarly this makes the expression for θ and ϕ simpler,

$$\phi(r) = -i\frac{2\pi}{L}\sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{e^{-ipx}}{p}(b_p^\dagger + b_{-p}) \tag{2.67}$$

$$\theta(r) = -i\frac{2\pi}{L}\sum_p \left(\frac{L|p|}{2\pi}\right)^{1/2} \frac{e^{-ipx}}{|p|}(b_p^\dagger - b_{-p}). \tag{2.68}$$

For those who did Problem 1.4, you should recognize this as exactly the expressions for

$$\rho(x) = -\frac{1}{\pi}\partial_x \phi(x) \tag{2.69}$$

$$\Pi(x) = \frac{1}{\pi}\partial_x \theta(x). \tag{2.70}$$

In particular, straightforward arithmetic gives

$$\hat{H}_0^b = \frac{v_f}{2\pi}\int dx\, \left[(\partial_x\theta)^2 + (\partial_x\phi)^2\right]. \tag{2.71}$$

F. Interactions

We now consider the role of interactions. A typical model for interactions is

$$H_{\rm int} = \int dx \int dx'\, V(x-x')\psi^\dagger(x)\psi^\dagger(x')\psi(x')\psi(x). \tag{2.72}$$

For electrons V could be a Coulomb potential, while for cold atoms it is some sort of short range interaction. Note, that because of the anti-commutation relations, one could also write this as

$$H_{\rm int} = \int dx \int dx'\, V(x-x')\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x') - V(0)\int dx\, \psi^\dagger(x)\psi(x). \tag{2.73}$$

The latter term just shifts the chemical potential – possibly by an infinite amount, depending on how V behaves at short distances. Thus some treatments use

$$H_{\rm int} = \int dx \int dx'\, V(x-x')\psi^\dagger(x)\psi(x)\psi^\dagger(x')\psi(x'). \tag{2.74}$$

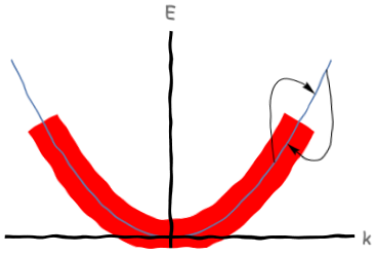
You should keep in mind that the strictly correct expression is Eq. (2.72) or (2.73), but as long as you don't worry about a few infinities, it is fine to work with Eq. (2.74).

We write Eq. (2.72) in momentum space as

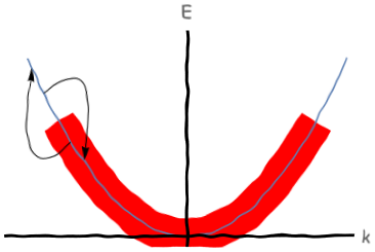
$$H_{\rm int} = \frac{1}{L^3}\sum_{kpq} V_q\, \psi^\dagger_{k+q}\psi^\dagger_{p-q}\psi_p\psi_k. \tag{2.75}$$

We think about this as a scattering process – where you have incoming particles with momentum k and p, which exchange momentum q. We only care about the cases where all of the momenta are near the Fermi surface. There are six possibilities:

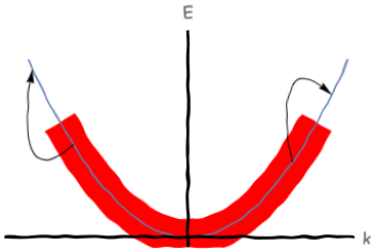
1. $k, p, k+q, p-q$ are all near k_f



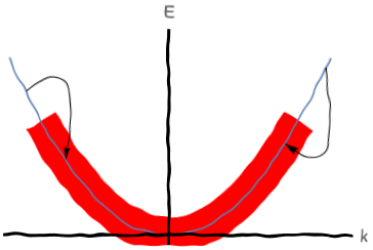
2. $k, p, k+q, p-q$ are all near $-k_f$



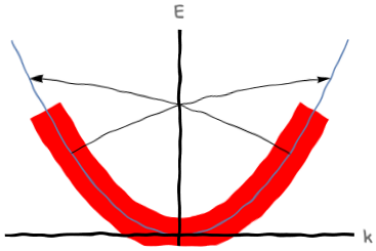
3. $k, k+p$ are all near k_f while q and $q-p$ are near $-k_f$



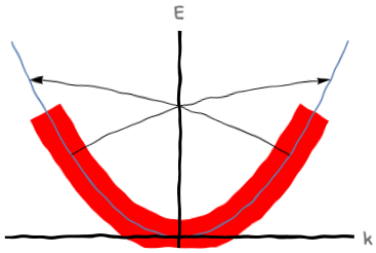
4. $k, k+p$ are all near $-k_f$ while q and $q-p$ are near k_f



5. $k, q-p$ are near k_f while $q, k+p$ are near $-k_f$



6. $k, q-p$ are near $-k_f$ while $q, k+p$ are near k_f



The first two are referred to as "g_4" processes – they involve just scattering amongst right movers or amongst left movers. For these $q \approx 0$, and one often just replaces V_q with V_0 for all these terms. The second two are referred to as "g_2" processes, and involve scattering between right and left movers. Again $q \approx 0$, and one often just replaces V_q with V_0. Finally, the last terms are "g_1"

processes, which involve scattering from one Fermi surface to the other. Here $q \approx 2k_f$, and one often just replaces V_q with V_{2k_f}.

If we make this approximation (replacing V_q with V_0 in the g_4 and g_2 processes, and replacing it with V_{2k_f} in the g_1 processes – then both the g_1 and g_2 terms are of the form

$$H_{1/2} = (\cdots) \int dx\, \rho_R(x)\rho_L(x). \tag{2.76}$$

The g_4 terms are of the form

$$H_4 = (\cdots) \int dx\, \rho_R(x)\rho_R(x) + \rho_L(x)\rho_L(x). \tag{2.77}$$

Following Giamarchi, we write the interaction Hamiltonian as

$$\begin{aligned} H_{\text{int}} &= \int dx\, \frac{g_4}{2} \left(\rho_R(x)\rho_R(x) + \rho_L(x)\rho_L(x)\right) + g_2\rho_R(x)\rho_L(x) && (2.78)\\ &= \frac{1}{2\pi}\int dx\, \frac{g_4 - g_2}{2\pi}(\partial_x\theta)^2 + \frac{g_4 + g_2}{2\pi}(\partial_x\theta)^2. && (2.79) \end{aligned}$$

Remarkably this is of exactly the same form as the free Hamiltonian, allowing us to write

$$H = \frac{1}{2\pi}\int dx\, u \left[K(\partial_x\theta)^2 + \frac{1}{K}(\partial_x\phi)\right], \tag{2.80}$$

which is readily diagonalized to produce

$$H = \sum_k uk\tilde{b}_k^\dagger \tilde{b}_k + \text{const}, \tag{2.81}$$

and the ground state has $\tilde{b}_k|\psi\rangle = 0$.

The effects of interactions are parameterized by the coefficients u and K. In particular, repulsive interactions correspond to $K < 1$, and attractive to $K > 1$.

