P653 HW11

Due Dec 1, 2005

Problem 1. Deterministic laser model
A laser involves a high-Q cavity containing a gain medium which is being pumped. Due to stimulated emission, the rate of photons produced by the gain medium is proportional to the number of photons present. However, at some point this gain must saturate. The maximum rate of photon production is limited by the rate of pumping. A simple model which captures this picture is

$$\left.\frac{dn}{dt}\right|_{\text{gain}} = f_{\text{gain}}(n) = \frac{A(n+1)}{1+(B/A)n},$$

where $A > 0$ is the gain coefficient in the linear regime, $1/B \gg 1$ is the rate of photon production when the process is saturated, and n is the number of photons in the cavity. The 1 in $n + 1$ is due to spontaneous emission into the cavity mode.

The cavity leaks photons (this is how the light gets out). Each photon has a certain probability to escape, leading to a loss

$$\left.\frac{dn}{dt}\right|_{\text{loss}} = f_{\text{loss}}(n) = -Cn,$$

where $C > 0$ is a constant.

1.1. In steady state $dn/dt = 0$. Find the possible steady state populations – note that $n > 0$. Separately consider the cases $A > C$ and $A < C$. Neglect spontaneous emission (ie. replace the $(1 + n)$'s by n's).

Solution 1.1. For $A > C$, the pumping is said to exceed threshold, and there are two steady-state solutions,

$$n = 0, A(A - C)/BC.$$

For $A < C$ there is only one physical steady-state solution,

$$n = 0.$$

This change in the number of solutions is sometimes called a "catastrophe". For $A < C$ the laser is not "lasing": no coherent light is coming out.

1.2. What is the stability of each of these solutions? *ie.* If n is slightly above/below each of them, does n get closer or farther away.

Solution 1.2. For $A > C$, the $n = 0$ solution is unstable, the $n = (A - C)/BC$ solution is stable

Note: The interesting parameter range is where $A \sim C$. This is referred to as "threshold". near threshold one can approximate

$$\left.\frac{dn}{dt}\right|_{\text{gain}} = \frac{A(n+1)}{1+(B/A)n} \approx A(n+1) - Bn^2.$$

Below theshold, where the number of photons quickly drops to 0, one can make the more drastic approximation

$$\left.\frac{dn}{dt}\right|_{\text{gain}} \approx A(n+1).$$

Far above threshold, where the equilibrium number of photons is large, one can make the approximation,

$$\left.\frac{dn}{dt}\right|_{\text{gain}} \approx A^2/B.$$

Problem 2. Stochastic laser model.
A more accurate laser model would instead consider a the probability distribution $p(n)$, which is the probability of having n photons in the cavity.

2.1. Write down a set of coupled differential equations which gives the rate of change of $p(n)$ in terms of $p(n+1)$, $p(n-1)$ and $p(n)$. This is known as a "master equation".

Solution 2.1. The rate of change of $p(n)$ is given by a sum of terms: (1) the probability of having $n-1$ photons times the probability of a stimulated emission event, (2) the probability of having $n+1$ photons times the probability of a photon leaving the cavity, (3) minus the probability of having n photons times the probability of either having an emission event or of loosing a particle:

$$\begin{aligned}
\frac{d}{dt}p(n) &= p(n-1)f_{\text{gain}}(n-1) + p(n+1)|f_{\text{loss}}(n+1)| - p(n)[f_{\text{gain}}(n) + |f_{\text{loss}}(n)|] \\
&= p(n-1)\frac{An}{1+(B/A)(n-1)} + p(n+1)C(n+1) - p(n)\left[\frac{A(n+1)}{1+(B/A)n} + Cn\right]
\end{aligned}$$

2.2. Use the principle of detailed balance to relate the steady state value of $p(n)$ to $p(n+1)$.

Solution 2.2. Detailed balance says that the rate of transitions from n to $n+1$ photons must equal the rate of transitions from $n+1$ to n:

$$\begin{aligned}
p(n+1)|f_{\text{loss}}(n+1)| &= p(n)f_{\text{gain}}(n) \\
C(n+1)p(n+1) &= \frac{A(n+1)}{1+(B/A)n}p(n),
\end{aligned}$$

One can simplify this to

$$\frac{p(n+1)}{p(n)} = \frac{A/C}{1+(B/A)n}$$

2.3. Consider the linear approximation, valid below threshold, where one sets $B = 0$. In this case, your expression from the last problem should read

$$\frac{p(n+1)}{p(n)} = \frac{A}{C}. \tag{1}$$

Assuming that $A < C$, and that $\sum_n p(n) = 1$, use equation (1) to calculate the steady-state probability distribution. Compare your result to the Planck black-body spectrum

$$p(n) = (1 - e^{-\hbar\nu/k_BT})e^{-n\hbar\nu/k_BT}.$$

Note that despite the superficial similarities, the Planck spectrum describes a system in thermal equilibrium, while here we have an open system with energy entering and leaving. Also note that this structure is very different from that seen in the deterministic laser model, where below threshold there were no photons in the system.

Solution 2.3. One can readily verify that

$$p(n) = \left(1 - \frac{A}{C}\right)\left(\frac{A}{C}\right)^n$$

is a solution. This looks like a Planck spectrum if one identifies A/C with $e^{-\hbar\nu/k_BT}$.

