P653 HW2

Due Sept 8, 2005

Problem 1. Information Theoretic Definition of Entropy

Entropy is a measure of the uncertainty in an ensemble. Here we will use some simple information theoretic ideas (due to Shannon) to calculate the entropy for a generic ensemble – even one which is not in equilibrium.

Our setup is that we know that our system is in one of N states, and that each state i occurs in the ensemble with probability p_i.

Suppose we measure that the system is in state i. Let us assume that we can quantify the amount that we learn from this measurement. Call this k_i. Clearly k_i is larger if the probability p_i is smaller. We will define k_i so that it is only a function of the probability p_i, ie

$$k_i = k(p_i).$$

We will define the entropy of information S_I (for the remainder of this question we drop the I) to be the average amount of information gained from a measurement:

$$S = \sum_i p_i k(p_i).$$

1.1. Entropy, as a state variable is extensive. Suppose we build a larger system out of two independant subsystems A and B. The entropy of the whole should be

$$S = S_A + S_B,$$

where S_A and S_B are the entropies of the parts.

Use this relationship to derive a functional relationship that k must obey.

[Hint 1: Recall that independance means that the probability of A being in state i and B being in state j is $p_{ij} = p_i + p_j$.]

[Hint 2: You will need to use that $\sum_i p_i = 1$.]

1.2. Show that this functional relationship is obeyed by

$$k(x) = -k \log(x),$$

for any constant k.

Prove that these are the only solutions to this relationship. [Feel free to assume that k(x) is differentiable, and you may use standard uniqueness theorems.]

1.3. We need that the entropy will be maximized in equilibrium. What constraint does that place on the sign of k?

As the notation might suggest, we will take the constant k to be Boltzmann's constant. We have therefore arrived at the definition

$$S_I = -k_b \sum_i p_i \log p_i,$$

where the I stands for information.

1.4. Show that the thermodynamic entropy in the microcannonical ensemble is equal to the information entropy.

1.5. Show that the thermodynamic entropy in the cannonical ensemble is equal to the information entropy.

Problem 2. Maximum Entropy Principle Consider the set of all possible states i of a system. Each of these state contains a given number of particles N_i and has energy E_i. We define an ensemble of these states by specifying the probability p_i for the system to be in a given state.

Find the p_i's for the ensemble which maximizes the entropy S_I subject to the constraint that the expectation value of the energy and particle number are fixed,

$$\langle E \rangle = \sum_i p_i E_i$$

$$\langle N \rangle = \sum_i p_i N_i$$

[Don't forget that the probabilities are constrained by $\sum_i p_i = 1$.]

Problem 3. Microscopic origin of dissipation

Here we look at a simple model of a dissipative classical system, namely an oscillator coupled to an oscillator bath. For a quantum treatment, see Feynman and Vernon, Ann. Phys. **24**, 118 (1963).

Consider a harmonic oscillator with position X, momentum P, and frequency Ω, linearly coupled to a bath of other oscillators with positions x_i, momenta p_i and frequencies ω_i. This system obeys a Hamiltonian

$$H = \frac{P^2}{2M} + \frac{1}{2} M \Omega^2 X^2 + \sum_i \left(\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 \right) + \sum_i \lambda_i x_i X.$$

3.1. Write the equations of motion for x_i, p_i, X, and P.

3.2. Fourier transform these equations of motion, using the convention

$$A(t) = \int \frac{d\omega}{2\pi} e^{-i\omega t} A(\omega).$$

3.3. Eliminate $p_i(\omega)$ to write $x_i(\omega)$ as a function of $X(\omega)$.

3.4. Eliminate $x_i(\omega)$ and P from the equations for X, to arrive at an equation of the form,

$$[\omega^2 - \Omega^2 - F(\omega)]X = 0.$$

What is $F(\omega)$?

3.5. It is convenient to introduce a "spectral density"

$$J(E) = \sum_i \frac{\lambda_i^2}{m_i \omega_i} 2\pi \delta(E - \omega_i)$$

which encodes all relevant information about the oscillator bath. Verify that

$$F(\omega) = \frac{1}{2M} \int \frac{dE}{2\pi} \frac{1}{\omega - E} \left[J(E) - J(-E) \right].$$

3.6. In the limit of a large bath, one can assume that $J(E)$ is smooth. We will concentrate on an *ohmic* bath, where $J(E) = \alpha E$ for small E. To be concrete, we will take

$$J(E) = \begin{cases} \alpha E & |E| < E_c \\ 0 & |E| > E_c \end{cases}$$

Furthermore, we will assume that E_c is the largest energy in the problem. Explicitly calculate $F(\omega)$, neglecting terms of order ω/E_c. It may be useful to recall that

$$\frac{1}{x - y} = \frac{P}{x - y} - i\pi\delta(x - y),$$

though this example has been cooked up so that one does not have to explicitly use this formula.

3.7. Substituting this expression for F into equation of motion for X should give the equations of motion for a damped harmonic oscillator. What is the frequency $\bar{\Omega}$ of this oscillator? Note that in most experimental realizations of this system there is no way to measure the bare frequency Ω. Only the renormalized frequency $\bar{\Omega}$ is physical.