The relationship between θ, ϕ and the $\tilde{b}$'s is the same as the relationship with the b's, except for a scaling. In particular, if we write $\tilde{\theta} = \sqrt{K}\theta$ and $\tilde{\phi} = \phi/\sqrt{K}$, these scaled fields have the same commutation relations as the original,

$$[\tilde{\phi}(x), \partial_x\tilde{\theta}(x)] = [\phi(x), \partial_x\theta(x)] = i\pi\delta(x-y). \tag{2.82}$$

But in these scaled variables, H looks like the free Hamiltonian,

$$H = \frac{1}{2\pi}\int dx\, u \left[(\partial_x\tilde{\theta})^2 + (\partial_x\tilde{\phi})\right], \tag{2.83}$$

and hence

$$\begin{aligned} \tilde{\phi} &= \frac{\phi}{\sqrt{K}} = \left(\frac{-i\pi}{L}\right)\sum_p \left(\frac{L}{2\pi}\right)^{1/2} \frac{e^{-ipx}}{\sqrt{|p|}\,\text{sign}\, p}\left(\tilde{b}_p^\dagger + \tilde{b}_{-p}\right) && (2.84)\\ \tilde{\theta} &= \sqrt{K}\theta = \left(\frac{i\pi}{L}\right)\sum_p \left(\frac{L}{2\pi}\right)^{1/2} \frac{e^{-ipx}}{\sqrt{|p|}}\left(\tilde{b}_p^\dagger - \tilde{b}_{-p}\right). && (2.85) \end{aligned}$$

Chapter 3

Applications – Mar 20, 2019

In this lecture we will use the results from the previous lecture to calculate various physical quantities. We will need the following facts. First, for an eigenstate of any quadratic Hamiltonian

$$\langle b_1 b_2 b_3 b_4 \rangle = \langle b_1 b_2 \rangle \langle b_3 b_4 \rangle + \langle b_1 b_3 \rangle \langle b_2 b_4 \rangle + \langle b_1 b_4 \rangle \langle b_2 b_3 \rangle, \tag{3.1}$$

where each b is either a $\tilde{b}_q$ or a $\tilde{b}_q^\dagger$. This fact, known as Wick's theorem, is proven in any standard book on many-body theory. It is a straightforward argument which we skip here. A related fact is that for an eigenstate of a quadratic Hamiltonian, and any operator $\hat{X}$ with $\langle \hat{X} \rangle = 0$, one has

$$\langle e^{\langle \hat{X} \rangle} \rangle = e^{\langle X^2 \rangle / 2}. \tag{3.2}$$

Finally, we need that for the ground state

$$\langle b_1 b_2 \rangle = 0 \tag{3.3}$$

unless there is some q such that $b_2 = \tilde{b}_q^\dagger$ and $b_1 = \tilde{b}_q$

A. Occupation Number

The first thing that we want to calculate is the occupation number $\langle \hat{n}_k \rangle = \langle c_k^\dagger c_k \rangle$. If we choose k near k_f, we can just consider

$$\hat{n}_{k_f+q} = c_{rq}^\dagger c_{rq} \tag{3.4}$$

$$= \frac{1}{L} \int dx \int dx' \, \psi_r^\dagger(x) \psi_r(x') e^{iq(x-x')}. \tag{3.5}$$

Taking the expectation value we have

$$\langle \hat{n}_{k_f+q} \rangle = \frac{1}{L} \int dx \int dx' \, \langle \psi_r^\dagger(x) \psi_r(x') \rangle e^{iq(x-x')}. \tag{3.6}$$

We then note that the expectation value should be translationally invariant, so

$$\langle \hat{n}_{k_f+q} \rangle = \int dx \, \langle \psi_r^\dagger(x) \psi_r(0) \rangle e^{iqx}. \tag{3.7}$$

We now use our bosonized expression

$$\psi_r(x) = A \hat{F}_r e^{-i(\phi - \theta)}, \tag{3.8}$$

which gives

$$\langle \hat{n}_{k_f+q} \rangle = A^2 \int dx \, \langle e^{i(\phi(x)-\theta(x))} e^{-i(\phi(0)-\theta(0))} \rangle e^{iqx}. \tag{3.9}$$

We can combine the two terms using "Baker-Campbell-Hausdorff," which says that if $[A, B] = C$, and C is a constant, then

$$e^A e^B = e^{A+B+(1/2)[A,B]}. \tag{3.10}$$

In the present case

$$[\phi(x), \theta(0)] = \frac{i\pi}{2} \operatorname{sign}(x), \tag{3.11}$$

$$[\theta(x), \phi(0)] = \frac{i\pi}{2} \operatorname{sign}(x), \tag{3.12}$$

and hence

$$\langle \hat{n}_{k_f+q} \rangle = A^2 \int dx \, \langle e^{i(\phi(x)-\phi(0))-i(\theta(x)-\theta(0))+i\pi \operatorname{sign}(x)} e^{-i(-\theta(0))} \rangle e^{iqx} \tag{3.13}$$

$$= A^2 \int dx \, \langle e^{i(\phi(x)-\phi(0))-i(\theta(x)-\theta(0))} \rangle e^{iqx} \tag{3.14}$$

At this point we use Eq. 3.2, to get

$$\langle \hat{n}_{k_f+q} \rangle = A^2 \int dx \, e^{-\frac{1}{2}\langle [\phi(x)-\phi(0))-i(\theta(x)-\theta(0)]^2 \rangle} e^{iqx}. \tag{3.15}$$

Thus we see that calculating the momentum distribution reduces to calculating the correlators of the fields ϕ and θ.

We start with

$$\langle [\phi(x) - \phi(x')]^2 \rangle = K \left(\frac{-i\pi}{L} \right) \left(\frac{L}{2\pi} \right) \sum_{pq} \left(\frac{e^{-ipx} - e^{-ipx'}}{\sqrt{|p|} \operatorname{sign}(p)} \right) \left(\frac{e^{-iqx} - e^{-iqx'}}{\sqrt{|q|} \operatorname{sign}(q)} \right) \times \langle (\tilde{b}_p^\dagger + \tilde{b}_{-p})(\tilde{b}_q^\dagger + \tilde{b}_{-q}) \rangle. \tag{3.16}$$

The operator expectation value is 1 if $q = -p$ and otherwise is 0, and hence

$$\langle [\phi(x) - \phi(x')]^2 \rangle = K \frac{\pi}{L} \sum_p \frac{\sin^2 \frac{p(x-x')}{2}}{|p|}. \tag{3.17}$$

In the thermodynamic limit we replace the sum with an integral, and,

$$\langle [\phi(x) - \phi(x')]^2 \rangle \quad = \quad K \int_0^\infty dp \, \frac{\sin^2 \frac{p(x-x')}{2}}{p}. \tag{3.18}$$

Unfortunately that integral is infinite. In particular, the $\sin^2$ is rapidly oscillating, and (at least in the tails of the integral) can be replaced by $1/2$. We then get the area under the $1/p$ curve, which is infinite.

The standard argument to save us is to say that we are only considering p near the Fermi surface, and hence we can introduce a cutoff. A better argument is to use a finite range interaction to cut things off. Nonetheless, we introduce a cutoff Λ, and replace the $\sin^2$ with $1/2$ – this latter substitution only works if $k(x - x') > 1$, and hence we also introduce a lower cutoff $1/(x - x')$. Thus

$$\langle [\phi(x) - \phi(x')]^2 \rangle \quad = \quad K \log[\Lambda(x - x')]. \tag{3.19}$$

Similarly we get

$$\langle [\theta(x) - \theta(x')]^2 \rangle \quad = \quad \frac{1}{K} \log[\Lambda(x - x')], \tag{3.20}$$

and

$$\langle [\phi(x) - \phi(x')] \, [\theta(x) - \theta(x')] \rangle \quad = \quad 0. \tag{3.21}$$

Plugging this into our expression for the occupation numbers yields

$$n_{k_f+q} \quad \sim \quad \int dx \, \frac{e^{iqx}}{|x|^{(K+K^{-1})/2}} \tag{3.22}$$

$$\sim \quad q^{(K+K^{-1})/2-1}. \tag{3.23}$$

And hence, the occupation number has a power law singularity at the Fermi surface.

In the non-interacting case, $K = 1$, instead we have that n_k is flat, and we have a discontinuity at the Fermi surface.

