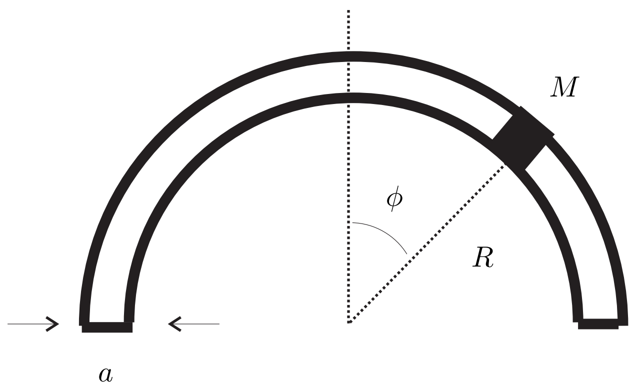
P653 HW3

Due Sept. 15, 2005

Problem 1. Alben model (Plischke and Bergersen 3.14)

The symmetry breaking aspect of second order phase transitions can be nicely illustrated in a simple mechanical model [R. Alben, American Journal of Physics **40**, 3 (1972)]. An airtight piston of mass M is inside a tube of cross sectional area a. The tube is bent into a semicircular shape of radius R.



The system is kept at temperature T. On each side of the piston there is an ideal gas consisting of N atoms. The volume to the right of the piston is $aR(\pi/2+\phi)$, while the volume to the left is $aR(\pi/2-\phi)$. Using the free energy of an ideal gas, $F = -Nk_B[1-\ln(N\lambda^3/V)]$, one finds

$$F = MgR\cos(\phi) - Nk_BT\left[2 + \ln\left(\frac{aR(\pi/2+\phi)}{N\lambda^3}\right) + \ln\left(\frac{aR(\pi/2-\phi)}{N\lambda^3}\right)\right],$$

where $\lambda^2 = \hbar^2/(2\pi m k_B T)$.

1.1. Show by minimizing the free energy that the system undergoes a symmetry breaking phase transition ($\phi \neq 0$) at a temperature

$$T_c = \frac{MgR\pi^2}{8Nk_B}$$

1.2. Plot the "order parameter" ϕ vs T/T_c for $T < T_c$.

1.3. Describe what happens to the phase transition if the number of atoms on the left and right side of the piston is $N(1-\delta)$ and $N(1+\delta)$ respectively.

1.4. At a certain temperature the right chamber (containing $N(1+\delta)$ molecules) is found to contain a puddle of liquid coexisting with its vapor. Which of the following statements may be true at equilibrium:

1. The left chamber will contain a liquid in coexistence with its vapor.
2. The left chamber contains only vapor.
3. The left chamber contains only liquid.

Problem 2. Bethe Approximation (Plischke and Bergersen sec. 3.4)

Here we work through a more general mean field theory for the Ising model, where we include some short range correlations. As before, we select out one site, with spin σ_0, but we also select out all q of its neighboring sites. We assume that none of the neighbors are neighbors to each-other. The Hamiltonian of these $q+1$ sites will be approximated by

$$H_c = -J\sigma_0 \sum_{j=1}^{q} \sigma_j - h\sigma_0 - h' \sum_{j=1}^{q} \sigma_j.$$

The h' is the effective field that each of the q neighbors feels, which includes the effect of all the spins (except σ_0) which are their neighbors.

2.1. Calculate the partition function for this cluster

$$Z_c = \sum_{\sigma_j=\pm 1} e^{-\beta H_c}.$$

2.2. Write an expression for $\langle \sigma_0 \rangle$ when $h=0$.

2.3. Write an expression for $\langle \sigma_j \rangle$ when $h=0$.

2.4. Since the ferromagnet is translationally invarient, we require $\langle \sigma_j \rangle = \sigma_0$. Show that this requirement leads to the formula

$$\frac{\cosh^{q-1}[\beta(J+h')]}{\cosh^{q-1}[\beta(J-h')]} = e^{2\beta h'}. \tag{1}$$

Equation (1) always has the solution $h'=0$. When $h' \to \infty$, the left hand side approaches a constant while the right hand side diverges. Therefore a sufficient condition to have a solution with $h' \neq 0$ is to have the slopes $\partial_{h'}$**L.H.S** $> \partial_{h'}$**R.H.S.** at $h'=0$.

2.5. Find the slopes at $h'=0$, and thereby find the critical temperature, at which the solutions for $h' \neq 0$ exist.

2.6. For the $1-D$ Ising model, $q=2$. What is the critical temperature predicted by this approximation? Is this better than our previous mean field approximation?

2.7. For the $2-D$ Ising model on a square lattice, $q=4$. What is the critical temperature?

2.8. What is the critical temperature in the limit of large q? How does this compare with our previous mean field approximation?

2.9. Near the transition temperature show that the magnetization obeys $m(h=0) \propto |T-T_c|^\beta$. Find β.

Problem 3. Latent Heat (Plischke and Bergersen 3.11)

Consider the Landau free energy

$$G(m,T) = a + \frac{b}{2}m^2 + \frac{c}{4}m^4 + \frac{d}{6}m^6$$

and assume that $b>0$, $c<0$, and $d>0$ so that a first-order transition takes place. Derive an expression for the latent heat of transition. [Hint: read section 3.7]