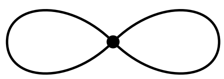
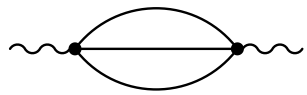
P653 HW8

Due Nov 3, 2005

Problem 1. Feynman Diagrams Here are a series of exercises intended to practice your proficiency with Feynman diagrams.

1.1. Write the integrals which correspond to the following diagrams


(a)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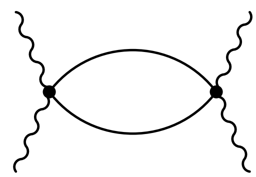
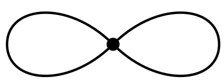
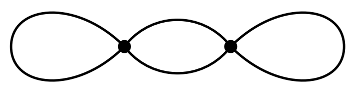

(c)

1.2. What are the multiplicities of the following diagrams?


(a)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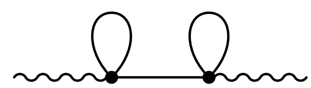
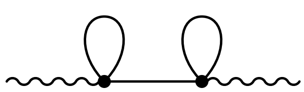
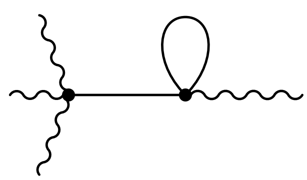
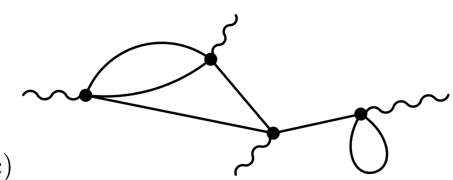

(c)

1.3. Prove that the following diagrams evaluate to zero. [The most straightforward approach is to convert the diagrams to integrals. Make sure to keep track of the limits of integration!! By the time you are on part (c), you should be able to give an argument that doesn't require writing out the integral. As an extra hint, note that there is a part of the diagram which is the same in each case.]


(a)


(b)


(c)

1.4. In class we explicitly showed that if you consider the terms which contain no ϕ''s (but do contain ψ's) that $\langle\langle V^2\rangle\rangle = \langle V^2\rangle - \langle V\rangle^2$ consists only of connected diagrams. Give the same construction for terms which contain two ϕ''s. i.e. construct all of the diagrams for $\langle V^2\rangle$ which contain two ϕ''s, and subtract off $\langle V\rangle^2$ to explicitly show that $\langle\langle V^2\rangle\rangle$ only contains connected diagrams.

Problem 2. Surface Area of a d-dimensional Sphere Here we calculate the area of a d-dimensional sphere by doing a Gaussian integral two ways. Let

$$F = \int d^d q \, e^{-q^2}.$$

First evaluate F by converting to Cartesian coordinates. Next evaluate F by using spherical coordinates, assuming that the surface area of a d-dimensional sphere of unit radius is S_d. Equating these two expressions gives you a formula for S_d.

Give your result in terms of the Gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Problem 3. n-vector model We can easily generalize our discussion of the Ising model to higher dimensional spins: for example, the x-y model uses 2-dimensional spins, and the Heisenberg model uses 3-dimensional spins. We will work with n dimensional spins, in which case the Landau free energy will be of the form

$$-H = \int d^d r \sum_{\alpha=1}^{n} \left\{ \left[\frac{1}{2} \nabla\phi^\alpha(r) \cdot \nabla\phi^\alpha(r) + \frac{r_0}{2}\phi^\alpha(r)^2 \right] + \sum_{\alpha,\beta=1}^{n} \frac{u_0}{4} \phi^\alpha(r)\phi^\alpha(r)\phi^\beta(r)\phi^\beta(r) \right\}$$

where α, β represent the component of the spin.

As before, we can produce the true free energy from a functional integral

$$e^{-F} = \int \prod_\alpha \mathcal{D}\phi^\alpha e^H.$$

Following the procedure we carried out in class, calculate the flow equations for r_0 and u_0 to first order in V. If you set $n = 1$ these should reduce to the equations we found in class.

Problem 4. Quantum-Classical Correspondence Here we will demonstrate that the quantum mechanical statistical mechanics of a single spin-1/2 is equivalent to the classical statistical mechanics of a 1-D Ising chain. This is a special case of the general result that the statistical mechanics of a d-dimensional quantum model is equivalent to the statistical mechanics of a d+1 dimensional classical model.

We will discuss the general argument later in class, but this example illustrates the basic idea.

Throughout we will consider a single spin with Hamiltonian

$$\hat{H} = E_0 - \frac{\Delta}{2}\hat{\sigma}_x - h\hat{\sigma}_z.$$

where $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the standard Pauli matrices.

4.1. Diagonalize H, to find the energy eigenvalues E_α. Use the definition of the partition function,

$$Z = \sum_\alpha e^{-\beta E_\alpha}$$

to show that

$$F = E_0 - T \log\left(2\cosh\beta\sqrt{(\Delta/2)^2 + h}\right).$$

4.2. We will now do the same calculation in a different basis. In an arbitrary basis the partition function is

$$Z = \sum_i \langle i | e^{-\beta \hat{H}} | i \rangle.$$

We will work in the standard basis aligned with the $\hat{z}$ direction.

In this language we need to calculate the matrix

$$e^{-\beta \hat{H}} = e^{-\beta \left(E_0 - \frac{\Delta}{2} \hat{\sigma}_x - h \hat{\sigma}_z \right)}.$$

We will use a simple trick to calculate this exponential of a matrix. We are going to write

$$e^{-\beta \hat{H}} = T^N,$$

where $T = e^{-\beta \hat{H}/N}$. Show that in the limit of large N,

$$T \approx \begin{pmatrix} e^{-\beta(E_0 - h)/N} & \beta\Delta/2N \\ \beta\Delta/2N & e^{-\beta(E_0 + h)/N} \end{pmatrix}$$

4.3. We now note that if we let $E_0 = (N/\beta)\log(\beta\Delta/2N)$, then T has the same form as the transfer matrix for the 1-D classical Ising model. Using this correspondence write down a classical model which is equivalent to the quantum problem of a single spin.

Verify that this classical system has the same free energy as the single quantum spin.

This quantum-classical correspondence lets us either use our knowledge of classical thermodynamics to solve problems in quantum thermodynamics, or it allows us to solve thermodynamics problems by studying quantum systems. For example, Onsager's famous solution of the 2-D Ising model simply involves mapping the 2-D Ising model onto a 1-D quantum mechanics problem. The quantum problem turns out to just involve non-interacting fermions, and is trivially solved.