A.1. Other Correlation Function

We can do the same arithmetic with the density-density correlation function. In particular,

$$\rho(x) \quad = \quad \rho_r + \rho_l + e^{2ik_f x} \psi_r^\dagger \psi_l + e^{-2ik_f x} \psi_l^\dagger \psi_r \tag{3.24}$$

$$= \quad -\frac{1}{\pi} \partial_x \phi + A^2 e^{2ik_f x} e^{2i\phi} + A^2 e^{-2ik_f x} e^{-2i\phi}. \tag{3.25}$$

The first part is smooth, while the latter show oscillations on the scale of $2k_f$. Turning the crank yields

$$\langle \rho(x)\rho(0)\rangle = \bar{\rho}^2 - \frac{K}{(2\pi)^2}\frac{1}{x^2} + (\cdots)\cos(2k_f x)\left(\frac{1}{x}\right)^{2K}. \tag{3.26}$$

We see that the larger K is, the faster the $2k_f$ oscillations decay: Attractive interactions make these oscillations decay faster, while repulsive makes them decay slower. One way to interpret this is that repulsive interactions lead to a fluctuating charge density wave.

A complementary variable is the "pairing operator,"

$$\Delta(x) = \pi^\dagger(x)\psi^\dagger(x+a), \tag{3.27}$$

where a is some small number. If a was zero, the Δ involves taking two spinless fermions from the same location in space, and hence Δ would vanish. For small, but finite a, Δ is related to how likely it is that two fermions are close together. Being sloppy about the arguments,

$$\Delta(x) \approx e^{2ik_f x}\psi_R^\dagger\psi_R^\dagger + e^{-2ik_f x}\psi_L^\dagger\psi_L^\dagger + \psi_R^\dagger\psi_L^\dagger + \psi_L^\dagger\psi_R^\dagger. \tag{3.28}$$

The first two terms are typically small, and

$$\Delta(x) \sim e^{2i\theta} + e^{-2i\theta}, \tag{3.29}$$

is obviously complementary to ρ – which involved ϕ's. Clearly

$$\langle \Delta(x)\Delta^\dagger(0)\rangle \sim \frac{1}{x^{1/2K}}. \tag{3.30}$$

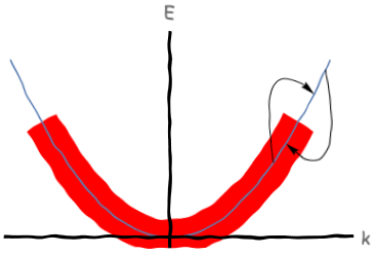
Attractive interactions make the pair correlations stronger.

A.2. Spin

I just want to very quickly sketch out what happens when one adds spin. In the absense of interactions the $\uparrow$ and $\downarrow$ fermions are independent, so everything we did goes through with a $\phi_\downarrow$, $\phi_\uparrow$, $\theta_\downarrow$ and $\theta_\uparrow$. The Free Hamiltonian is

$$H_0 = \frac{1}{2\pi}\int dx\, v_f\left[(\partial_x\theta_\uparrow)^2 + (\partial_x\theta_\downarrow)^2 + (\partial_x\phi_\uparrow)^2 + (\partial_x\phi_\downarrow)^2\right]. \tag{3.31}$$

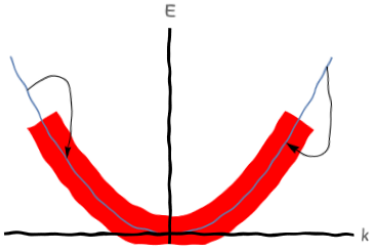
The same interaction terms as before occur, but now we should be careful about the spin. The g_4 processes



can occur with both spins, and gives terms

$$\begin{aligned} H_4 &\propto \int dx\,[\rho_r^2 + \rho_l^2] \\ &\propto \int dx\,[\partial_x(\phi_\uparrow + \theta_\uparrow + \phi_\downarrow + \theta_\downarrow)]^2 + [\partial_x(\phi_\uparrow - \theta_\uparrow + \phi_\downarrow - \theta_\downarrow)]^2 \end{aligned} \tag{3.32}$$

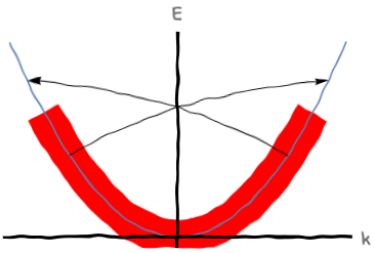
The g_2 processes



give

$$\begin{aligned} H_2 &\propto \int dx\,[\rho_r \rho_l] \\ &\propto \int dx\,\partial_x(\phi_\uparrow + \theta_\uparrow + \phi_\downarrow + \theta_\downarrow)\partial_x(\phi_\uparrow - \theta_\uparrow + \phi_\downarrow - \theta_\downarrow) \end{aligned} \tag{3.33}$$

Finally, the g_1 processes



separate into sets of terms. First if the spins are all the same we get

$$\begin{aligned} H_1^a &\propto \int dx\,\rho_{r\uparrow}\rho_{l\uparrow} + \rho_{r\downarrow}\rho_{l\downarrow} \\ &\propto \int dx\,\partial_x(\phi_\uparrow + \theta_\uparrow)\partial_x(\phi_\uparrow - \theta_\uparrow) + \partial_x(\phi_\downarrow + \theta_\downarrow)\partial_x(\phi_\downarrow - \theta_\downarrow). \end{aligned} \tag{3.34}$$

The terms where the spins are different have another structure

$$H_1^b \propto \int dx\, \psi_{r\uparrow}^\dagger \psi_{l\downarrow}^\dagger \psi_{r\downarrow}\psi_{l\uparrow} + \psi_{r\downarrow}^\dagger \psi_{l\uparrow}^\dagger \psi_{r\uparrow}\psi_{l\downarrow} \tag{3.35}$$

$$\propto \int dx\, \cos(2i[\phi_\uparrow - \phi_\downarrow]). \tag{3.36}$$

These terms can be combined to give a Hamiltonian which separately involves spin and charge degrees of freedom. In particular we define

$$\phi_\rho = \frac{1}{\sqrt{2}}(\phi_\uparrow + \phi_\downarrow) \tag{3.37}$$

$$\phi_\sigma = \frac{1}{\sqrt{2}}(\phi_\uparrow - \phi_\downarrow) \tag{3.38}$$

$$\theta_\rho = \frac{1}{\sqrt{2}}(\theta_\uparrow + \theta_\downarrow) \tag{3.39}$$

$$\theta_\sigma = \frac{1}{\sqrt{2}}(\theta_\uparrow - \theta_\downarrow), \tag{3.40}$$

in terms of which

$$H = H_\rho + H_\sigma \tag{3.41}$$

where

$$H_\rho = \int dx(\cdots)(\partial_x\theta_\rho)^2 + (\cdots)(\partial_x\phi_\rho)^2 \tag{3.42}$$

$$H_\sigma = \int dx(\cdots)(\partial_x\theta_\sigma)^2 + (\cdots)(\partial_x\phi_\sigma)^2 + (\cdots)\cos(2\sqrt{2}\phi_\sigma). \tag{3.43}$$

The charge degrees of freedom can be analyzed using the techniques we have already discussed. The spin degrees of freedom are a bit more complicated: It turns out there is a phase transition between a phase where ϕ_σ is frozen (which it turns out corresponds to pairing) and one where it fluctuates. The phase transition can be mapped onto the classical Kosterlitz-Thouless transition of a 2D XY model.

B. HW 2 – Not Due

These problems are just to help understanding. You neither need to do them nor to hand them in.

Problem 3.1. Squeezed States

We can "solve" the Harmonic oscillator

$$H_0 = \frac{\hat{p}^2}{2} + \frac{\hat{x}^2}{2}, \tag{3.44}$$

with $[\hat{x}, \hat{p}] = i$, by the transformation:

$$\hat{a} = \frac{1}{\sqrt{2}}(\hat{x} + i\hat{p}) \tag{3.45}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\hat{x} - i\hat{p}), \tag{3.46}$$

in terms of which

$$H_0 = a^\dagger a + \frac{1}{2}. \tag{3.47}$$

The operators $\hat{a}, \hat{a}^\dagger$ obey the standard ladder operator commutators, and are ladder operators for this Hamiltonian. The ground state is defined by

$$\hat{a}|\psi_0\rangle = 0. \tag{3.48}$$

Consider now the Hamiltonian

$$H = \alpha\left(\lambda^2\frac{\hat{p}^2}{2} + \frac{1}{\lambda^2}\frac{\hat{x}^2}{2}\right), \tag{3.49}$$

3.1.1. Show that $\bar{x} = \hat{x}/\lambda$ and $\bar{p} = \lambda\hat{p}$ obey the same commutation relations as $\hat{x}$ and $\hat{p}$.

