P653 HW6

Due Oct 6, 2005

Problem 1. Scaling

Here we explore how the renormalization group leads to scaling for coupling constants which are not aligned with the eigendirections of the linearized flow equations. For concreteness we will imagine that the two coupling constants of interest are t and h, the reduced temperature and magnetic field of a spin system. We imagine that these are linearly related to coupling constants K_1 and K_1,

$$\begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} t \\ h \end{pmatrix}. \tag{1}$$

The constants K_1 and K_2 correspond to eigendirections of the linearized flow equations, so under rescaling by length ℓ,

$$\begin{aligned} K_1' &= \ell^{y_1} K_1 \\ K_2' &= \ell^{y_2} K_2. \end{aligned}$$

Under rescaling the coherence length is reduced by a factor of ℓ so

$$\xi(K_1', K_2') = \frac{1}{\ell}\xi(K_1, K_2).$$

Assume both constants are relevant, so $y_1, y_2 > 0$.

1.1. Prove that ξ can be written in the form

$$\xi(K_1, K_2) = K_2^{-1/y_2}\phi(K_1 K_2^{-y_1/y_2}). \tag{2}$$

Find the asymptotic behavior of $\phi(x)$ as $x \to 0$ and $x \to \infty$.

Solution 1.1. The desired expression comes from setting $\ell = K_2^{-1/y_2}$, and defining $\phi(x) = \xi(x, 1)$. When $K_1 = 0$, and $K_2 \neq 0$, the coherence length, ξ, should be well behaved and finite, which requires $\phi(0)$ is simply a constant. Similarly, when $K_2 = 0$ and $K_1 \neq 0$, the coherence length, ξ, should be well behaved and finite, which requires $\phi(x) \propto x^{-1/y_1}$ as $x \to \infty$.

1.2. Setting $h = 0$, use equation (**??**) and (**??**) to write ξ as a function of t. Using the asymptotic properties of ϕ, determine how ξ behaves as $t \to 0$.

Solution 1.2. Simple substitution yields

$$\xi(t) = c^{-1/y_2} t^{-1/y_2} \phi(t^{1-y_1/y_2} a c^{-y_1/y_2}).$$

As $t \to 0$ there are two possibilities. If $y_1 < y_2$, then the argument of ϕ goes to zero and $\xi(t) \sim t^{-1/y_2}$. On the other hand, if $y_2 < y_1$ then the argument of ϕ goes to infinity and $\xi(t) \sim t^{-1/y_1}$.

1.3. How would this argument change if one of these coupling constants, say K_1, was irrelevant?

Solution 1.3. If $y_1 < 0$, the argument goes through to the point where we write

$$\xi(t) = c^{-1/y_2} t^{-1/y_2} \phi(t^{1-y_1/y_2} b c^{-y_1/y_2}).$$

At this point we note that if $y_1 < 0$ and $y_2 > 0$ the the argument of ϕ vanishes as $t \to 0$, and therefore $\xi(t) \sim t^{-1/y_2}$. The irrelevant exponent is unimportant (irrelevant?).
Note that the argument about the asymptotic behavior of ϕ is a little subtle here. First, as $K_1 \to 0$ at fixed K_2, the result cannot depend on K_1, so $\phi(0)$ must be a constant. Now, if we take $K_2 \to 0$ zero at fixed K_1 we also have $x \to 0$. Therefore $\xi \to \infty$ as $K_2 \to 0$ at fixed K_1. This is exactly what you expect. All points on the critical manifold have $\xi = \infty$.
Your should also note that that we know nothing about the behavior of $\phi(x)$ as $x \to \infty$. However, for the structure near the critical point this behavior is unimportant.

Problem 2. Method of Auxilliary Fields

Consider an Ising model of the form

$$H = -\frac{1}{2} \sum_{ij} J_{ij} \sigma_i \sigma_j - \sum_i H_i \sigma_i,$$

where the spins lie on a lattice in d dimensional space, $J_{ij} = J > 0$ if i and j are nearest neighbors, but $J_{ij} = 0$ otherwise. The spins take on values $\sigma = \pm 1$.

