

P653 HW7

Due Oct 20, 2005

Problem 1. Ising model near the upper critical dimension On the last homework we showed that the Ising model in d dimensions can be mapped onto a field theory, which near the critical point had a free energy

$$e^{-\beta F} = \int \mathcal{D}\phi e^{-\beta S} \quad (1)$$

$$S = \int d^d r \frac{a}{2} \phi^2 + b \phi^4 + \frac{1}{2} |\nabla \phi|^2, \quad (2)$$

where a and b are temperature dependant. We also showed that if you neglect b , you recover the mean field result (which gives the correct critical behavior for sufficiently large dimension). Thus we expect that b will be irrelevant above the upper critical dimension for the Ising model $d = 4$. We will prove that this is the case by using dimensional analysis. We will then look at the behavior near this upper critical dimension.

1.1. The argument of an exponent must be dimensionless. Deduce the dimensions of $[a]$, $[b]$ and $[\phi]$ in terms of Energy $[E]$ and Length $[L]$?

Solution 1.1. The dimensions of $[S]$ is $[E]$. From the gradient term we then see $[\phi] = [E^{1/2}/L^{d/2-1}]$. Looking sequentially at the other terms, we then find $[a] = [L^{-2}]$ and $[b] = [L^{d-4}/E]$.

1.2. One can produce an expression for βS which only involves dimensionless quantities by writing it in terms of $x = r/r_0$ and $\psi = \phi/\phi_0$, where r_0 and ϕ_0 have the same dimensions as r and ϕ . Choose the rescaling to produce an expression

$$\beta S = \int d^d x \frac{1}{2} \psi(x)^2 + \gamma \psi(x)^4 + \frac{1}{2} |\nabla_x \psi(x)|^2.$$

How is the dimensionless coupling γ related to a and b ?

Solution 1.2. Under the rescaling we require $a r_0^d \phi_0^2 = k_B T$, $r_0^{d-2} \phi_0^2 = k_B T$, and $\gamma = b r_0^d \phi_0^4 k_B T$. This gives $r_0 = 1/\sqrt{a}$, $\phi_0^2 = k_B T/a^{d-2}$, and

$$\gamma = b (k_B T)^3 a^{(d-4)/2}.$$

1.3. Near the critical point $a \propto T - T_c$. To investigate critical properties we therefore let $a \rightarrow 0$. Show that $\gamma \rightarrow 0$ if $d > 4$, but $\gamma \rightarrow \infty$ if $d < 4$. Therefore one can ignore the ψ^4 term in higher dimension.

Solution 1.3. This result is self-evident.

It turns out that for $d = 4 - \epsilon$ one can do perturbation theory in ϵ and derive a renormalization group flow. We will do this in class soon. Upon rescaling by a factor ℓ , the constants a and b obey

$$\begin{aligned}\ell \frac{da}{d\ell} &= 2a + 12K_d \frac{b}{1+a} \\ \ell \frac{db}{d\ell} &= \epsilon b - 36K_d \frac{b^2}{(1+a)^2},\end{aligned}$$

where K_d is a positive constant which depends slowly upon the dimension d . These hold for sufficiently small a and b . Note this is a notation change from class, where what we are calling ℓ here was called b . The change is necessary since we have something else called b this time.

1.4. To lowest order in ϵ , find the fixed points and sketch the flow diagram for $\epsilon < 0$ (corresponding to $d > 4$) and $\epsilon > 0$ (corresponding to $d < 4$).

Solution 1.4. In all cases there is a fixed point at $a = b = 0$. Linearizing about this fixed point one finds

$$\ell\partial_\ell \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2 & 12K_d \\ 0 & \epsilon \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}. \quad (3)$$

This has eigenvalues $\lambda = 2$ and $\lambda = \epsilon$, corresponding to eigenvectors $(\delta a, \delta b) = (1, 0)$ and $(12K_d, -2 + \epsilon)$.

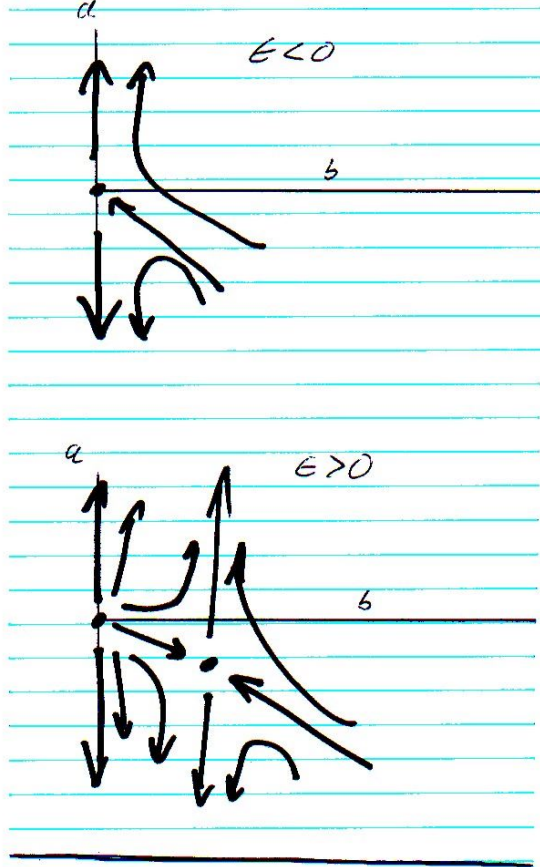
For $\epsilon > 0$ there is a second fixed point at $a = a^* = -\epsilon/6$ and $b = b^* = \epsilon/36K_d$. Linearizing about this point one finds

