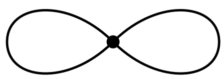
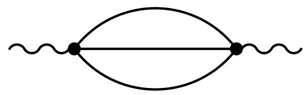
P653 HW8

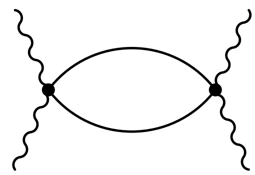
Due Nov 3, 2005

Problem 1. Feynman Diagrams Here are a series of exercises intended to practice your proficiency with Feynman diagrams.

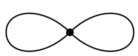
1.1. Write the integrals which correspond to the following diagrams


(a)


(b)

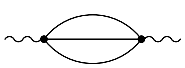

(c)

Solution 1.1.



$$= \frac{u_0}{4} \int_{\frac{\Lambda}{\ell}<|q_j|<\Lambda} \frac{d^d q_1\, d^d q_2}{(2\pi)^{2d}} \frac{1}{(r_0+q_1^2)(r_0+q_2^2)}$$

$$= \frac{u_0}{4} \left[\int_{\frac{\Lambda}{\ell}<|q|<\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{r_0+q_1^2} \right]^2$$

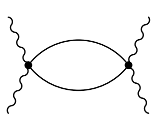


$$= \left(\frac{u_0}{4}\right)^2 \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^{2d}} |\phi_k|^2 \int_{\frac{\Lambda}{\ell}<|q_j|<\Lambda} \frac{d^d q_1\, d^d q_2\, d^d q_3}{(2\pi)^{3d}} (2\pi)^d \delta(k+q_1+q_2+q_3)$$

$$\times \frac{1}{(r_0+q_1^2)(r_0+q_2^2)(r_0+q_3^2)}$$

$$= \left(\frac{u_0}{4}\right)^2 \int_{|k|<\Lambda} \frac{d^d k}{(2\pi)^{2d}} |\phi_k|^2$$

$$\times \int_{\frac{\Lambda}{\ell}<|q_j|<\Lambda} \frac{d^d q_1\, d^d q_2}{(2\pi)^{2d}} \frac{1}{(r_0+q_1^2)(r_0+q_2^2)(r_0+(k-q_1-q_2)^2)}$$



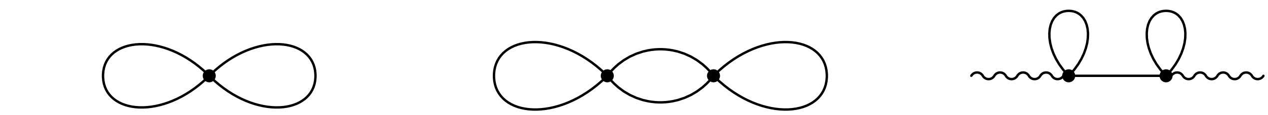
$$= \left(\frac{u_0}{4}\right)^2 \int_{|k_j|<\Lambda} \frac{d^d k_1\, d^d k_2\, d^d k_3\, d^d k_4}{(2\pi)^{4d}} \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4} \int_{\frac{\Lambda}{\ell}<|q_j|<\Lambda} \frac{d^d q_1\, d^d q_2}{(2\pi)^{2d}}$$

$$\times (2\pi)^d \delta(k_1+k_2+q_1+q_2)(2\pi)^d \delta(k_3+k_4-q_1-q_2) \frac{1}{(r_0+q_1^2)(r_0+q_2^2)}$$

$$= \left(\frac{u_0}{4}\right)^2 \int_{|k_j|<\Lambda} \frac{d^d k_1\, d^d k_2\, d^d k_3\, d^d k_4}{(2\pi)^{4d}} \phi_{k_1}\phi_{k_2}\phi_{k_3}\phi_{k_4}\, (2\pi)^d \delta(k_1+k_2+k_3+k_4)$$

$$\times \int_{\frac{\Lambda}{\ell}<|q|<\Lambda} \frac{d^d q}{(2\pi)^d} \frac{1}{(r_0+q^2)(r_0+(k_1+k_2-q)^2)}$$

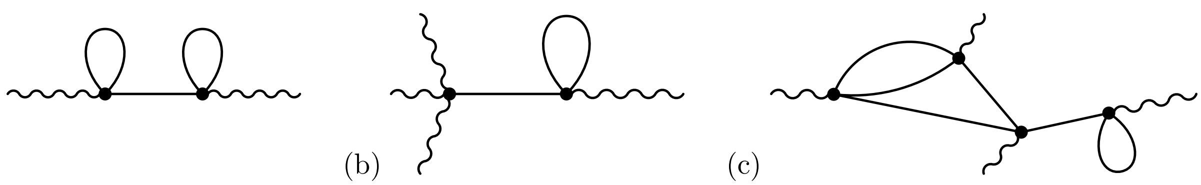
1.2. What are the multiplicities of the following diagrams?