If we use Einsteins summation notation, where repeated indices are summed over, this becomes

$$H = -\frac{1}{2} J_{ij} \sigma_i \sigma_j - H_i \sigma_i,$$

2.1. Why is the factor of 1/2 in this Hamiltonian? Show that this is the same as

$$H = -J \sum_{\langle ij \rangle} \sigma_i \sigma_j - \sum_i H_i \sigma_i,$$

Solution 2.1. The 1/2 is because we are counting each bond twice now.

2.2. Prove that for any $N \times N$ symmetric matrix A, and any length N vector B, that

$$\int_{-\infty}^{\infty} \frac{dx_1}{\sqrt{2\pi}} \frac{dx_2}{\sqrt{2\pi}} \cdots \frac{dx_N}{\sqrt{2\pi}} e^{-x_i A_{ij} x_j/2 + x_i B_i} = \frac{1}{(\det A)^{1/2}} e^{B_i (A^{-1})_{ij} B_j/2}. \tag{3}$$

Hint: make a change of variables $y_i = x_i - (A^{-1})_{ij} B_j$.

Solution 2.2. Performing the suggested substitution we have

$$-x_i A_{ij} x_j/2 + x_i B_i = -y_i A_{ij} y_j/2 + B_i (A^{-1})_{ij} B_j/2.$$

The measures change trivially, $dy_i = dx_i$, so we only need to prove that

$$I = \int_{-\infty}^{\infty} \frac{dy_1}{\sqrt{2\pi}} \frac{dy_2}{\sqrt{2\pi}} \cdots \frac{dy_N}{\sqrt{2\pi}} e^{-y_i A_{ij} y_j/2} = \frac{1}{(\det A)^{1/2}}.$$

For this step, we use that a symmetric matrix A can be diagonalized with a orthonormal matrix Λ. This lets us transform the y's into a new set of coordinates z_i in which A is diagonal. The Jacobian for the transformation is 1. We then have a product of N independent gaussian integrals yielding

$$I = \prod_\alpha \frac{1}{\alpha^{1/2}},$$

where α runs over the eigenvalues of A. One readily sees that this is the determinant.

2.3. We want to calculate the partition function

$$Z = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_N} e^{-\beta H}.$$

By using equation (**??**) with $(A^{-1})_{ij} = \beta J_{ij}$ and $B_i = \sigma_i$, show that

$$Z = \int_{-\infty}^{\infty} d\psi_1\, d\psi_2 \cdots d\psi_N e^{-\beta S}, \tag{4}$$

where S, is given by

$$e^{-\beta S} = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_N} \sqrt{\frac{\det((\beta J)^{-1})}{(2\pi)^N}} \exp\left[-\frac{1}{2}(\psi_i - \beta H_i)[(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j) + \psi_i \sigma_i\right].$$

Solution 2.3. Direct substitution gives

$$e^{-\beta S} = \sum_{\sigma_1, \sigma_2, \ldots, \sigma_N} \sqrt{\frac{\det((\beta J)^{-1})}{(2\pi)^N}} \exp\left[-\frac{1}{2} x_i [(\beta J)^{-1}]_{ij} x_j + (x_i + \beta H_i)\sigma_i\right].$$

Changing variables $\psi_i = x_i + \beta H_i$ gives the desired expression.

2.4. Do the sum over the σ's.

Solution 2.4. The sum is now easy, and we find

$$e^{-\beta S} = \sqrt{\frac{\det((\beta J)^{-1})}{(2\pi)^N}} \exp\left[-\frac{1}{2}(\psi_i - \beta H_i)[(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j)\right] \prod_i 2\cosh\psi_i.$$

The partition function for this discrete spin model is thus mapped onto the partition function of a theory where a continuous variable ψ_i lives on each site. This is known as a lattice field theory.

If we take the continuum limit we would have a field $\psi(r)$, and the multiple integral in equation (**??**) would be a functional integral (an integral over the space of functions). We would then have a continuum field theory (usually just called a field theory).

2.5. Saddle point approximation

Assuming that S is a strongly peaked function of ψ, one can approximate $Z \approx e^{-\beta S_0}$, where S_0 is the value of S at the minimum [ie. where $\partial S/\partial \psi_j = 0$].

Let $\bar{\psi}_i$ be the value of ψ_i at the minimum. Find the equation satisfied by $\bar{\psi}_i$ and show that

$$m_i = \langle \sigma_i \rangle = -\frac{\partial F}{\partial H_i} \approx -\partial S_0/\partial H_i$$

is given by $m_i = \tanh \bar{\psi}_i$. Hence find H_i as a function of $\{m_i\}$.