3.1.2. Use that result to argue that

$$H = \alpha\left(b^\dagger b + \frac{1}{2}\right), \tag{3.50}$$

where b and $b^\dagger$ obey the standard commutation relations. Express b and $b^\dagger$ in terms of $\hat{x}$ and $\hat{p}$.

3.1.3. Express a and $a^\dagger$ in terms of b and $b^\dagger$.

3.1.4. Consider the state $|\psi\rangle$ defined by

$$b|\psi\rangle = 0. \tag{3.51}$$

This is a "squeezed state" in the original variables. Find

$$\langle\psi|a^\dagger a|\psi\rangle \tag{3.52}$$

$$\langle\psi|aa^\dagger|\psi\rangle \tag{3.53}$$

$$\langle\psi|aa|\psi\rangle \tag{3.54}$$

$$\langle\psi|a^\dagger a^\dagger|\psi\rangle. \tag{3.55}$$

The relevance to the Luttinger liquid is that the ground state of the interacting system is exactly this sort of squeezed state.

3.1.5. A more tedious approach to the same problem is to consider the Hamiltonian

$$H = A(a^\dagger a + aa^\dagger) + B(aa + a^\dagger a^\dagger). \tag{3.56}$$

Show that Eq. (3.56) is of the same form as Eq. (3.49), and relate α and λ to A and B.

I am not going to make you go through it, but the "Bogoliubov" approach to diagonalizing Eq. (3.56) is to make the general transformation:

$$a = ub + vb^\dagger \tag{3.57}$$
$$a^\dagger = ub^\dagger + vb, \tag{3.58}$$

with $u^2 - v^2 = 1$. One can easily verify that the b's obey the standard ladder operator commutation relations. Substituting into Eq. (3.56) yields

$$H = \left[A(u^2+v^2) + B(2uv)\right](b^\dagger b + bb^\dagger) + \left[B(u^2+v^2) + A(2uv)\right](b^\dagger b^\dagger + bb) \tag{3.59}$$

To diagonalize the Hamiltonian, one chooses u and v to obey

$$u^2 - v^2 = 1 \tag{3.60}$$
$$B(u^2+v^2) + A(2uv) = 0. \tag{3.61}$$

One traditionally writes $u = \cosh(\xi)$ and $v = \sinh(\xi)$, in which case

$$u^2 + v^2 = \cosh(2\xi) \tag{3.62}$$
$$2uv = \sinh(2\xi), \tag{3.63}$$

and

$$\tanh(2\xi) = -B/A. \tag{3.64}$$

Clearly we require $|A| \geq |B|$. Simple "trig" then gives

$$u^2 + v^2 = \frac{A}{\sqrt{A^2 - B^2}} \tag{3.65}$$
$$2uv = -\frac{B}{\sqrt{A^2 - B^2}}, \tag{3.66}$$

and

$$H = \sqrt{A^2 - B^2}(b^\dagger b + bb^\dagger) \tag{3.67}$$

which should agree with the less tedious "scaling" method.

Problem 3.2. This is mostly an exercise in book-keeping. Feel free to skip it if it seems too dull. Nonetheless going through this once is useful – and if you end

up doing more 1D stuff, getting fast at it is useful, as you need to keep doing this to find all your π's and $-$ signs.

3.2.1. In the main text we defined:

$$\phi_r(x) = i\frac{2\pi}{L}\sum_p \frac{e^{ipx}}{p}\rho_r(-p) \tag{3.68}$$

$$\phi_\ell(x) = i\frac{2\pi}{L}\sum_p \frac{e^{ipx}}{p}\rho_\ell(-p), \tag{3.69}$$

and we have

$$\hat{b}_p = \theta(p)\sqrt{\frac{2\pi}{L|p|}}\rho_r(-p) + \theta(-p)\sqrt{\frac{2\pi}{L|p|}}\rho_\ell(p) \tag{3.70}$$

or for $p > 0$

$$\rho_r(-p) = \sqrt{\frac{Lp}{2\pi}}b_p \tag{3.71}$$

$$\rho_\ell(p) = \sqrt{\frac{Lp}{2\pi}}b_{-p} \tag{3.72}$$

$$\rho_r(p) = \sqrt{\frac{Lp}{2\pi}}b_p^\dagger \tag{3.73}$$

$$\rho_\ell(-p) = \sqrt{\frac{Lp}{2\pi}}b_{-p}^\dagger. \tag{3.74}$$

Write ϕ_r and ϕ_ℓ in terms of the b's.

3.2.2. Write $\phi = (\phi_r - \phi_\ell)/2$ and $\theta = -(\phi_r + \phi_\ell)/2$ in terms of the b's.

3.2.3. As argued in the main text $\rho = \partial_x\phi/\pi$, and hence ϕ has the physical meaning of a displacement field. Show that

$$[\phi(x), \partial_y\theta(y)] = i\pi\delta(x-y), \tag{3.75}$$

and hence we can interpret $\Pi(x) = \partial_x\theta(x)/\pi$ as the conjugate momentum to θ.

3.2.4. The Hamiltonian is

$$\hat{H}_0^b = \sum_p v_f p b_p^\dagger b_p. \tag{3.76}$$

Show that

$$\hat{H}_0^b = \frac{v_f}{2\pi}\int dx\, \left[(\partial_x\theta)^2 + (\partial_x\phi)^2\right]. \tag{3.77}$$

Chapter 4

Tonks Gas and the Leib-Lininger Model – Mar 27, 2019

In the previous chapters we explored the "bosonization" techniques, largely in the context of spin-free fermions. The remarkable result there was that the low energy physics was describable in terms of a 1D elastic medium, and that physical quantities, such as the momentum occupation space occupation numbers, could be expressed in terms of corelators of the fields describing this elastic medium. Both a positive and negative feature of this bosonization approach is that it is constructed in terms of operators. In the next two chapters we consider a complementary point of view, and explicitly construct the wavefunction of a 1D system. This is clearly more concrete, but it means that calculating physical quantities is a little more difficult.

The approach, widely known as the "Bethe Ansatz" is applicable to a number of systems with "point interactions." It has been applied to bosons (the Lieb-Lininger model), spin-1/2 fermions (the Gaudin-Yang model), spin-1/2 lattice fermions (the Hubbard model), and spin chains (the XXZ model). We will explicitly discuss only the simplest of these, namely the boson system.

.1. Permutations

Permutations play an important role in our arguments, so it is good to briefly introduce some notation that we will use. Consider the permutations of n objects. These form a group, S_n, consisting of $n!$ elements, and known as the "symmetric group". We think about the permutations as functions which map

the integers $1, 2, \cdots, n$ onto themselves. For example, with $n = 5$, we might have $\sigma(1) = 3, \sigma(2) = 2, \sigma(3) = 4, \sigma(4) = 1, \sigma(5) = 5$. The composition of two permutations is a permutation, and we just use standard "multiplication" notation for composition. So if $\tau(1) = 5, \tau(2) = 2, \tau(3) = 1, \tau(4) = 3, \tau(5) = 4$, then we can define their product $\lambda = \sigma\tau$ by $\lambda(1) = \sigma(\tau(1)) = \sigma(5) = 5$, and so on. A permutation is uniquely defined by how it scrambles the ordered set, for example $1, 2, 3, 4, 5$. Thus we can write $\sigma = [3, 2, 4, 1, 5]$ or $\tau = [5, 2, 1, 3, 4]$.