(a) (b) (c)

Solution 1.2. (a) 3; (b) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}^2 2 = 72$; (c) $\begin{pmatrix} 4 \\ 2 \end{pmatrix}^2 4 = 144$

1.3. Prove that the following diagrams evaluate to zero. [The most straightforward approach is to convert the diagrams to integrals. Make sure to keep track of the limits of integration!! By the time you are on part (c), you should be able to give an argument that doesn't require writing out the integral. As an extra hint, note that there is a part of the diagram which is the same in each case.]

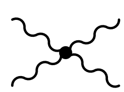
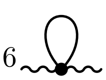
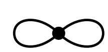


(a) (b) (c)

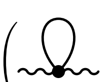
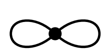
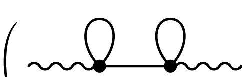
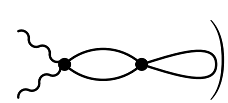
Solution 1.3. All of these diagrams have an external line ϕ'_k which has the same momentum as an internal line ψ_k. Since the external lines must have $k < \Lambda/\ell$, but the internal lines require $\Lambda/\ell < k < \Lambda$, the integrals are zero.

1.4. In class we explicitly showed that if you consider the terms which contain no ϕ''s (but do contain ψ's) that $\langle\langle V^2 \rangle\rangle = \langle V^2 \rangle - \langle V \rangle^2$ consists only of connected diagrams. Give the same construction for terms which contain two ϕ''s. i.e. construct all of the diagrams for $\langle V^2 \rangle$ which contain two ϕ''s, and subtract off $\langle V \rangle^2$ to explicitly show that $\langle\langle V^2 \rangle\rangle$ only contains connected diagrams.

Solution 1.4. This is an exercise in combinatorics. Keeping all terms,

$\langle V \rangle^2 = \Big[$  $+ 6$  $+ 3$  $\Big]^2.$

On the other hand, the requested terms of the second moment are

$\langle V^2 \rangle \;=\; 36 \Big($   $\Big) + 144 \Big($  $\Big) + 144 \Big($  $\Big).$

Problem 2. Surface Area of a d-dimensional Sphere Here we calculate the area of a d-dimensional sphere by doing a Gaussian integral two ways. Let

$$F = \int d^d q \, e^{-q^2}.$$

First evaluate F by converting to Cartesian coordinates. Next evaluate F by using spherical coordinates, assuming that the surface area of a d-dimensional sphere of unit radius is S_d. Equating these two expressions gives you a formula for S_d.

Give your result in terms of the Gamma function,

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt.$$

Solution 2.1.

$$\begin{aligned} F &= \pi^{d/2} && (1) \\ &= S_d \int_0^\infty dq\, q^{d-1} e^{-q^2}. && (2) \end{aligned}$$

We then change variables: $x = q^2$ to arrive at

$$F = S_d \Gamma(d/2)/2,$$

which gives

$$S_d = 2\pi^{d/2}/\Gamma(d/2)$$

Problem 3. n-vector model We can easily generalize our discussion of the Ising model to higher dimensional spins: for example, the x-y model uses 2-dimensional spins, and the Heisenberg model uses 3-dimensional spins. We will work with n dimensional spins, in which case the Landau free energy will be of the form

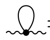
$$-H = \int d^d r \sum_{\alpha=1}^{n} \left\{ \left[\frac{1}{2} \nabla\phi^\alpha(r) \cdot \nabla\phi^\alpha(r) + \frac{r_0}{2} \phi^\alpha(r)^2 \right] + \sum_{\alpha,\beta=1}^{n} \frac{u_0}{4} \phi^\alpha(r)\phi^\alpha(r)\phi^\beta(r)\phi^\beta(r) \right\}$$

where α, β represent the component of the spin.

As before, we can produce the true free energy from a functional integral

$$e^{-F} = \int \prod_\alpha \mathcal{D}\phi^\alpha e^H.$$

Following the procedure we carried out in class, calculate the flow equations for r_0 and u_0 to first order in V. If you set $n = 1$ these should reduce to the equations we found in class.

