P653 Take-Home Exam

Due Dec 2, 2005: 5:00pm

Please work alone on this take-home examination. Feel free to use any textbooks/notes that you wish.

Problem 1.

1.1. An experiment measures the susceptibility $\chi(T)$ in a magnet for temperatures T slightly above the ferromagnetic transition temperature T_c. They find their data is fit well by the form

$$\chi(T) = A(T - T_c)^{-1.25} + B + C(T - T_c) + D(T - T_c)^{1.77}.$$

What is the critical exponent γ?

Solution 1.1. As $T \to T_c$ only the most singular part contributes, so $\gamma = 1.25$

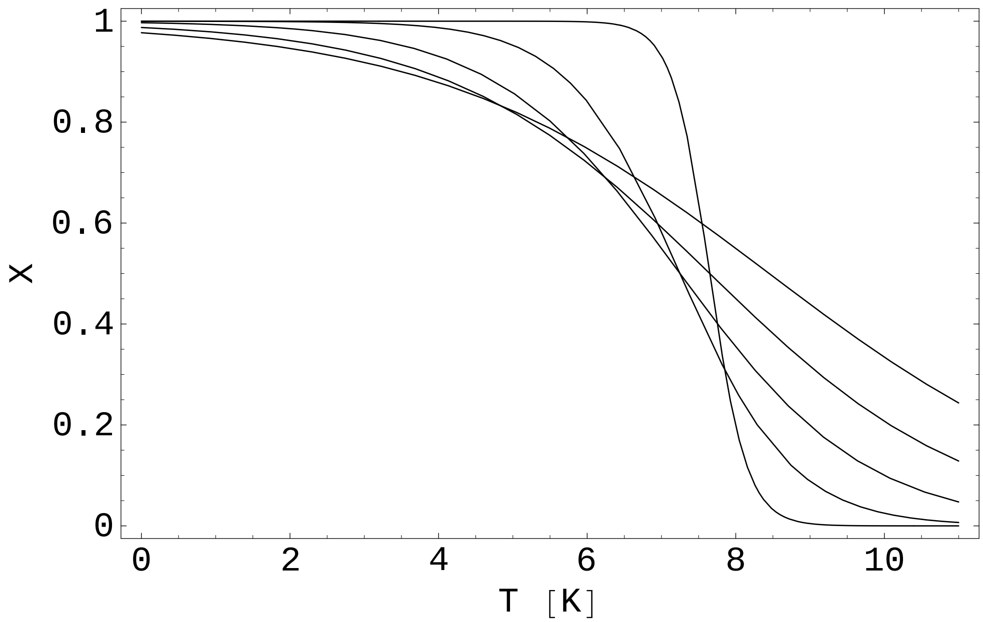
1.2. A different experiment, on a different (three dimensional) material, finds that the spin-spin correlation function is

$$C(r, T) = \langle S(x)S(x + r)\rangle = r^{-1.026} g(r(T - T_c)^{0.65}).$$

What is the critical exponent ν? The exponent η?

Solution 1.2. One expects $C(r, t) \sim r^{-(2-d+\eta)} g(r/\xi)$, where $\xi \sim (T - T_c)^{-\nu}$, so $\nu = 0.65$ and $\eta = 0.026$.

Problem 2. A numerical experiment calculates the ratios of the fourth and second moment of the order parameter $X = \langle m^4\rangle/\langle m^2\rangle^2$ as a function of temperature for different system sizes. Their data is shown below. Is this a first or second order phase transition? Give your reasoning.