Solution 2.5. The stationarity condition $\partial S/\partial \psi_i = 0$ yields

$$-[(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j) + \tanh \psi_i = 0.$$

The definition of m_i gives

$$m_i = -\partial S_0/\partial H_i = [(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j).$$

Comparing terms we have

$$m_i = \tanh \bar{\psi}_i$$

.

2.6. Assuming that the $F \approx S_0$, and substituting H_i with m_i, show that

$$F \approx F_0 + \frac{1}{2} J_{ij} m_i m_j - k_B T \sum_i \log \left(\frac{2}{\sqrt{1 - m_i^2}} \right),$$

where F_0 is a constant independent of m_i.

Solution 2.6. Several of you noticed that there is something fundamentally wrong about this free energy – see the end of this problem for the explanation. The wrong argument goes as follows: We make two substitutions. First, using

$$\tanh \psi_i = m_i,$$

we write

$$\cosh \psi_i = \frac{1}{\sqrt{1 - m_i^2}}.$$

Second, using

$$\psi_i - \beta H_i = \beta J_{ij} m_j,$$

we write

$$(\psi_i - \beta H_i)[(\beta J)^{-1}]_{ij}(\psi_j - \beta H_j) = \beta m_i J_{ij} m_j.$$

This then gives the desired expression.

2.7. The equilibrium value of m_i is found by minimizing this Free energy. Assuming that $m_i = m$ is uniform, and that each spin has q nearest neighbors, find the temperature at which the system undergoes a phase transition.

Solution 2.7. Again, this argument is not quite right:
The transition temperature is found by expanding F to quadratic order and finding where the quadratic coefficient vanishes. Taking m to be uniform this yields

$$\frac{qJN}{2} - \frac{Nk_BT}{2} = 0.$$

Which gives $T_c = qJ/k_B$.

Note, this saddle point approximation is equivalent to the mean field approximation that we had made in class.

2.8. What went wrong?

Solution 2.8. So what went wrong? The problem is that when we substituted $\tanh\psi_i = m_i$, we needed to add a constraint. The fact that something is wrong should be clear from the fact that we somehow lost a thermodynamic variable (H). Also, the free energy went from concave to convex.

There are several ways to fix the problem. The most physical one is to notice that what we wanted to do was a Legendre transformation. I'll leave it as an exercise to work out how that goes.

The most straightforward approach, however is to simply not use the constraint, in which case one instead makes the substitution

$$\cosh\psi_i = \cosh\left(\beta J_{ij} m_j + \beta H_i\right),$$

which gives

$$F \approx F_0 + \frac{1}{2} J_{ij} m_i m_j - k_B T \sum_i \log\left(\cosh\left(\beta J_{ij} m_j + \beta H_i\right)\right),$$

which we recognize as our mean-field free energy. taking all m_i's to be equal, and setting $H_i = 0$ gives

$$\begin{aligned} F &\approx F_0 + \frac{qN}{2} J m^2 - N k_B T \log\cosh(q\beta J m) \\ &= F_0 + \frac{qNJ}{2}\left(1 - q\beta J\right) m^2 + \cdots \end{aligned}$$

which has the correct curvature and which recovers the mean-field transition temperature.

Problem 3. Free Energy of a Continuum model

To prep ourselves for discussions in class, it will be useful to calculate the exact free energy of a simple continuum model: the Gaussian model. We consider at each place in space there is a real valued field $\phi(r)$, and the free energy is given by a sum over all configurations of the field. Formally we can write

$$Z = e^{-\beta F} = \int \mathcal{D}\phi\, e^{-\beta S[\phi]}.$$

Where $S[\phi]$ is a functional.

The simplest way to define such a *functional integral* is to work in a finite volume so that we may write $\phi(r)$ as a Fourier sum,

$$\phi(r) = \frac{1}{V} \sum_k e^{ik\cdot r} \phi_k.$$

We then define

$$\int \mathcal{D}\phi = \int \prod_k d\phi_k.$$

We will use a simple model where

$$S[\phi] = \int dr\, \frac{\gamma}{2} |\nabla\phi(r)|^2 + \frac{at}{2} |\phi(r)|^2,$$

where $t = (T - T_c)/T_c$ is linear in temperature, and $a, \gamma > 0$.