Solution 3.1. As in class we divide momenta between $k < \Lambda' = \Lambda/\ell$ and $\Lambda' < k < \Lambda$, and let $\phi_k \to \varphi_k$ or $\phi_k \to \psi_k$ depending upon which of these regions we are in. We then write $H = H_0 + H_1 + V$, with $-H_0 = \sum_{k<\Lambda'} \sum_\alpha |\varphi_k^\alpha|^2 (r_0 + k^2)/2$, $-H_1 = \sum_{\Lambda'<k<\Lambda} \sum_\alpha |\psi_k^\alpha|^2$, and $-V = \sum_{k_1,k_2,k_3,k_4<\Lambda} (u_0/4)\delta_{k_1+k_2+k_3+k_4} \sum_{\alpha,\beta} \phi_{k_1}^\alpha \phi_{k_2}^\alpha \phi_{k_3}^\beta \phi_{k_4}^\beta$, where we have used the compact notation $\sum_k \to (2\pi)^{-d} \int d^d k$ and $\delta_k \to 2\pi^d \delta(k)$. We denote $\langle X \rangle = \int \mathcal{D}\psi X e^{H_1}$, in which case $e^{-F} = \int \mathcal{D}\phi e^H = \int \mathcal{D}\varphi e^{H'} \approx \int \mathcal{D}\varphi e^{H_0 + \langle V \rangle}$. Our first task is then the calculate $\langle V \rangle$. We will use that H_1 is Gaussian so that we can use Wick's theorem. A straightforward Gaussian integral gives $\langle \psi_q^\alpha \psi_k^\beta \rangle = (2\pi)^d \delta(q+k)\delta_{\alpha\beta}/(r_0 + k^2)$. As before there are three sorts of terms in $\langle V \rangle$: those with all ψ's, those with two φ's and those with four φ's. These terms respectively supply a constant, a renormalization of r_0 and the bare u_0 term. The only calculation we need to do is the second term. It should be clear there is only one integral we need to consider; ie $=(2\pi)^{-d} \int d^d k \sum_\alpha |\varphi_k^\alpha|^2 (u_0/4) I_1$, where $I_1 = (2\pi)^{-d} \int_{\Lambda'<q<\Lambda} d^d q/(q^2 + r_0)$ is the same integral we encountered in the Ising case. The only difference here is the multiplicity of this term. Before we found that there were six of these terms in the expansion of $\langle V \rangle$. Now however we have to consider the spin multiplicity. Of these six contractions, two of them [those of the form $\langle \psi_{k_1}^\alpha \psi_{k_2}^\alpha \rangle \varphi_{k_3}^\beta \varphi_{k_4}^\beta$ and $\langle \psi_{k_3}^\beta \psi_{k_4}^\beta \rangle \varphi_{k_1}^\alpha \varphi_{k_2}^\alpha$] each contribute n terms because of the sum over the spins of the ψ's. On the other hand, the other four contractions just contribute 1 term each because once you specify the spin of the φ's it fixes the spins of the ψ's. Thus the multiplicity here is $2n + 4$, and to this level of approximation $H' = \sum_{k<\Lambda'} \sum_\alpha |\varphi_k^\alpha|^2 (k^2 + r_0 + (n+2)u_0 I_1)/2 + \sum_{k_1,k_2,k_3,k_4<\Lambda'} (u_0/4)\delta_{k_1+k_2+k_3+k_4} \sum_{\alpha,\beta} \varphi_{k_1}^\alpha \varphi_{k_2}^\alpha \varphi_{k_3}^\beta \varphi_{k_4}^\beta$. As before we must rescale k and φ so that we are integrating over the domain $k < \Lambda$ and that the k^2 term has coefficient 1/2. One then finds

$$
\begin{aligned}
r_0' &= \ell^2 [r_0 + (n+2) I_1] \\
u_0' &= \ell^(4 - d) u_0.
\end{aligned}
$$

If we set $n = 1$ we clearly recover the results from class.

Problem 4. Quantum-Classical Correspondence

Solution 4.0. I told everyone that they did not need to do this problem – there was a logical flaw in it. I've included the solutions to show how if you ignore part of my advice you can get the right answer. Apologies!

Here we will demonstrate that the quantum mechanical statistical mechanics of a single spin-1/2 is equivalent to the classical statistical mechanics of a 1-D Ising chain. This is a special case of the general result that the statistical mechanics of a d-dimensional quantum model is equivalent to the statistical mechanics of a d+1 dimensional classical model.

We will discuss the general argument later in class, but this example illustrates the basic idea.

Throughout we will consider a single spin with Hamiltonian

$$\hat{H} = E_0 - \frac{\Delta}{2}\hat{\sigma}_x - h\hat{\sigma}_z.$$

where $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the standard Pauli matrices.

4.1. Diagonalize H, to find the energy eigenvalues E_α. Use the definition of the partition function,

$$Z = \sum_\alpha e^{-\beta E_\alpha}$$

to show that

$$F = E_0 - T\log\left(2\cosh\beta\sqrt{(\Delta/2)^2 + h}\right).$$

Solution 4.1. The eigenvalues are $E_\pm = E_0 + \pm\sqrt{(\Delta/2)^2 + h^2}$, which gives the desired expression.

