P653 HW6

Due Oct 6, 2005

Problem 1. Scaling

Here we explore how the renormalization group leads to scaling for coupling constants which are not aligned with the eigendirections of the linearized flow equations. For concreteness we will imagine that the two coupling constants of interest are t and h, the reduced temperature and magnetic field of a spin system. We imagine that these are linearly related to coupling constants K_1 and K_1,

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix}. \tag{1}$$

The constants K_1 and K_2 correspond to eigendirections of the linearized flow equations, so under rescaling by length ℓ,

$$\begin{aligned} K_1' &= \ell^{y_1} K_1 \\ K_2' &= \ell^{y_2} K_2. \end{aligned}$$

Under rescaling the coherence length is reduced by a factor of ℓ so

$$\xi(K_1', K_2') = \frac{1}{\ell}\xi(K_1, K_2).$$

Assume both constants are relevant, so $y_1, y_2 > 0$.

1.1. Prove that ξ can be written in the form

$$\xi(K_1, K_2) = K_2^{-1/y_2}\phi(K_1 K_2^{-y_1/y_2}). \tag{2}$$

Find the asymptotic behavior of $\phi(x)$ as $x \to 0$ and $x \to \infty$.

1.2. Setting $h = 0$, use equation (1) and (2) to write ξ as a function of t. Using the asymptotic properties of ϕ, determine how ξ behaves as $t \to 0$.

1.3. How would this argument change if one of these coupling constants, say K_1, was irrelevant?

Problem 2. Method of Auxilliary Fields

Consider an Ising model of the form

$$H = -\frac{1}{2}\sum_{ij} J_{ij}\sigma_i\sigma_j - \sum_i H_i\sigma_i,$$

where the spins lie on a lattice in d dimensional space, $J_{ij} = J > 0$ if i and j are nearest neighbors, but $J_{ij} = 0$ otherwise. The spins take on values $\sigma = \pm 1$.

If we use Einsteins summation notation, where repeated indices are summed over, this becomes

$$H = -\frac{1}{2} J_{ij}\sigma_i\sigma_j - H_i\sigma_i,$$

2.1. Why is the factor of 1/2 in this Hamiltonian? Show that this is the same as

$$H = -J\sum_{\langle ij\rangle}\sigma_i\sigma_j - \sum_i H_i\sigma_i,$$

2.2. Prove that for any $N \times N$ symmetric matrix A, and any length N vector B, that

$$\int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{2\pi}}\frac{dx_2}{\sqrt{2\pi}}\cdots\frac{dx_N}{\sqrt{2\pi}} e^{-x_iA_{ij}x_j/2 + x_iB_i} = \frac{1}{(\det A)^{1/2}} e^{B_i(A^{-1})_{ij}B_j/2}. \tag{3}$$

Hint: make a change of variables $y_i = x_i - (A^{-1})_{ij}B_j$.

2.3. We want to calculate the partition function

$$Z = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} e^{-\beta H}.$$

By using equation (3) with $(A^{-1})_{ij} = \beta J_{ij}$ and $B_i = \sigma_i$, show that

$$Z = \int_{-\infty}^{\infty} d\psi_1\, d\psi_2 \cdots d\psi_N e^{-\beta S}, \tag{4}$$

where S, is given by

$$e^{-\beta S} = \sum_{\sigma_1,\sigma_2,\ldots,\sigma_N} \sqrt{\frac{\det((\beta J)^{-1})}{(2\pi)^N}} \exp\left[-\frac{1}{2}(\psi_i - \beta H_i)[(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j) + \psi_i\sigma_i\right].$$

2.4. Do the sum over the σ's.

The partition function for this discrete spin model is thus mapped onto the partition function of a theory where a continuous variable ψ_i lives on each site. This is known as a lattice field theory.

If we take the continuum limit we would have a field $\psi(r)$, and the multiple integral in equation (4) would be a functional integral (an integral over the space of functions). We would then have a continuum field theory (usually just called a field theory).

2.5. Saddle point approximation

Assuming that S is a strongly peaked function of ψ, one can approximate $Z \approx e^{-\beta S_0}$, where S_0 is the value of S at the minimum [ie. where $\partial S/\partial\psi_j = 0$].

Let $\bar{\psi}_i$ be the value of ψ_i at the minimum. Find the equation satisfied by $\bar{\psi}_i$ and show that

$$m_i = \langle \sigma_i \rangle = -\frac{\partial F}{\partial H_i} \approx -\partial S_0 / \partial H_i$$

is given by $m_i = \tanh \bar{\psi}_i$. Hence find H_i as a function of $\{m_i\}$.

2.6. Assuming that the $F \approx S_0$, and substituting H_i with m_i, show that

$$F \approx F_0 + \frac{1}{2} J_{ij} m_i m_j - k_B T \sum_i \log\left(\frac{2}{\sqrt{1-m_i^2}}\right),$$

where F_0 is a constant independent of m_i.

2.7. The equilibrium value of m_i is found by minimizing this Free energy. Assuming that $m_i = m$ is uniform, and that each spin has q nearest neighbors, find the temperature at which the system undergoes a phase transition.

Note, this saddle point approximation is equivalent to the mean field approximation that we had made in class.

Problem 3. Free Energy of a Continuum model

To prep ourselves for discussions in class, it will be useful to calculate the exact free energy of a simple continuum model: the Gaussian model. We consider at each place in space there is a real valued field $\phi(r)$, and the free energy is given by a sum over all configurations of the field. Formally we can write

$$Z = e^{-\beta F} = \int \mathcal{D}\phi \, e^{-\beta S[\phi]}.$$

Where $S[\phi]$ is a functional.

The simplest way to define such a *functional integral* is to work in a finite volume so that we may write $\phi(r)$ as a Fourier sum,

$$\phi(r) = \frac{1}{V} \sum_k e^{ik\cdot r} \phi_k.$$

We then define

$$\int \mathcal{D}\phi = \int \prod_k d\phi_k.$$

We will use a simple model where

$$S[\phi] = \int dr \, \frac{\gamma}{2} |\nabla \phi(r)|^2 + \frac{at}{2} |\phi(r)|^2,$$

where $t = (T - T_c)/T_c$ is linear in temperature, and $a, \gamma > 0$.

[Note, one could derive this free energy by taking the continuum limit of problem 2, and expanding to quadratic order.]

Once $t < 0$ this theory is ill-defined, but we can still consider what happens as we approach the critical temperature $t = 0$ from the disordered phase.

We will be thinking of ϕ as a coarse-grained variable, so it only makes sense to talk about it on sufficiently long length-scales. Thus we will also have a momentum scale Λ, and set $\phi_k = 0$ for all $|k| > \Lambda$.

3.1. Write S in terms of the ϕ_k, and perform the integrals over r to arrive at

$$e^{-\beta F} = \int \prod_k d\phi_k \exp - \left[\frac{\beta}{2V} \sum_k (at + \gamma k^2) \phi_k \phi_{-k}) \right]$$

3.2. By noting that $\phi(r)$ is real, show that $\phi_{-k} = \phi_k^*$.

3.3. Let w be a complex number. Calculate the Gaussian integral

$$I = \int dw\, dw^* \, e^{-a|w|^2} = 2i \int \, dx\, dy\, e^{-a(x^2+y^2)},$$

where $w = x + iy$. You can take this to be the definition of the measure $dw\, dw^*$.

3.4. Use this result to calculate the partition function. Do not worry about multiplicative constants (such as 2π's) which will play no role in any thermodynamic derivatives.

3.5. With what power of t does the specific heat diverge as $t \to 0^+$? How does this compare with mean-field theory results.