[Note, one could derive this free energy by taking the continuum limit of problem 2, and expanding to quadratic order.]

Once $t < 0$ this theory is ill-defined, but we can still consider what happens as we approach the critical temperature $t = 0$ from the disordered phase.

We will be thinking of ϕ as a coarse-grained variable, so it only makes sense to talk about it on sufficiently long length-scales. Thus we will also have a momentum scale Λ, and set $\phi_k = 0$ for all $|k| > \Lambda$.

3.1. Write S in terms of the ϕ_k, and perform the integrals over r to arrive at

$$e^{-\beta F} = \int \prod_k d\phi_k \exp - \left[\frac{\beta}{2V} \sum_k (at + \gamma k^2) \phi_k \phi_{-k} \right]$$

Solution 3.1. Performing the substitution,

$$\begin{aligned} S &= \int dr \, \frac{1}{V^2} \sum_{k\,k'} \left(-\frac{\gamma}{2} k k' + \frac{at}{2} \right) \phi_k \phi_{k'} e^{i(k-k')\cdot r} \\ &= \frac{1}{2V} \sum_k \left(\gamma k^2 + at \right) \phi_k \phi_{-k}, \end{aligned}$$

which is of the required form.

3.2. By noting that $\phi(r)$ is real, show that $\phi_{-k} = \phi_k^*$.

Solution 3.2. Using the Fourier expansion of $\phi(r) = \phi^*(r)$,

$$\sum_k e^{ik\cdot r} \phi_k = \sum_k e^{-ik\cdot r} \phi_k^*.$$

Equating coefficients of $e^{ik\cdot r}$ gives $\phi_k = \phi_{-k}^*$.

3.3. Let w be a complex number. Calculate the Gaussian integral

$$I = \int dw \, dw^* \, e^{-a|w|^2} = 2i \int \, dx \, dy \, e^{-a(x^2+y^2)},$$

where $w = x + iy$. You can take this to be the definition of the measure $dw \, dw^*$.

Solution 3.3. This is a standard integral

$$I = \frac{2\pi i}{a}.$$

3.4. Use this result to calculate the partition function. Do not worry about multiplicative constants (such as 2π's) which will play no role in any thermodynamic derivatives.

Solution 3.4. Neglecting multiplicative constants,

$$Z = \prod_k \left(\frac{k_B T V}{\gamma k^2 + at} \right).$$

3.5. With what power of t does the specific heat diverge as $t \to 0^+$? How does this compare with mean-field theory results.

Solution 3.5. The free energy is

$$F = -k_B T \log Z = -k_B T \sum_k \log \left(\frac{k_B T V}{\gamma k^2 + at} \right).$$

The heat capacity is

$$c = -T \frac{\partial^2 F}{\partial T^2}.$$

There are three T's that we need to differentiate with respect to. Since we are only interested in the most singular behavior, however, we can save ourselves some work. We note that every time we differentiate with respect to the t we bring a k^2 into the denominator, making the expression more singular. Differentiating with respect to the other T's has no such effect. Thus the most singular part of c is

$$c_s = -\frac{T}{T_c^2} \frac{\partial^2 F}{\partial t^2} = \frac{T}{T_c^2} \sum_k \frac{a^2}{(at + \gamma k^2)^2}.$$

Converting the sum to an integral

$$c_s = \frac{TV}{T_c^2} \int \frac{d^d k}{(2\pi)^d} \frac{a^2}{(at + \gamma k^2)^2}.$$

There are several ways to estimate this integral. The easiest one is to break it into two parts: large k and small k. For large k the integrand becomes independent of t, and because of the cutoff Λ, just integrates to a constant. For small k, we need to worry about t. If we set $t = 0$ we have a problem. We are then trying to integrate $d^d k / k^4$. The small k part of the integral diverges if $d \leq 4$. Finite t effectively cuts off the integral at $k = \sqrt{at/\gamma}$, resulting in

$$c_s \sim \int_{\sqrt{at/\gamma}} \frac{d^d k}{k^4} \sim t^{(d-4)/2},$$

for $d < 4$ and

$$c_s \sim \text{constant}$$

for $d > 4$.
Thus in $d > 4$ we recover the mean field result $\alpha = 0$ and in lower dimensions we find $\alpha = 2 - d/2$, where

$$c \sim t^{-\alpha}.$$