4.2. We will now do the same calculation in a different basis. In an arbitrary basis the partition function is

$$Z = \sum_i \langle i|e^{-\beta\hat{H}}|i\rangle.$$

We will work in the standard basis aligned with the $\hat{z}$ direction.

In this language we need to calculate the matrix

$$e^{-\beta\hat{H}} = e^{-\beta\left(E_0 - \frac{\Delta}{2}\hat{\sigma}_x - h\hat{\sigma}_z\right)}.$$

We will use a simple trick to calculate this exponential of a matrix. We are going to write

$$e^{-\beta\hat{H}} = T^N,$$

where $T = e^{-\beta\hat{H}/N}$. Show that in the limit of large N,

$$T \approx \begin{pmatrix} e^{-\beta(E_0-h)/N} & \beta\Delta/2N \\ \beta\Delta/2N & e^{-\beta(E_0+h)/N} \end{pmatrix}$$

Solution 4.2.

$$\begin{aligned} T &\approx 1 - \beta\hat{H}/N = \begin{pmatrix} 1 - \frac{\beta}{N}(E_0 - h) & \frac{\beta}{2N}\Delta \\ \frac{\beta}{2N}\Delta & 1 - \frac{\beta}{N}(E_0 + h) \end{pmatrix} \\ &\approx \begin{pmatrix} e^{-\beta(E_0-h)/N} & \beta\Delta/2N \\ \beta\Delta/2N & e^{-\beta(E_0+h)/N} \end{pmatrix} \end{aligned}$$

4.3. We now note that if we let $E_0 = (N/\beta)\log(\beta\Delta/2N)$, then T has the same form as the transfer matrix for the 1-D classical Ising model. Using this correspondence write down a classical model which is equivalent to the quantum problem of a single spin.

Verify that this classical system has the same free energy as the single quantum spin.

Solution 4.3. It seems my advice of setting $E_0 = (N/\beta)\log(\beta\Delta/2N)$ causes all sorts of probems. To get things to work out, one instead needs to add a constant term to our Ising model.
With this extra term, the equivalent Ising model is

$$H = \sum_i (A - J\sigma_i\sigma_{i+1}) - \tilde{h}\sum_i \sigma_i,$$

with transfer matrix

$$T = e^{-\beta(A-J)} \begin{pmatrix} e^{\beta\tilde{h}} & e^{-2\beta J} \\ e^{-2\beta J} & e^{-\beta\tilde{h}} \end{pmatrix}$$

Equating terms gives $e^{-2\beta J} = e^{\beta E_0/N}\beta\Delta/2N$, $\tilde{h} = h/N$, and $(A-J) = E_0/N$. Note that $N(A-J)$ is the energy of the ground state, so this last identification makes sense.
The eigenvalues of the transfer matrix are

$$\lambda_\pm = e^{-\beta(A-J)}\left[\cosh(\beta\tilde{h}) \pm \sqrt{\sinh^2(\beta\tilde{h}) + e^{-4\beta J}}\right],$$

giving a free energy

$$\begin{aligned} F &= -\frac{1}{\beta}\log(\lambda_+^N + \lambda_-^n) \\ &= N(A-J) - \frac{1}{\beta}\log\left[\left(\cosh(\beta\tilde{h}) + \sqrt{\sinh^2(\beta\tilde{h}) + e^{-4\beta J}}\right)^N + \left(\cosh(\beta\tilde{h}) - \sqrt{\sinh^2(\beta\tilde{h}) + e^{-4\beta J}}\right)^N\right] \end{aligned}$$

We now make the appropriate substitutions, keeping only the leading corrections in $1/N$,

$$\begin{aligned} F &= E_0 - \frac{1}{\beta}\log\left[\left(1 + \frac{\sqrt{\beta^2 h^2 + (\beta\Delta/2)^2}}{N}\right)^N + \left(1 + \frac{\sqrt{h^2 + (\Delta/2)^2}}{N}\right)^N\right] \\ &\approx E_0 - \frac{1}{\beta}\log(2\cosh\beta\sqrt{h^2 + (\Delta/2)^2}), \end{aligned}$$

where we have used that $(1 + x/N)^N \to e^x$.

This quantum-classical correspondence lets us either use our knowledge of classical thermodynamics to solve problems in quantum thermodynamics, or it allows us to solve thermodynamics problems by studying quantum systems. For example, Onsager's famous solution of the 2-D Ising model simply involves mapping the 2-D Ising model onto a 1-D quantum mechanics problem. The quantum problem turns out to just involve non-interacting fermions, and is trivially solved.