Another useful notation for permutations is to decompose them into disjoint cycles. For example $\sigma = (134)(2)(5)$, meaning that $\sigma(1) = 3$ and $\sigma(3) = 4$ and $\sigma(4) = 1$. Up to cyclical permutations of each cycle, this decomposition is unique. We can safely ignore the 1-cycles (or unicycles as Veit suggested) as they don't do anything. Hence $\sigma = (1, 2, 3)$ and $\tau = (1, 5, 4, 3)$.

A further refinement is that each cycle can be written as a product of non-disjoint 2-cycles, also known as transpositions. For example, $(1, 2, 3) = (1, 2)(2, 3)$. The decomposition into transpositions is not unique, but the parity of the number of transpositions is. The permutations consisting of even numbers of transpositions forms a subgroup known as the "alternating group," A_n.

Clearly the 2-cycles generate S_n. It turns out that a minimal set of generators is just the "neighboring" transpositions $(1, 2), (2, 3), \cdots (n-1, n)$. For example $(1, 3) = (1, 2)(2, 3)(1, 2)$. This decomposition is not unique.

.2. 1D Hard Core Gas

Let us now think about bosons with hard-core interactions. This is a bit of a warm-up, that will introduce some of the basic concepts. The hard core Bose gas is often known as the Tonks-Girardeau model, and sometimes it seems that keeping track of the names of the various models is half the work. [For those who care, Tonks actually studied the classical model, but nonetheless his name is still associated with this system.]

As we know from solving the "particle in an infinite well," infinitely strong potentials can be replaced by a boundary condition on the wavefunction. By the same argument, the hard core Bose gas can be described by a free-particle Schrodinger equation,

$$-\frac{1}{2}\sum_j \partial_j^2 \psi(x_1, \cdots, x_n) = E\psi(x_1, \cdots, x_n), \tag{4.1}$$

with the boundary condition that $\psi = 0$ if any of the two x's are equal. We should also probably add the Bose symmetry requirement that ψ is unchanged by permuting any two of the particles. Finally, if we want a finite particle density, we should put this in a box of length L. Lets use periodic boundary

conditions, so $\psi(x_1 + L) = \psi(x_1)$, where I have held all the other coordinates fixed.

There is a wonderful trick for solving this problem. Namely we introduce a function $A(x_1, x_2, \cdots x_n)$ which is always either 1 or -1. It equals 1 if $x_1 < x_2 < \cdots < x_N$, or if the ordering of the particles is an even permutation. It is -1 if it is an odd permutation. Given A, and a bosonic wavefunction ψ_B that obeys our hard core boundary conditions, we can always generate a fermionic wavefunction $\psi_F = A\psi_B$. Conversely, given an arbitrary fermionic wavefunction ψ_F we can generate a hard-core Bose wavefunction vie $\psi_B = A\psi_F$. Clearly ψ_B obey EQ. (4.1) if and only if ψ_F also does. Thus we have established a one-to-one mapping between hard core Bose wavefunctions, and non-interacting Fermi wavefunctions in one dimension.

In particular, the fermionic wavefunction can always be expressed as a Slater determinant

$$\psi_F = \begin{vmatrix} e^{ik_1x_1} & e^{ik_1x_2} & \dots & e^{ik_1x_n} \\ e^{ik_2x_1} & e^{ik_2x_2} & \dots & \\ \vdots & & & \end{vmatrix} \tag{4.2}$$

$$= \sum_{\sigma \in S_n} \mathrm{sig}(\sigma) \prod_j e^{ik_{\sigma(j)}x_j} \tag{4.3}$$

where $k_1, k_2, \cdots k_n$ are distinct wave-vectors, satisfying $k_j = n_j(2\pi/L)$, and n_j are distinct integers. The energy is simply $E = \sum_j k_j^2/2$, and the ground state consists of choosing all of the n_j's to have as small as possible magnitudes – ie forming a Fermi surface.

There is a minor subtlety worth mentioning. Namely, if there are an even number of particles, and we want the bosonic wavefunction to have periodic boundary conditions, then the fermions need to have antiperiodic boundary conditions, ie $k_j = (n_j + 1/2)(2\pi/L)$.

It is also useful to note that one can explicitly write down the ground state wavefunction by using the fact that for numbers $z_1, z_2, \cdots z_n$, the Vandermond determinant can be evaluated as

$$\begin{vmatrix} 1 & 1 & 1 & \cdots \\ z_1 & z_2 & z_3 & \cdots \\ z_1^2 & z_2^2 & z_3^2 & \dots \\ \vdots & & & \end{vmatrix} = \prod_{i<j} (z_i - z_j). \tag{4.4}$$

This can be proven by noting that both sides vanish when $z_i = z_j$, and both

are n'th order homogeneous polynomials. We then write

$$\begin{aligned}
\psi_f &= e^{-i\frac{n\pi}{L}\sum_j x_j} \begin{vmatrix} 1 & 1 & \cdots & 1 \\ e^{2\pi i x_1/L} & e^{2\pi i x_2/L} & \cdots & e^{2\pi i x_n/L} \\ e^{4\pi i x_1/L} & e^{44\pi i x_2/L} & \cdots & e^{4\pi i x_n/L} \\ \vdots & & & \end{vmatrix} && (4.5)\\
&= e^{-i\frac{n\pi}{L}\sum_j x_j} \prod_{i<j} (e^{2\pi i x_i/L} - e^{2\pi i x_j/L}(&& (4.6)\\
&\propto \prod_{i<j} \sin\frac{\pi}{L}(x_i - x_j) && (4.7)
\end{aligned}$$

The bosonic wavefunction is then

$$\psi_b = \prod_{i<j} \sin\frac{\pi}{L}|x_i - x_j|. \tag{4.8}$$

The remarkable thing is that all thermodynamic properties of the hard core bosons and the free fermions are identical. Also the density, or the density-density correlations functions should be identical. I should caution that other properties are not the same. For example, the k's in the determinant, referred to as rapidities, do not correspond to the momenta of the bosons. The momentum distribution is

$$n_k = \propto \int dx_1' \int dx_1\, dx_2 \cdots dx_n e^{-ik(x-x')} \psi^*(x_1', x_2, \cdots x_n)\psi(x_1, x_2, \cdots x_n). \tag{4.9}$$

At small k, $n_k \sim k^{-1/2}$, while for large k, $n_k \sim k^{-4}$.

A. Spin Chains

To further illustrate the principle of mapping onto a Fermi system, we can consider a chain of spin-1/2 objects,

$$H = \sum_j J_x(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y) + J_z(S_j^z S_{j+1}^z). \tag{4.10}$$

This is referred to as the XXZ model, and represents some sort of anisotropic magnetic interactions. The isotropic case $J_x = J_z$ is the Heisenberg model, while the extremes case $J_x = 0$ and $J_z = 0$ are the Ising model and XY models.

The operators S^x, S^y, S^z are Pauli operators. It is convenient to introduce ladder operators S^+ and S^-, in terms of which

$$H = \sum_j \frac{J_x}{2}(S_j^+ S_{j+1}^- + S_j^- S_{j+1}^+) + J_z(S_j^z S_{j+1}^z). \tag{4.11}$$

We map this onto a Fermi gas by identifying a down-spin site with an empty site, and an up-spin site with an occupied site. One naturally expects something like

$$S_j^+ = f_j^\dagger \tag{4.12}$$
$$S_j^- = f_j \tag{4.13}$$
$$S_j^z = 2f_j^\dagger f_j - 1. \tag{4.14}$$

One can readily verify that the $\{f_j, f_j^\dagger\} = 1$. Unfortunately the f_j's on different sites commute. We can fix this with what is known as a "Jordan-Wigner" transform

$$a_j^\dagger = (-1)^{\sum_{k<j} f_k^\dagger f_k} f_j^\dagger \tag{4.15}$$
$$a_j = (-1)^{\sum_{k<j} f_k^\dagger f_k} f_j, \tag{4.16}$$

which gives

$$H = \frac{J_x}{2} \sum_i \left(a_i^\dagger a_{i+1} + a_{i+1}^\dagger a_i \right) + J_z \sum_i (2n_i - 1)(2n_{i+1} - 1). \tag{4.17}$$

This is the Hubbard model. Note the special case $J_z = 0$ corresponds to free Fermions, and we can trivially solve that case.