$$\ell\partial_\ell \begin{pmatrix} \delta a \\ \delta b \end{pmatrix} = \begin{pmatrix} 2 - \epsilon/3 & 12K_d \\ 0 & -\epsilon \end{pmatrix} \begin{pmatrix} \delta a \\ \delta b \end{pmatrix}. \quad (4)$$

Here the eigenvalues are $\lambda = 2 - \epsilon/3$ and $\lambda = -\epsilon$, corresponding to eigenvectors $(\delta a, \delta b) = (1, 0)$ and $(18K_d, -3 - \epsilon)$.

Note that for $\epsilon < 0$ there is also a fixed point at negative b , but this unstable fixed point is unphysical. This model with negative b is unstable.

Using this behavior near the fixed points, we make the following sketch,



As with our expansion about the lower critical dimension ($d = 1$ – the dimension at which $T_c=0$) we have one fixed point splitting into two. We know that when $\epsilon < 0$ the one fixed point is a

critical point, corresponding to $\xi = \infty$. Therefore all flows that terminate on this fixed point are on the critical manifold with $\xi = \infty$. By continuity, when $\epsilon > 0$ all flows which terminate at either of these fixed points must still correspond to $\xi = \infty$. Therefore these are both critical points. [This linearized theory does not produce the fixed points which describe the ordered phases.]

1.5. Show that for $\epsilon > 0$ the critical point at $a = b = 0$ is unstable in all directions and hence the system can only be near this fixed point if both a and b are carefully tuned.

Solution 1.5. This is easily shown by examining the eigenvalues of (3), which are described above. When $\epsilon > 0$ the eigenvalues are both positive.

Because of this need for careful tuning, generically the system will flow by the other fixed point. This second fixed point therefore determines the critical properties.

1.6. Linearize the flow equations about this second fixed point. Using that a is proportional to $T - T_c$, deduce the critical exponent ν defined by $\xi \sim |T - T_c|^{-\nu}$.

Solution 1.6. We have already shown the linearization, and the relevant (in both the technical and colloquial sense) eigenvalue is $\lambda = 2 - \epsilon/3$. Now, since $t \sim \ell^\lambda$ and $\xi \sim \ell^{-1}$, we should have $\xi \sim t^{-1/\lambda}$, which gives $\nu = 1/\lambda = 1/(2 - \epsilon/3)$.

Problem 2. Using the Migdal-Kadanoff recursion relations, calculate the specific heat of the Ising model in dimension $d = 1 + \epsilon$. Using notation from class, the recursion relations are

$$b \frac{dK}{db} = (d-1)k + \frac{1}{2} \sinh(2K) \log(\tanh K). \quad (5)$$

Hint: Under renormalization by scale b , the free energy remains fixed, but the volume of space is reduced. Therefore the singular part of the free energy scales as

$$F'_s = b^{-d} F_s(t') = F_s(t). \quad (6)$$

(cf. Plischke and Bergersen section 6.2). Use the recursion relationship to relate t' to t , and choose b intelligently.

Solution 2.1. Linearizing the recursion relations about the critical point yields

$$b \frac{\partial t}{\partial b} = \epsilon t, \quad (7)$$

which is integrated to find

$$t' = b^\epsilon t. \quad (8)$$

Using the hint, we then have

$$F_s(t) = b^{-d} F_s(b^\ell t).$$

This is true for all b . In particular, we can take $b = e^{-1/\epsilon}$ to see that

$$F_s(t) \propto t^{d/\epsilon}$$

The specific heat is the second derivative of the free energy with respect to temperature, so

$$C \propto t^{d/\epsilon-2} = t^{1/\epsilon-1} = t^{-\alpha}.$$

The specific heat exponent is therefore $\alpha = 1 - 1/\epsilon$.

This should be compared to the exact value $\alpha = 0$ in two dimensions, and the numerical result $\alpha = 0.11$ in three dimensions.

Problem 3. Wick's Theorem

3.1. Let A be a real symmetric matrix. Define

$$\langle x_q x_r \rangle = \frac{\int_{-\infty}^{\infty} d^n x x_q x_r e^{-\sum_{ij} A_{ij} x_i x_j}}{\int_{-\infty}^{\infty} d^n x e^{-\sum_{ij} A_{ij} x_i x_j}}.$$

Prove that $\langle x_q x_r \rangle$ is a matrix element of the inverse matrix A^{-1} , namely, $\langle x_q x_r \rangle = (1/2) A_{qr}^{-1}$. (Note this is different than $(A_{qr})^{-1}$..

