P653 HW9

Due Nov 10, 2005

Problem 1. Susceptability of O(n) model

Consider a system of spins which sit in d-dimensional space and which can point in n dimensions, with Landau Free energy

$$F = \int d^d r (c/2)(\nabla_\mu S_i(r))(\nabla_\mu S_i(r)) + (a/2)S_i(r)S_i(r) + (b/4)(S_i(r)S_i(r))(S_j(r)S_j(r)) - h_i(r)S_i(r),$$

where Einstein summation is assumed with μ running from 1 to d and i, j running from 1 to n. We will flip between using integers and letters to denote the directions. For example if I say $\mathbf{h} = h\hat{x}$, that is equivalent to saying $h_j = \delta_{j1}h$.

In class we considered the case where $h = 0$, here we will consider $h \neq 0$.

1.1. First consider the case where $\mathbf{h}(r) = h_1\hat{x}$ is uniform and points in the $\hat{x}$ direction [ie $h_i = h\delta_{j1}$]. Minimize F, and show that $\mathbf{S} = S\hat{x}$, and S satisfied the cubic equation

$$aS + bS^3 - h = 0.$$

1.2. Fix $b > 0$, and find the boundary in the $h - a$ plane between where this equation has one and three solutions. For $h \neq 0$ this defines the spinodal.

1.3. Let $\mathbf{h} = h\hat{x} + \delta h_\| e^{ik\cdot r}\hat{x}$, and let $\mathbf{S} = S\hat{x} + \delta S e^{i\cdot r}\hat{x}$ minimize the free energy. Calculate δS to linear order in $\delta h_\|$. Your expression may contain S.

The longitundinal susceptability is defined as

$$\chi_\| = \frac{\delta S}{\delta h_\|}$$

Verify that when $h \to 0$ you recover the expression from class

$$\chi_\| \Big|_{h=0} = \frac{1}{ck^2 + 2|a|}.$$

1.4. What happens to the longitudinal susceptibility in the metastable state at the spinodal? [ie. at the spinodal, one of the minima disapears. Evaluate the susceptibility of that metastable state.]

1.5. Show that even in the presence of nonzero $\delta h_\|$ that $S_j = 0$ for all $j \neq 1$, and hence that

$$\chi_{yx} = \frac{\delta S_y}{\delta h_x} = 0,$$

where $h_x = h_\|$.

1.6. Now lets consider a transverse perturbation. Let $\mathbf{h} = h\hat{x} + \delta h_\perp e^{ik\cdot r}\hat{y}$, and let $\mathbf{S} = S\hat{x} + \delta S_y e^{i\cdot r}\hat{y}$ minimize the free energy. To linear order in $\delta h_\perp$, calculate δS_y. Show that as $h \to 0$ one recovers the result from class that

$$\chi_\perp|_{h=0} = \frac{1}{ck^2},$$

Problem 2. Continuum limit of x-y model Consider a microscopic x-y model on a square lattice in two dimensions,

$$H = -J \sum_{\langle i \langle j} \mathbf{S_i} \cdot \mathbf{S_j} = -JS^2 \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j),$$

where S is the length of the spins, and θ defines their directions. We will derive a continuum version of this model, and evaluate the energy of some important quantities.

2.1. Suppose that θ_i varies slowly from one site to the next. Let $\theta(r)$ be a smooth function for which $\theta(r_i) = \theta_i$. Show that

$$H \approx \int d^2r \frac{-JS^2}{2} |\nabla\theta(r)|^2,$$

independent of the lattice spacing.

2.2. There can be spin configurations which are not smooth. An example is a vortex: $\theta(r) = \arctan(y/x) = \operatorname{Im}\log(x + Iy)$. This configuration is smooth except for a region near $r = 0$. Let $\xi \sim a$ be a length for which $\theta(r)$ is smooth when $r > \xi$. If the size of the system is L, estimate the contribution to the energy of a vortex configuration from all spins at $r > \xi$. This is described as the region "outside the vortex core".

You should find that this energy diverges as $L \to \infty$.

Hint 1: The continuum approximation works in this region.

Hint 2: Take the sample to be circular in shape.

2.3. Estimate the energy contributions from outside the vortex cores of a vortex-antivortex pair separated by a distance $d(\gg \xi)$: $\theta(r) = \operatorname{Im}\left[\log(x - d/2 + Iy) - \log(x + d/2 + Iy)\right]$.

Hint 1: Take the limit of an infinite system, this energy is finite in that limit.

Hint 2: Use Stoke's Theorem (ie integrate by parts):

$$\int_\Omega d^2r |\nabla\theta|^2 = \int_{\partial\Omega} d\ell \cdot \theta\nabla\theta - \int_\Omega d^2r \theta\nabla^2\theta.$$

Note that $\nabla^2\theta = 0$. Look out for branch cuts.

Problem 3. Correlation functions in harmonic crystal

As a simple model of a crystal, consider a system of particles that want to for a square lattice in d dimensions, with lattice constant a. If we only consider the interaction between neighboring atoms one can approximate the Hamiltonian as

$$H = \sum_{\langle ij \rangle} V(\mathbf{r}_i - \mathbf{r}_j) + \sum_i \frac{\mathbf{p}_i^2}{2m},$$

where r_i is the position of the i'th particle and p_i is the momentum of that particle. We now assume that each particle stays near its equilibrium position, $r_i^{(0)}$, in which case $r_i = r_i^{(0)} + \delta_i$. Presumably V has a minimum at this point, so we can expand and get to an Einstein model,

$$H = \sum_{\langle ij \rangle} \frac{m\omega_0^2}{2} (\delta_i - \delta_j)^2 + \sum_i \frac{\mathbf{p}_i^2}{2m}.$$

3.1. Find the normal modes and their frequencies. What is the energy cost of exciting each of these modes with some given amplitude.

This is a system with a spontaneously broken continuous symmetry. How is Goldstone's theorem manifested in these modes?

3.2. The equipartition theorem says that at finite temperature each degree of freedom should have an energy $kT/2$. Use the equipartition theorem and the normal modes to estimate $\langle |\delta_i|^2 \rangle$ as the system size becomes large.

What happens for $d = 1, 2$?

Hint 1: This result is independant of i, so you might as well take $r_i^0 = 0$.

Hint 2: Turn the sum into an integral. The integral is dominated by the modes of lowest energy. Approximate $\cos(x) \approx 1 - x^2/2$.

3.3. Use the same method to write down an integral for $g_{ij} = \langle \delta_i \delta_j \rangle$ as a function of the distance $r_i^{(0)} - r_j^{(0)}$. How does this integral behave in the infinite system as $r_i^{(0)} - r_j^{(0)} \to \infty$.

How is this related to the Mermin-Wagner theorem?