Chapter 5

Bethe Ansatz – Mar 29, 2019

The Bethe ansatz is a generalization of the approach in Sec. .2 – and is best illustrated using a generalization of the model discussed their, namely a gas of 1D bosons with point interactions,

$$-\frac{1}{2}\sum_j \partial_j^2\psi + c\sum_{i<j}\delta(x_i - x_j)\psi = E\psi. \tag{5.1}$$

As before, we impose bosonic symmetry, and periodic boundary conditions in a box of size L. The limit $c \to \infty$ corresponds to the Tonks gas, which we solved by mapping onto free fermions. The limit $c \to 0$ corresponds to the free bosons, with $\psi = 1$.

We will approach this problem, by first solving the two-body problem, then the three-body problem, then generalizing to arbitrary particle number.

A. Two-Body Problem

The two-body problem is a classic undergraduate quantum mechanics problem. In lecture I jumped to the answer, but in these notes I will go through all of the details (since I never remember them, and have to do it from scratch anyways. We begin with

$$-\frac{\partial_1^2 + \partial_2^2}{2}\psi(x_1, x_2) + c\delta(x_1 - x_2)\psi(x_1, x_2) = E\psi(x_2, x_2). \tag{5.2}$$

We change variables to center-of-mass and relative:

$$y = x_1 - x_2 \tag{5.3}$$
$$s = \frac{x_1 + x_2}{2}. \tag{5.4}$$

We then use the chain rule to find:

$$\left(\frac{\partial\psi}{\partial_{x_1}}\right)_{x_2} = \left(\frac{\partial\psi}{\partial_y}\right)_s \left(\frac{y}{\partial_{x_1}}\right)_{x_2} + \left(\frac{\partial\psi}{\partial_s}\right)_y \left(\frac{s}{\partial_{x_1}}\right)_{x_2} \tag{5.5}$$
$$= \left(\frac{\partial\psi}{\partial_y}\right)_s + \frac{1}{2}\left(\frac{\partial\psi}{\partial_s}\right)_y. \tag{5.6}$$

We then repeat to get

$$\partial_1^2 = \partial_y^2 + \partial_y\partial_s + \frac{1}{4}\partial_s^2. \tag{5.7}$$

To get ∂_2^2, we just flip the sign of y, so

$$\partial_2^2 = \partial_y^2 - \partial_y\partial_s + \frac{1}{4}\partial_s^2. \tag{5.8}$$

Hence, we can rewrite Eq. (5.9) as

$$-\partial_y^2\psi - \frac{1}{4}\partial_s^2\psi + c\delta(y)\psi = E\psi. \tag{5.9}$$

We then integrate over y around $y = 0$ to yields

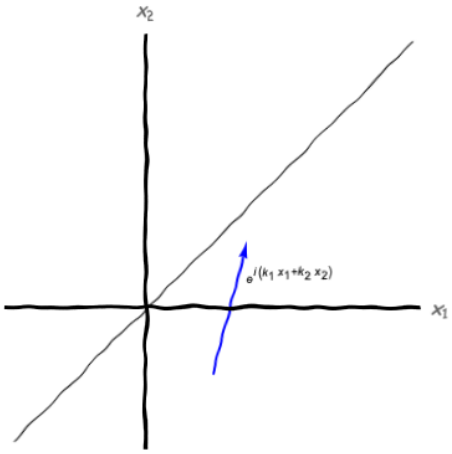
$$\partial_y\psi|_{y=0^-}^{y=0^+} = c\,\psi|_{y=0}\,. \tag{5.10}$$

We now convert this back to our original variables to get

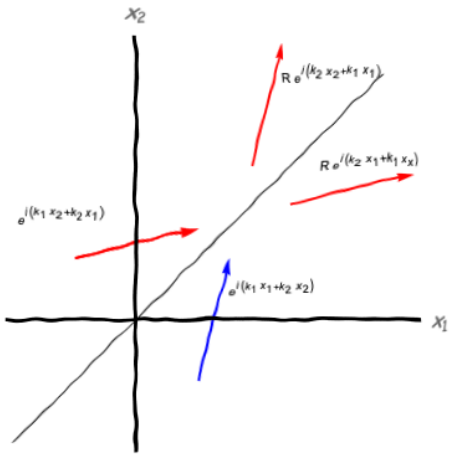
$$(\partial_1 - \partial_2)\psi|_{x_1=x_2-\epsilon}^{x_1=x_2+\epsilon} = 2c\,\psi|_{x_1=x_2}\,. \tag{5.11}$$

The only tricky thing with that whole argument is getting the factors of 2 right. We can think of Eq. (5.9) as a free Schrodinger equation with the boundary condition in Eq. (5.11).

Lets now think about this as a scattering problem in the $x_1 - x_2$ plane. In our elementary classes, we solve scattering problems by imagining an incoming plane wave, and an outgoing plane wave, then choosing amplitudes and phases so that the boundary conditions are satisfied. Thus we imagine an incoming wave, $e^{i(k_1x_1+k_2x_2)}$, in the regime $x_1 > x_2$:



By Bose symmetry, one must also have an incoming wave $e^{i(k_1 x_2 + k_2 x_1)}$ with $x_1 < x_2$. One will also end up with reflected and transmitted rays from each of these two incoming waves. The reflected ray from the $x_1 > x_2$ wave has the same wave-vector as the transmitted ray from the $x_1 < x_2$ wave, so we only need one coefficient R to describe both these processes. And hence we draw the various rays as:



To be explicit, the wavefunction for $x_1 > x_2$ is

$$\psi_{12}(x_1, x_2) = e^{ik_1 x_1 + ik_2 x_2} + Re^{ik_2 x_1 + ik_1 x_2}, \tag{5.12}$$

and the wavefunction for $x_2 > x_1$ is

$$\psi_{21}(x_1, x_2) = e^{ik_1 x_2 + ik_2 x_1} + Re^{ik_2 x_2 + ik_1 x_1}. \tag{5.13}$$

We now implement the boundary condition in Eq. (5.11), and calculate

$$\begin{aligned}
(\partial_1 - \partial_2)\psi|_{x_1=x_2+\epsilon} &= (\partial_1 - \partial_2)\psi_{12}|_{x_1=x_2} && (5.14)\\
&= i(k_1 - k_2)(1 - R)e^{i(k_1+k_2)x} && (5.15)\\
(\partial_1 - \partial_2)\psi|_{x_1=x_2-\epsilon} &= (\partial_1 - \partial_2)\psi_{21}|_{x_1=x_2} && (5.16)\\
&= -i(k_1 - k_2)(1 - R)e^{i(k_1+k_2)x} && (5.17)\\
\psi_{x_1=x_2} &= (1 + R)e^{i(k_1+k_2)x}. && (5.18)
\end{aligned}$$

For the boundary condition to be satisfied, we need

$$i(1 - R)(k_1 - k_2) = c(1 + R) \tag{5.19}$$

which can be manipulated into

$$\begin{aligned}
R &= -e^{i\phi} && (5.20)\\
&= -\left(\frac{c - i(k_1 - k_2)}{c + i(k_1 - k_2)}\right), && (5.21)
\end{aligned}$$

which defines the angle

$$\phi = 2\arctan\frac{k_2 - k_1}{c}. \tag{5.22}$$

The first observation is that it makes sense for ϕ to simply be a phase: this is consistent with conservation of probability. One may be curious about why I included the minus sign in $R = -e^{i\phi}$. The reason is that with this definition, in the hard core case $\phi = 0$. It turns out that the expansion about this hard core limit is better behaved than the expansion about the non-interacting limit, so it is convenient to set up the notation in anticipation.

