P653 HW1

Due Sept. 1, 2005

This first homework is mostly mechanical, giving you some intuition about some of the distribution functions that you will encounter throughout the term.

Problem 1. One Sided Distributions
Here we explore the differences between one sided and two sided distributions. This distinction will become important when we are discussing phase transitions.

Consider a random variable $X \in (-\infty, \infty)$, distributed by $P(X) \propto \exp(-(X - X_0)^2/(2W^2))$, where X_0 and W are constants.

1.1. What is the mean $\langle X \rangle$

Solution 1.1. X_0

1.2. What is the variance $\langle X^2 \rangle - \langle X \rangle^2$?

Solution 1.2. W^2

1.3. What is the most probable value of X?

Solution 1.3. X_0

Now consider $Y \in [0, \infty)$, distributed by $P(Y) \propto \exp(-(Y - Y_0)^2/W^2)$.

1.4. If $Y_0 > 0$, what is the most probable value of Y?

Solution 1.4. Y_0

1.5. If $Y_0 < 0$, what is the most probable value of Y?

Solution 1.5. 0

1.6. In terms of the error function

$$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}\, dx,$$

what is the mean $\langle Y \rangle$

Solution 1.6. If one defines

$$f = \int_0^\infty dY\, e^{-\frac{(Y-Y_0)^2}{W^2}},$$

then by differentiating under the integral, on sees that

$$\langle Y \rangle = Y_0 + \left(\frac{W^2}{2}\right) \frac{\partial \log(f)}{\partial Y_0}$$

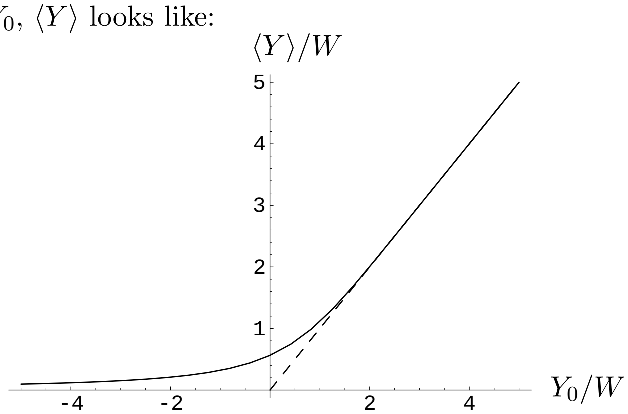
. Breaking the integral into two parts, and shifting the integration variable, one sees that

$$f = \int_0^{Y_0} dY\, e^{-Y^2/W^2} + \int_0^\infty dY\, e^{-Y^2/W^2},$$

implying that

$$\langle Y \rangle = Y_0 + \frac{W}{\sqrt{\pi}} \frac{1}{e^{Y_0^2/W^2}[1 + \mathrm{erf}(Y_0/W)]}$$

Note that this solution is valid for all values of Y_0, both positive and negative. As a function of Y_0, $\langle Y \rangle$ looks like:



The dashed line is $Y = Y_0$.

1.7. If $t = Y_0/W$ satisfies $|t| \ll 1$, what is the leading order expression for $\langle Y \rangle$ and $\langle Y^2 \rangle$?

Hint:

$$e^{-(Y-Y_0)^2/W^2} = e^{-Y^2/W^2} + \mathcal{O}(t)$$

Solution 1.7. Going one step beyond the approximation in the hint,

$$e^{-(Y-Y_0)^2/W^2} = e^{-Y^2/W^2}\,(1 + 2tY/W) + \mathcal{O}(t^2).$$

A few Gaussian integrals then give,

$$\begin{aligned} \langle Y \rangle &= \frac{W}{\sqrt{\pi}} + \left(1 - \frac{2}{\pi}\right) Y_0 + \cdots \\ \langle Y^2 \rangle &= \frac{W^2}{2} + \frac{W Y_0}{\sqrt{\pi}} + \cdots \end{aligned}$$

The thing to note is that the variance $\langle Y^2 \rangle - \langle Y \rangle^2$ is of order $\langle Y \rangle^2$

1.8. If $Y_0 \ll -W$, what is the leading order expression for $\langle Y \rangle$ and $\langle Y^2 \rangle - \langle Y \rangle^2$?

Hint: Expand about the most probable value of Y.

Solution 1.8. I apologize for my notation. I think I confuse a lot of you. What I meant was that $Y_0 < 0$ and $|Y_0| \gg W$.
Since $P(Y)$ is strongly peaked about $Y = 0$, we expand

$$P(y) \propto e^{2YY_0/W^2}.$$

Finding the moments of this approximate distribution give

$$\begin{aligned} \langle Y \rangle &= -W^2/2Y_0 \\ \langle Y^2 \rangle - \langle Y \rangle^2 &= W^4/4Y_0^2, \end{aligned}$$

so that $\langle Y^2 \rangle - \langle Y \rangle^2$ is of order $\langle Y \rangle^2$.

1.9. If $Y_0 \gg W$, what is the leading order expression for $\langle Y \rangle$ and $\langle Y^2 \rangle - \langle Y \rangle^2$.

Hint: Try extending the domain.