2.4. Consider the opposite limit, valid far above threshold, where $f_{\text{gain}} \approx A^2/B$. In this limit, your result from problem 2.2 should reduce to

$$\frac{p(n+1)}{p(n)} = \frac{A^2}{BCn}.$$

Find the steady state distribution. Note: this equation only makes sense if one takes $p(0) = 0$.

Solution 2.4. The easiest way to solve this relationship is to define $p(n) = g(n)/(n-1)!$. Then $g(n)$ obeys

$$\frac{g(n+1)}{g(n)} = \frac{A^2}{BC},$$

which is easily solved to give $g(n) = g(1)(A^2/BC)^{n-1}$. Converting to p and normalizing gives the poissonian

$$p(n) = e^{-A^2/BC}\frac{1}{(n-1)!}\left(\frac{A^2}{BC}\right)^{n-1}$$

Having arrived at some understanding of the limits $A \ll C$ and $A \gg C$, we can now look at the general case. In both 2.3 and 2.4 the distribution was strongly peaked. In the former case it was peaked about zero, while in the latter case it was peaked about a finite value. We will assume that the distribution is always strongly peaked, and expand about the peak. As usual in statistical mechanics, it makes most sense to expand $\log p$, rather than p itself.

2.5. Let $p(n) = \exp(f(n))$. Use your expression from problem 2.2 to relate $f(n+1)$ and $f(n)$

Solution 2.5.

$$f(n+1) - f(n) = \log(A/C) - \log(1 + (B/A)n).$$

2.6. We can replace this difference equation with a differential equation if we assume that f is smooth. To lowest order, $f(n+1) = f(n) + f'(n)$. Derive an equation for $f'(n)$.

Solution 2.6.

$$f'(n) = \log(A/C) - \log(1 + (B/A)n).$$

2.7. Find the most probable n. (Separately consider $A > C$ and $A < C$ – recall $n > 0$.) How does this compare to the results of the deterministic laser model.

Solution 2.7. The most probable n occurs when $f'(n) = 0$, which gives the same result as we had before, If $A > C$,

$$n_{\text{peak}} = A(A - C)/BC,$$

while if $A < C$,

$$n_{\text{peak}} = 0.$$

2.8. Linearize your equation for $f'(n)$ about these peak values. Solve for $f(n)$, and thus calculate $p(n)$ (you don't need to calculate the normalization). You may want to look at homework 1 to remind yourself what these one-sided Gaussian distributions mean.

Solution 2.8. For $A < C$, we expand about $n = 0$, finding

$$f'(n) = \log(A/C) - (B/A)n,$$

so

$$p(n) \propto \exp\left(-\left(\frac{B}{2A}\right)\left[n + \left(\frac{A}{B}\right)\log\left(C/A\right)\right]^2\right).$$

For $A > C$, we expand about $n_{\text{peak}} = A(A - C)/BC$, so that

$$f'(n) = -\frac{BC}{A^2}(n - n_{\text{peak}}).$$

This results in

$$p(n) \propto \exp\left(-\left(\frac{BC}{2A^2}\right)(n - n_{\text{peak}})^2\right)$$

Problem 3. Thomas-Fermi Approximation

In typical experiment on cold atoms the atoms are trapped in some sort of potential [typically by magnetic dipole forces]. On general grounds the trapping force is harmonic.

If the potential is sufficiently smooth one can assume that at every point in space the system is locally homogeneous. That is, at each point in space one defines a local chemical potential

$\mu(r) = \mu_0 - V(r)$, where $V(r)$ is the external potential. The density at that location $n(r)$ is then simply the density of a uniform gas with that chemical potential. This approximation is known as the "Thomas-Fermi" approximation.

Here we will illustrate this approximation by considering a one dimensional gas of noninteracting Fermions in a harmonic trap.

3.1. What is the density of a zero temperature 1D fermi gas in free space with chemical potential μ?

Solution 3.1.

$$n = \int_{-k_f}^{k_f} \frac{dk}{2\pi} = \frac{k_f}{\pi} = \frac{\sqrt{2m\mu}}{\pi}$$

3.2. Using the result from (3.1) and the Thomas-Fermi approximation, $\mu(x) = \mu_0 - V(x)$, calculate the density profile of a trapped 1D gas with chemical potential μ_0 in a trap of the form $V(x) = m\omega^2 x^2/2$.

Solution 3.2.

$$n(x) = \frac{\sqrt{2m}}{\pi}\sqrt{\mu_0 - \frac{1}{2}m\omega^2 x^2}$$

3.3. Given that $N = \int dx\, n(x)$, relate μ_0 to N.

Solution 3.3.

$$N = \int dx n(x) = \mu_0/\omega$$

3.4. The exact density profile for N fermions in a 1-D harmonic trap is

$$\begin{aligned} n(x) &= \sum_{j=0}^{N-1} |\phi_j(x)|^2 \\ \phi_j(x) &= \frac{1}{(\pi d)^{1/4}} \frac{1}{\sqrt{2^n n!}} H_n(x/d) e^{-x^2/(2d^2)}, \end{aligned}$$

where $d = \sqrt{\hbar/m\omega}$ and $H_n(y)$ is the n'th Hermite polynomial.

Numerically graph the exact 5-particle density and compare it with the Thomas-Fermi prediction for $N = 5$.

Solution 3.4. The Thomas-Fermi solution (dashed line) is remarkably close to the exact result (solid line).