A second observation is that the boundary condition can be satisfied by any k_1 and k_2. This is analogous to the observation in section .2, that the hard core constraints and Bose symmetry could be satisfied by any Slater determinant. In both that case, and this one, the quantization of the k's comes from imposing periodic boundary conditions:

$$\psi(x_1 + L, x_2) = \psi(x_1, x_2). \tag{5.23}$$

If we take $x_1 < x_2$, this requires

$$\psi_{12}(x_1 + L, x_2) = \psi_{21}(x_1, x_2) \tag{5.24}$$

which means

$$e^{ik_1L}e^{ik_1x_1+ik_2x_2} + Re^{ik_2L}e^{ik_2x_1+ik_1x_2} = e^{ik_1x_2+ik_2x_1} + Re^{ik_2x_2+ik_1x_1} \tag{5.25}$$

This must be true for all x_1 and x_2, so we need

$$Re^{ik_2L} = 1 \tag{5.26}$$
$$e^{ik_1L} = R. \tag{5.27}$$

If I multiply these two equations I find

$$e^{i(k_1+k_2)L} = 1. \tag{5.28}$$

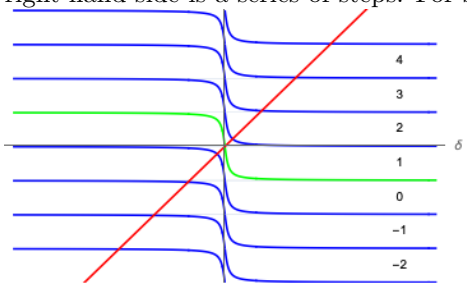
This is just quantization of the center of mass momentum. If I divide the equations, I get

$$e^{i(k_1-k_2)} = R^2 = e^{2i\phi} \tag{5.29}$$

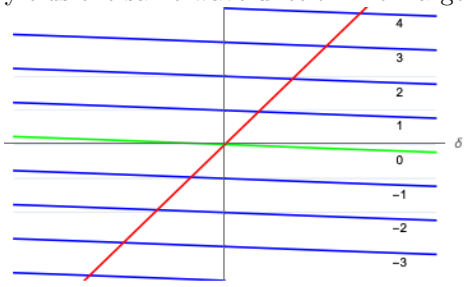
Taking the logarithm, and defining $\delta = k_1 - k_2$, we have

$$\delta = 2\log\frac{c-i\delta}{c+i\delta} + 2\pi n \tag{5.30}$$
$$= 2\pi n - 4\arctan\frac{\delta k}{c}. \tag{5.31}$$

We can solve this graphically. The left hand side is a straight line, while the right hand side is a series of steps. For smallish c, it looks like



The arctangent lines are labeled by n. The $n = 0$ case gives $\delta = 0$, which gives $R = -1$, and $\psi = 0$. Thus we should neglect the $n = 0$ line, which I colored green. As $c \to 0$, the remaining solutions approach $\delta = 2\pi n/L, n = \pm 0, \pm 1, \pm 2, \cdots$. They come in plus-minus pairs because permuting k_1 and k_2 yields the same wavefunction. For large c, the curves become flat:



Again as $c \to \infty$ we have $\delta = 2\pi n/L$, but this time $n = \pm 1, \pm 2, \cdots$. [Recall, the green curve should be ignored.]

If $c < 0$ we also have complex solutions, which correspond to bound states. We will not consider these here.

B. Three Particles

We now solve the three-particle problem. We simply follow the same approach as the two-particle problem, and the only challenge is book-keeping. Instead of dividing space into two sectors we divide it into six sectors: $x_1 < x_2 < x_3$, $x_2 < x_1 < x_3$,$\cdots$. I denote the wavefunction when $x_1 < x_2 < x_3$ as ψ_{123}, and when $x_2 < x_1 < x_3$ as ψ_{213}, $\cdots$. More formally, for a permutation σ we define ψ_σ to describe the sector where $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}$. We then postulate that in each sector the wavefunction is a sum of plane waves, which we can think of as all the rays which happen when one of the particles scatter off one-another,

$$\begin{aligned}\psi_{123}(x_1,x_2,x_3) \quad = \quad & A^{123}_{123}e^{i(k_1x_1+k_2x_2+k_3x_3)} + A^{123}_{213}e^{i(k_2x_1+k_1x_2+k_3x_3)} \\ & + A^{123}_{132}e^{i(k_1x_1+k_3x_2+k_2x_3)} + A^{123}_{231}e^{i(k_2x_1+k_3x_2+k_1x_3)} \\ & + A^{123}_{312}e^{i(k_3x_1+k_1x_2+k_2x_3)} + A^{123}_{321}e^{i(k_3x_1+k_2x_2+k_1x_3)}.\end{aligned} \tag{5.32}$$

That is A^{123}_τ is the coefficient of $e^{i(k_{\tau(1)}x_1+k_{\tau(2)}x_2+k_{\tau(3)}x_3)}$. In other sectors one would define

$$\begin{aligned}\psi_{213}(x_1,x_2,x_3) \quad = \quad & A^{213}_{123}e^{i(k_1x_2+k_2x_1+k_3x_3)} + A^{213}_{213}e^{i(k_2x_2+k_1x_1+k_3x_3)} \\ & + A^{213}_{132}e^{i(k_1x_2+k_3x_1+k_2x_3)} + A^{213}_{231}e^{i(k_2x_2+k_3x_1+k_1x_3)} \\ & + A^{213}_{312}e^{i(k_3x_2+k_1x_1+k_2x_3)} + A^{213}_{321}e^{i(k_3x_2+k_2x_1+k_1x_3)}.\end{aligned} \tag{5.33}$$

That is, in the sector where $x_{\sigma(1)} < x_{\sigma(2)} < x_{\sigma(3)}$, the factor A^σ_τ is the coefficient of $e^{i(k_{\tau(1)}x_{\sigma(1)}+k_{\tau(2)}x_{\sigma(2)}+k_{\tau(3)}x_{\sigma(3)})}$. The generalization to more particles should be clear.

We now consider Bose symmetry, which says that $\psi_{123}(x,y,z) = \psi_{213}(y,x,z)$. Writing things out, we find $A^{123}_\tau = A^{213}_\tau$. Clearly the generalization is that A^σ_τ is independent of σ. This generalization works for all particle numbers.

We now turn to the boundary conditions, for example, lets take x_3 to be bigger than x_1 and x_2, and consider the condition

$$(\partial_1 - \partial_2)\psi\big|^{x_1=x_2+\epsilon}_{x_1=x_2-\epsilon} = 2c\,\psi\big|_{x_1=x_2}\,. \tag{5.34}$$

This can be expressed as

$$(\partial_1-\partial_2)\psi_{213}(x,x,x_3)-(\partial_1-\partial_2)\psi_{123}(x,x,x_3) = 2c\psi_{213}(x,x,x_3) = 2c\psi_{123}(x,x,x_3). \tag{5.35}$$

There are three sorts of terms, proportional to $e^{i(k_1+k_2)x+k_3x_3}$, $e^{i(k_2+k_3)x+k_1x_3}$, and $e^{i(k_1+k_3)x+k_2x_3}$. One can equate the coefficient of each of these. Lets look at the terms proportional to $e^{i(k_1+k_2)x+k_3x_3}$:

$$\left[i(k_2-k_1)A^{213}_{123}+i(k_1-k_2)A^{213}_{213}\right]-\left[i(k_1-k_2)A^{123}_{123}+i(k_2-k_1)A^{213}_{123}\right]=2c(A_{123}+A_{213}) \tag{5.36}$$

We added the superscripts just to make it more obvious where each term came from. We can then group terms, and find

$$\frac{A_{213}}{A_{123}}=R_{12}=-e^{i\theta_{12}}=-\frac{c+i(k_1-k_2)}{c-i(k_1-k_2)}. \tag{5.37}$$

We can similarly define

$$R_{ij}=-e^{i\theta_{ij}}=-\frac{c+i(k_i-k_j)}{c-i(k_i-k_j)}, \tag{5.38}$$

which relates

$$\frac{A_{jik}}{A_{ijk}}=\frac{A_{kji}}{A_{kij}}=R_{ij}. \tag{5.39}$$

That is, because the boundary conditions relate permutations of neighboring particles, we get relations between A's where neighboring indices are exchanged. Because the neighboring transpositions generate all the permutations, we can find a general A_σ in terms of A_{123}. We can further always choose $A_{123}=1$. That is

$$\begin{aligned}
A_{123} &= 1 && (5.40)\\
A_{213} &= R_{12} && (5.41)\\
A_{132} &= R_{23} && (5.42)\\
A_{231} &= R_{31}R_{12} && (5.43)\\
A_{312} &= R_{13}R_{23} && (5.44)\\
A_{321} &= R_{12}R_{13}R_{23}. && (5.45)
\end{aligned}$$

These are clearly consistent with one-another: There are two distinct ways to reach A_{321} from A_{123}, but they give the same R's. The generalization to more particles will also be consistent.

