P653 HW5

Due Sept 29, 2005

Problem 1. Upper Critical Dimension

By comparing the fluctuations in one coherence volume to the mean field, calculate the upper critical dimension for a system with free energy

$$f = am^2/2 + bm^6,$$

where m is the order parameter. As usual, we assume that a changes sign at the critical point and b is positive. Such a free energy corresponds to a tricritical point.

Solution 1.1. First we need to calculate how big the order parameter is near T_c. Minimizing f gives

$$m = \begin{cases} 0 & a > 0 \\ (\frac{-a}{6b})^{1/4} & a < 0 \end{cases}$$

the first corresponding to $T > T_c$, the latter $T < T_c$. In particular, assuming that a is linear near T_c we have $m \sim |T - T_c|^{1/4}$.

Second we need to calculate the correlation function. To do that we add a term $c|\nabla m|^2$ to the free energy, and look at linear response to a perturbation $h_0 \delta(r)$.

$$G(r) = \frac{\delta \langle m(r) \rangle}{\delta h_0} = \beta \left[\langle m(r) m(0) \rangle - \langle m(r) \rangle \langle m(0) \right] = \beta \Gamma(r).$$

For $a > 0$, the response function obeys a differential equation of the form

$$\nabla^2 G(r) - \frac{a}{2c} G(r) = \frac{-h_0}{2c} \delta(r),$$

which as we showed in class in d dimensions gives something like

$$G(r) \sim r^{2-d} e^{-r/\xi},$$

where the coherence length $\xi = \sqrt{a/2c}$, and thus $\xi \sim |T - T_c|^{-1/2}$.

The fluctuations in a coherence volume are then

$$F = \int_{\Omega(\xi)} d^d r \, \Gamma(r) \sim \int_{\Omega(\xi)} d^d r \, r^{2-d} \sim \xi^2.$$

The strength of the order in a coherence volume is

$$A = \int_{\Omega(\xi)} d^d r \, m_0^2 \sim \xi^d m_0^2.$$

For mean field theory to work, the ratio G/F must be large,

$$\frac{A}{F} \sim m_0^2 \xi^{d-2} \sim |T - T_c|^{(3-d)/2}.$$

Near the critical temperature this requires that $d \geq 3$, and the upper critical dimension is therefore $d_c = 3$.

Problem 2. One dimensional s-state Potts model

Calculate the transfer matrix and free energy of the one-dimensional s-state Potts model with periodic boundary conditions. This is a generalization of the Ising model where the spin on each site σ_j can take on the values $1, 2, \ldots, s$, and has a Hamiltonian

$$H = -K \sum_{j=1}^{N} [s \delta_{\sigma_j \sigma_{j+1}} - 1].$$

Solution 2.1. The partition function is

$$
\begin{aligned}
Z &= \sum_{\sigma_1,\sigma_2,\ldots} e^{-\beta H} \\
&= \sum_{\sigma_1,\sigma_2,\ldots} e^{\beta K[s\delta_{\sigma_1,\sigma_2}-1]} e^{\beta K[s\delta_{\sigma_2,\sigma_3}-1]} \cdots e^{\beta K[s\delta_{\sigma_1,\sigma_2}-1]} \\
&= \mathrm{Tr} P^N,
\end{aligned}
$$

where P is the $s \times s$ transfer matrix with components

$$P_{\sigma\tau} = e^{\beta K[s\delta_{\sigma,\tau}-1]}.$$

That is

$$P = \begin{pmatrix} e^{\beta K(s-1)} & e^{-\beta K} & e^{-\beta K} & \cdots \\ e^{-\beta K} & e^{\beta K(s-1)} & e^{-\beta K} & \\ e^{-\beta K} & e^{-\beta K} & \ddots & \\ \vdots & & & \end{pmatrix}.$$

The easiest way to find the eigenvalues of this matrix is to guess the eigenvectors:

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ \vdots \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ \vdots \end{pmatrix}, \cdots$$

If you don't like guessing you can note that if you subtract off the diagonal and rescale, P is a projection matrix. Regardless, one comes up with eigenvalues

$$
\begin{aligned}
\lambda_1 &= e^{\beta K(s-1)} + (s-1)e^{-\beta k} \\
\lambda_{j\neq 1} &= \left(e^{\beta K(s-1)} - e^{-\beta k}\right).
\end{aligned}
$$

The net result is that

$$
\begin{aligned}
F &= -k_B T \log Z = -k_B T \log\left[\lambda_1^N + (s-1)\lambda_2^N\right] \\
&\approx -N k_B T \log\left[e^{\beta K(s-1)} + (s-1)e^{-\beta k}\right].
\end{aligned}
$$

Problem 3. Critical Exponents

Here we explore if critical exponents can be different above and below the transition temperature.

We will concentrate the critical exponent for the coherence length, taking $\xi \sim (T - T_c)^{\nu}$ $(T > T_C)$ and $\xi \sim (T_c - T)^{\nu'}$ $(T < T_c)$. As shown in class, the scaling hypothesis tells us that the singular part of the free energy density is

$$f = |t|^{d\bar{\nu}} F_f^{\pm}\left(\frac{h}{|t|^{\bar{\nu} y_h}}\right),$$

with $\bar{\nu} = \nu$ or ν' depending on if $T > T_c$ or $T < T_c$. For $h \neq 0$, f should be a smooth function of $t = (T - T_c)/T_c$, because the only singularity is the one coming from the critical point at $t = h = 0$. Show that f can be written in the form

$$f = h^{d/y_h} \phi_\pm \left(\frac{h}{|t|^{\bar{\nu} y_h}} \right),$$

and explain how the smoothness assumption mentioned above constrains the analytic form of the functions $\phi_\pm$. Hence show that $\nu = \nu'$.

Solution 3.1. We define

$$\phi_\pm(x) = x^{-d/y_h} F_f^\pm(x).$$

As long as $x \neq= 0$ or ∞, this is a smooth transformation. With this definition,

$$f = h^{d/y_h} \phi_\pm \left(\frac{h}{|t|^{\bar{\nu} y_h}} \right).$$

If $h \neq 0$ then f must be smooth. For example, continuity requires $\phi_+(\infty) = \phi_-(\infty)$. Looking at the derivative with respect to t,

$$\begin{aligned}
\frac{\partial f}{\partial t} &= -\nu y_h \frac{h^{d/y_h+1}}{t^{\nu y_h+1}} \phi'_+ \left(\frac{h}{|t|^{\nu y_h}} \right) \quad t > 0 \\
&= -\bar{\nu} y_h h^{\frac{1}{y_h}(d-\frac{1}{\nu})} \chi_+ \left(\frac{h}{|t|^{\nu y_h}} \right) \\
\frac{\partial f}{\partial t} &= \nu' y_h \frac{h^{d/y_h+1}}{|t|^{\nu' y_h+1}} \phi'_- \left(\frac{h}{|t|^{\nu' y_h}} \right) \quad t < 0 \\
&= \nu' y_h h^{\frac{1}{y_h}(d-\frac{1}{\nu'})} \chi_- \left(\frac{h}{|t|^{\nu' y_h}} \right),
\end{aligned}$$

where $\chi_\pm(x) = x^{1+1/\bar{\nu} y_h} \phi'_\pm(x)$. Since there can be no singularity at $t = 0$ when $h \neq 0$, one must have that $\nu \chi_+(x) + \nu' \chi_(x) \to 0$ as $x \to \infty$. Moreover, continuity of this derivative requires $\chi_+ = \chi_-$ and $\nu = \nu'$.