Solution 3.1. The really slick solution is to note that

$$\langle x_q x_r \rangle = \frac{\partial^2}{\partial \eta_r \partial \eta_s} \frac{\int_{-\infty}^{\infty} d^n x e^{-\sum_{ij} A_{ij} x_i x_j + \sum_j \eta_j x_j}}{\int_{-\infty}^{\infty} d^n x e^{-\sum_{ij} A_{ij} x_i x_j}},$$

evaluated at $\eta = 0$. Completing the square, this gives

$$\langle x_q x_r \rangle = \frac{\partial^2}{\partial \eta_r \partial \eta_s} e^{(1/2)(A^{-1})_{ij} \eta_i \eta_j},$$

which trivially gives the desired result.

3.2. Using the obvious generalization of this notation, prove that

$$\langle x_a x_b x_c x_d \rangle = \langle x_a x_b \rangle \langle x_c x_d \rangle + \langle x_a x_c \rangle \langle x_b x_d \rangle + \langle x_a x_d \rangle \langle x_b x_c \rangle$$

This result is a special case of the linked-cluster theorem, known as "Wick's theorem". We will use it for developing a perturbation theory.

Solution 3.2. Using the slick approach to the last problems

$$\langle x_a x_b x_c x_d \rangle = \left(\frac{\partial^4}{\partial \eta_a \partial \eta_b \partial \eta_c \partial \eta_d} e^{\langle x_i x_j \rangle \eta_i \eta_j} \right)_{\eta=0}.$$

Expanding the exponential, one sees that the only nonzero terms are the ones shown.

Problem 4. Kosterlitz-Thouless flows

We will talk at length about the two-dimensional x-y model soon. Here you will play with the renormalization group flow equations that we will derive.

For the two dimensional x-y model, it turns out there are two important coupling constants. The obvious one is the spin stiffness $K = J/k_b T$, which in a discrete model could be defined by

$$H = -J \sum_{\langle ij \rangle} \mathbf{S}_i \cdot \mathbf{S}_j$$

while in a continuum model would be

$$H = J_{\text{cont}} \int d^2 r |\nabla \phi(r)|^2,$$

There is some numerical factor (which we don't care about) which connects J to J_{cont} . In each of these cases $\mathbf{S}$ is the direction of the spin (of magnitude 1) and ϕ is the angle which defines the direction of the spin.

It turns out that a second coupling constant is important; namely the energy E_c to create a vortex [if you are unsure about what exactly this means – don't worry about it, we will talk about the physics in class]. We parameterize this energy by the fugacity $y = e^{\beta E_c}$.

A nontrivial argument (which we will do in class) yields flow equations

$$\begin{aligned} \ell \frac{d(K^{-1})}{d\ell} &= 2\pi^3 y^2 \\ \ell \frac{dy}{d\ell} &= (2 - \pi K)y, \end{aligned}$$

where ℓ is the length by which the system is scaled. These equations only hold for sufficiently small y and K^{-1} .

These have a structure which is very different from the flow equations that we have seen so far.

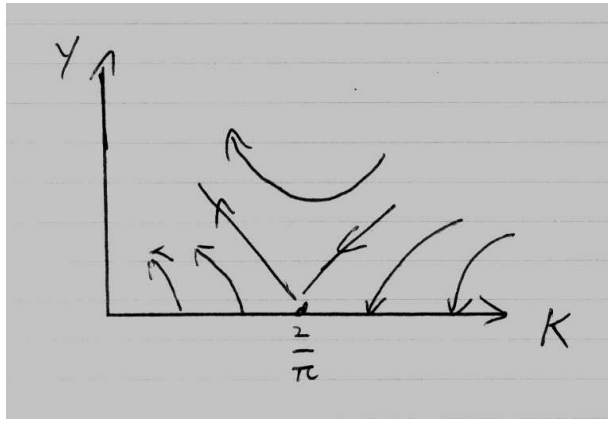
4.1. Show that there exists a fixed line at $y = 0$.

Solution 4.1. Setting $y = 0$ shows that this is true.

4.2. Show that for $K^{-1} = k_B T/J < \pi/2$ this fixed line is stable, while for $K^{-1} = k_B T/J > \pi/2$ it is unstable. This means that at sufficiently low temperature, vortices are irrelevant (in the technical sense) and that they play no role in the physics, while at high temperature vortices drive the system to a high temperature fixed point which is not described by this linearized theory.

Solution 4.2. $dy/d\ell > 0$ if $K^{-1} > \pi/2$ and $dy/d\ell < 0$ if $K^{-1} < \pi/2$.

4.3. Draw a flow diagram which shows typical renormalization group trajectories in the K - y plane.



Solution 4.3.