For any set of k's, our ansatz solves Schrodinger's equation (as long as no two k's are not identical – for otherwise the wavefunction vanishes). Adding periodic boundary conditions will then restrict the allowed k's. Take $0 <= x_1 < x_2 < x_3 < L$, then

$$\psi(x_1+L,x_2,x_3)=\psi(x_1,x_2,x_3). \tag{5.46}$$

The wavefuction of the left is clearly ψ_{231}, and the wavefunction on the right is ψ_{123}. There are several ways to do the arithmetic, but I like to use the Bose symmetry so that these are both in the 123 sector,

$$\psi_{123}(x_2, x_3, x_1 + L) = \psi_{123}(x_1, x_2, x_3). \tag{5.47}$$

Equating terms with the same x dependence yields

$$e^{ik_3 L} A_{123} \quad = \quad A_{312} = R_{13} R_{23} A_{123} \tag{5.48}$$

and permutations. The way I like to think of it, is with periodic boundary conditions you can go from 123 to 312 in two ways: You do two transpositions, or you simply slip particle number 1 off the left. Equating these gives Eq. (5.48). We consequently find three equations

$$\begin{aligned} R_{13} R_{23} &= e i k_3 L & (5.49)\\ R_{12} R_{32} &= e^{ik_2 L} & (5.50)\\ R_{21} R_{31} &= e^{ik_1 L} & (5.51) \end{aligned}$$

Taking logs yields

$$\begin{aligned} k_3 L &= (\pi + \theta_{13}) + (\pi + \theta_{23}) + 2\pi n_3 & (5.52)\\ k_2 L &= (\pi + \theta_{12}) + (\pi + \theta_{32}) + 2\pi n_2 & (5.53)\\ k_1 L &= (\pi + \theta_{21}) + (\pi + \theta_{31}) + 2\pi n_1, & (5.54) \end{aligned}$$

where the n_j are integers, and

$$\theta_{ij} = -i \log \left(\frac{c + i(k_i - k_j)}{c - i(k_i - k_j)} \right) = 2 \tan^{-1} \frac{k_i - k_j}{c}. \tag{5.55}$$

If we define $\theta_{ii} = 0$, then the general many-particle result is clearly

$$k_j L = \sum_i \theta_{ij} + 2\pi (n_j + \xi/2) \tag{5.56}$$

where $xi = 0$ if there are an odd number of particles, but $\xi = 1$ for an even.

I claim that Eq. (5.56) has a unique solution once we specify the n_j's. The best way that I know to prove this uniqueness is to first show that Eq. (5.56) is equivalent to extremizing the function

$$F = \sum_j \left(\frac{k_j^2}{2} - \frac{2\pi(n_j + \xi/2)}{L} k_j \right) - \frac{1}{2} \sum_{ij} \Lambda_{ij} \tag{5.57}$$

with

$$\Lambda_{ij} = -\frac{i}{L} \left[(c + (k_i - k_j)) \log (c + (k_i - k_j)) + (c - (k_i - k_j)) \log (c - (k_i - k_j)) \right]. \tag{5.58}$$

We then prove that F is concave up. Thus we calculate the second derivatives

$$\frac{\partial^2 F}{\partial k_i^2} = 1 + \frac{2}{L}\sum_{j\neq i}\frac{c}{c^2+(k_i-k_j)^2} \tag{5.59}$$

$$\frac{\partial^2 F}{\partial k_i \partial k_j} = -\frac{1}{L}\frac{2c}{c^2+(k_i-k_j)^2}. \tag{5.60}$$

We now consider the quadratic form

$$G = \sum_{ij}\frac{\partial^2 F}{\partial k_i \partial k_j}x_i x_j \tag{5.61}$$

$$= \sum_i x_i^2 + \frac{1}{L}\sum_j \frac{c}{c^2+(k_i-k_j)^2}(x_i-x_j)^2. \tag{5.62}$$

Clearly, as the sum of positive quantities, G is positive – hence F is concave up.

C. Thermodynamic Limit

While numerically it is quite efficient to simply solve Eq. (5.56) for fixed N, it is appealing to understand how to take the thermodynamic limit. To do so, we consider an odd number of particles, and order the n's such that $n_1 < n_2 < \cdots n_N$. The ground state will have

$$n_j = -(N+1)/2 + j. \tag{5.63}$$

We then try to find the spacing between subsequent k's:

$$(k_j - k_{j-1})\,L = \sum_i (\theta_{ij} - \theta_{ij-1}) + 2\pi. \tag{5.64}$$

In the thermodynamic limit the spacing will be small, and we define the density of k's to be

$$Lf(k_j) = \frac{1}{k_j - k_{j-1}}. \tag{5.65}$$

We then divide Eq. (5.64) by $(k_j - k_{j-1})\,L$ to arrive at

$$1 = \sum_i \frac{\theta_{ij} - \theta_{i,j-1}}{L(k_j - k_{j-1})} + 2\pi f(k_j). \tag{5.66}$$

We then note that in the thermodynamic limit, the term in the sum is a derivative:

$$1 = \sum_i \frac{1}{L}\frac{\partial \theta_{ij}}{\partial k_j} + 2\pi f(k_j). \tag{5.67}$$

Finally we convert the sum to an integral,

$$1 = \int dq\, f(q)\partial_k \theta(q-k) + 2\pi f(k), \tag{5.68}$$

where we have replaced k_j with k and k_i with q. We have also introduced the function

$$\theta(q) = -i \log \frac{c + iq}{c - iq}. \tag{5.69}$$

Taking the derivative, we find the integral equation

$$2\pi f(k) - 1 = \int dq \, \frac{2c}{c^2 + (q-k)^2} f(q). \tag{5.70}$$

This should be supplemented with the constraint

$$\int f(k) dk = \frac{N}{L}. \tag{5.71}$$

The traditional approach to solving this integral equation involves first assuming that in the thermodynamic limit there is a maximum k, called k_0. This is certainly true in the hard core case, so it seems plausible. Thus $f(k) = 0$ for all $|k| > k_0$. We rescale by this maximum, writing $x = k/k_0$. We also rescale c, writing $\lambda = c/k_0$. We also introduce $\gamma = cL/N$, which is a dimensionless measure of the interaction strength. In the thermodynamic limit, one expects that the interactions only enter in this combination. Finally to avoid confusion we write $g(x = k/k_0) = f(k)$.

Equation (5.71) relates λ to g and the physical quantity γ via

$$\lambda = \gamma \int_{-1}^{1} g(x) dx. \tag{5.72}$$

The other equation becomes

$$2\pi g(y) = 1 + 2\lambda \int_{-1}^{1} \frac{g(x)}{\lambda^2 + (x-y)^2} dx. \tag{5.73}$$

One, simple approach to solving these equations is to begin with some trial g_0. one then uses Eq. (5.72) to calculate λ_0. One substitutes this into Eq. (5.73) to get a new g,

$$2\pi g_1(y) = 1 + 2\lambda_0 \int_{-1}^{1} \frac{g_0(x)}{\lambda^2 + (x-y)^2} dx. \tag{5.74}$$

one then iterates.