Solution 2.1. This is a first order phase transition. For a second order phase transition all of these curves should cross at one point

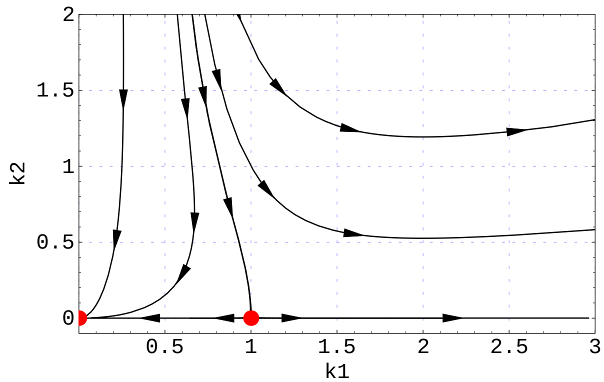
Problem 3. In the following RG flow equations, $K_1 = \beta J_1$ and $K_2 = \beta J_2$ are both externally controlled coupling constants.

$$\ell \frac{\partial K_1}{\partial \ell} = K_1^2 - \frac{K_1}{1 + K_2^2}$$
$$\ell \frac{\partial K_2}{\partial \ell} = K_2(K_1 - 2)$$

3.1. Find the fixed points of this flow for $K_j \geq 0$. [Remember to include the possibility of a fixed-point at infinity.] How many unstable directions do each of them have.

Solution 3.1. There are five fixed points: $(K_1, K_2) = (0,0), (1,0), (\infty, 0), (0, \infty)$, and (∞, ∞). These respectively have $0, 1, 1, 2, 0$ unstable directions.

3.2. Sketch the flow diagram in the $K_1 - K_2$ plane for $K_1 > 0$ and $K_2 > 0$.



Solution 3.2.

3.3. Which fixed point corresponds to the critical point? Which to the high temperature phase? Which to the low temperature phase?

Solution 3.3. The critical point is at $(K_1, K_2) = (1, 0)$, the low temperature phase flows to $(K_1, K_2) = (\infty, \infty)$, and the high temperature phase flows to $(K_1, K_2) = (0, 0)$.

3.4. What is the critical exponent ν?

Solution 3.4. Linearizing about the critical point one writes $K_1 = 1 + k_1$, $K_2 = k_2$, and finds [to linear order]

$$\begin{aligned} \ell\frac{dk_1}{d\ell} &= k_1 \\ \ell\frac{dk_2}{d\ell} &= -k_2, \end{aligned}$$

and k_1 is relevant, while k_2 is irrelevant, with $k_1(\ell) = \ell k_1(1)$, and $k_2(\ell) = \ell^{-1} k_2(1)$. Under scaling the coherence length evolves as $\xi(\ell) = \ell^{-1}\xi(1) \propto k_1(\ell)^{-1}$. Since K_1 is inversely proportional to T, k_1 is proportional to the deviation of T from T_c, implying that $\xi \propto (T - T_c)^{-1}$, and that $\nu = 1$.

3.5. In an experiment where $J_1/k_B = 3$ Kelvin and $J_2/k_B = 1$ Kelvin, what is the transition temperature? [Try for at least 2 significant figures]

Solution 3.5. Given these values of J, the physical system must lie on the line $K_2 = K_1/3$. One needs to find the intersection of this line with the critical manifold. There are really only two reasonable ways that I know how to do this: numerically or with a series expansion around the critical point.

For the series expansion, one begins by noting that any flow line obeys the differential equation

$$\frac{\partial K_1}{\partial K_2} = \frac{\left(\ell \frac{dK_1}{d\ell}\right)}{\left(\ell \frac{dK_1}{d\ell}\right)} = \frac{K_1^2 - \frac{K_1}{1+K_2^2}}{K_2(K_1 - 2)}.$$

As we already saw from linearizing about the critical point, for small K_2, the critical manifold is the line $K_1 = 1$, which gives a first estimate: $T_c = J_1 = 3$ Kelvin. To improve this result we substitute our zeroth order form into the differential equation to get

$$\left.\frac{\partial K_1}{\partial K_2}\right|_{K_1=1} \approx -K_2,$$

which is integrated to find the approximation $K_1 \approx 1 - K_2^2/2$. This intersects the line $K_2 = K_1/2$ when $K_1 = 1 - K_1^2/18 \approx 17/18$. Which gives $T_c = (18/17)J_1 = 3.2$ Kelvin. If you continue in this direction you can get more accuracy.

The exact answer (from integrating the full differential equation and numerically computing the intersection) is that the intersection occurs at $K_1 = 0.968128$, which gives $T_c = 3.09877$ Kelvin.