Solution 1.9. Since $P(Y)$ is strongly peaked about $Y = Y_0$, the probability of having $Y < 0$ is exponentially small. In particular

$$\int_0^\infty dY\,(Y - Y_0)^n e^{-(Y-Y_0)^2/W^2} = \int_{-\infty}^\infty dx\, x^n e^{-x^2/W^2} - \int_{Y_0}^\infty dx\, x^n e^{-x^2/W^2}$$

The first integral is elementary, and gives the leading order behavior. A perturbative expansion of the second integral is calculated by making the substitution $t = x/W^2$ and then sequentially integrating by parts. The leading order behavior plus the first correction is

$$\begin{aligned} \langle Y \rangle &= Y_0 + \frac{1}{2\sqrt{\pi}} \frac{Y_0^2}{W} e^{-Y_0^2/W^2} \\ \langle Y^2 \rangle - \langle Y \rangle^2 &= \frac{W^2}{2} + \frac{Y_0^3}{3\sqrt{\pi}W} e^{-Y_0^2/W^2}. \end{aligned}$$

In this case, the variance is small compared to the mean.

Problem 2. Bimodal Distributions

Let X be a random variable distributed by $P(x) = (1/2)\delta(x-a) + (1/2)\delta(x+a)$.

2.1. What is the characteristic function?

Solution 2.1.

$$\phi(s) = \cos(as)$$

2.2. What are the moments $\langle X^j \rangle$

Solution 2.2.

$$\langle X^j \rangle = \begin{cases} a^j & j \text{ even} \\ 0 & j \text{ odd} \end{cases}$$

2.3. What are the first four cumulants: $\langle\langle X \rangle\rangle$, $\langle\langle X^2 \rangle\rangle$, $\langle\langle X^3 \rangle\rangle$, $\langle\langle X^4 \rangle\rangle$,

Solution 2.3.

$$\begin{aligned} \langle\langle X \rangle\rangle &= 0 \\ \langle\langle X^2 \rangle\rangle &= a^2 \\ \langle\langle X^3 \rangle\rangle &= 0 \\ \langle\langle X^4 \rangle\rangle &= -2a^4 \end{aligned}$$

Problem 3. Random Number Generators

A typical computer random number generator gives a random number with uniform probability. In this problem we think about generating more general distributions. Throughout, we imagine that we have a random number generator which generates numbers $x \in [0,1]$ with uniform probability.

3.1. Let $y(x)$ be a monotonic function of x. How is the probability distribution $P(y)$ related to the probability distribution $P(x)$?

Solution 3.1. Noting that $\int_X P(x)dx = \int_{y(X)} P(y)dy$, one must have

$$P(y) = P(x)dx/dy.$$

3.2. Construct a function $y(x) : [0,1] \to (-\infty, \infty)$ such that y is distributed by a Gaussian,

$$P(y) = \sqrt{\frac{1}{\pi}}\sigma \exp\left(\frac{-y^2}{\sigma^2}\right).$$

hint: Find $x(y)$, and define y as the inverse of that function.

Solution 3.2. Some of you noticed that I screwed up the normalization. It should be

$$P(y) = \sqrt{\frac{1}{\pi}}\sigma^{-1} \exp\left(\frac{-y^2}{\sigma^2}\right).$$

The function $x(y)$ obeys the differential equation

$$\frac{dx}{dy} = \frac{P(y)}{P(x)} = \sqrt{\frac{1}{\pi}}\sigma^{-1} \exp\left(\frac{-y^2}{\sigma^2}\right)$$

with boundary condition that $x(y = -\infty) = 0$. This is solved by

$$x = [\mathrm{erf}(y/\sigma) + 1]/2,$$

so that

$$y = \sigma \mathrm{erf}^{-1}(2x - 1)$$

.

3.3. (Bonus) Try it out using your favorite programming language: 1. calculate a few hundred thousand y's; 2. bin them; 3. plot a histogram. Is it Gaussian?

Solution 3.3. I will illustrate this with Mathematica. First, lets look at the histogram for the random number generator. First, one loads built in routines for plotting histograms

```
<< Graphics‘Graphics‘
```

We can then produce a histogram of ten thousand random x's with

```
hx = Histogram[Table[Random[], {10000}], HistogramScale -> 1]
```

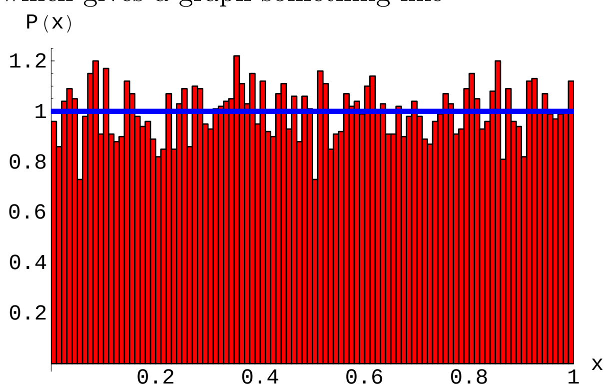
The `HistogramScale->1` scales the plot so that it gives a probability density. One then can compare this with the expected distribuion

```
px = Plot[1, {x, 0, 1}, PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 1]}}]
```

The two graphs are plotted together with

```
cx = Show[hx, px]
```

which gives a graph something like



To get the histogram of y's, one uses

```
hy = Histogram[Table[InverseErf[2Random[] - 1], {10000}], HistogramScale -> 1]
```

The expected distribution is

```
py = Plot[ 1/Sqrt[Pi] Exp[-y^2], {y, -3, 3}, PlotStyle -> {{Thickness[0.01], RGBColor[0, 0, 1]}}]
```

Compare them with

```
cx = Show[hx, px]
```

which gives